DISCRETE MATH 37181 TUTORIAL WORKSHEET 11

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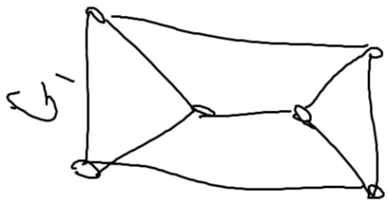
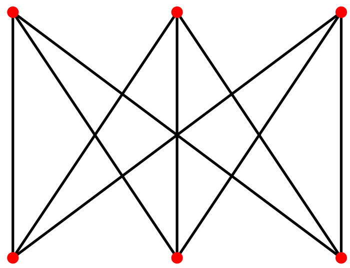
Instructions. Complete these problems in groups of 3-4 at the whiteboard. Partial solutions at the end of the PDF.

1. For each of the following graphs
   (a) find an Euler circuit or explain why none exists
   (b) find a Hamilton cycle or explain why none exists





2. Either show that the two graphs in the previous question are isomorphic or explain why they are not isomorphic.

3. Draw a planar, simple connected graph that has a Hamilton cycle but no Euler path.

4. Consider the degree sequence 44222111111. Draw as many non-isomorphic trees with this degree sequence as you can.

5. Draw three non-isomorphic trees with degree sequence

$$3322111$$

6. Draw all non-isomorphic trees with $n \leqslant 5$ vertices.

7. The number of non-isomorphic trees with 5 vertices is

   A. 1 **B**. 2 **C**. 3 **D**. 4 **E**. none of the above.

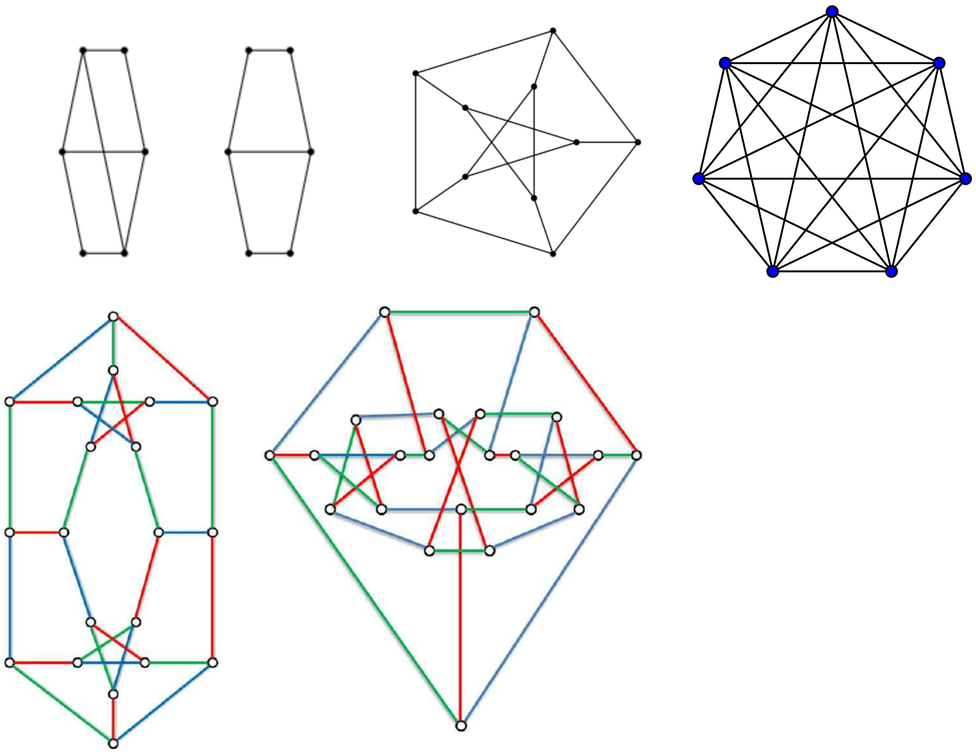
Date: Week 11 workshop (Wednesday 11, Thursday 12, Friday 13 May).

8. The number of non-isomorphic trees with 6 vertices is

A. 1 **C**. 3 **E**. none of the above.

B. 2 **D**. 4

9. Find spanning trees for each of these graphs:



10. How many distinct (non-isomorphic) spanning trees does K_n have? [1]

11. Prove that for a tree with n vertices with some vertex of degree $k < n$, the longest simple path has length at most $n - k + 1$. Give examples to show this bound is achieved sometimes, but not all of the time.

12. (a) Under what conditions does K_n have a Hamilton cycle?

(b) The following theorem gives an easy (polynomial time) method to say "Yes" when a graph satisfies the conditions. In some sense it says if a graph is pretty close to being a complete graph, then it will have a Hamilton cycle. Try to prove the theorem.

Theorem 1. *If $G = (V, E)$ is a simple graph (no loops or multi-edges) with $|V| = n \geqslant 3$ vertices, and each pair of vertices $a, b \in V$ with a, b distinct and non-adjacent satisfies*

$$\deg(a) + \deg(b) \geqslant n,$$

then G has a Hamilton cycle.

END OF WORKSHEET

[1] Try $n = 1, 2, 3, 4$ first, to see a pattern. The final answer will be some formula involving n, and probably you will have to guess it then prove by induction.

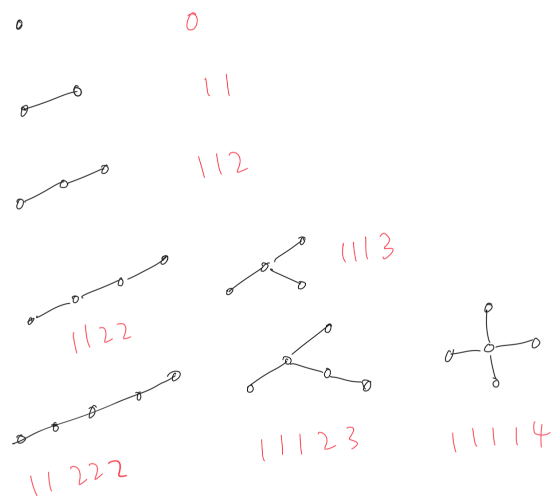
Brief solutions:

2. They are not isomorphic: one has a simple circuit of length 3 and the other doesn't (all circuits in $K_{3,3}$ have even length because it is complete bipartite (and the shortest simple circuit should be preserved by isomorphism). Another reason: G_1 is planar and $K_{3,3,}$ is not.

3. Anything with exactly one or > 2 vertices of odd degree, say draw a circle with some dots on it (so it has a Hamilton cycle) then put a few extra edges across, making sure to keep it planar and simple.

5. Degree sum is odd. No graphs (or trees) exist.

6.



Must be connected. Non-isomorphic is shown by giving the degree sequences (they are all different).

11. $n = 1$: single vertex of degree 0, bound is $n - k + 1 = 1 + 1 = 2$ and the longest simple path has length 0. For $n = 2$, only tree is a single edge, both vertices degree 1, so $n - k + 1 = 2 - 1 + 1 = 2$ and longest simple path has length 1. To prove the bound always holds, induction.

For $n > 2$ we always have a tree which is a path of length $n - 1$ (degree sequence $1122 \dots 2$) so the bound is sharp for these trees, and there is always a tree with degree sequence $11 \dots 1(n-1)$ which looks like a star, so the bound is not sharp for these trees.

12. (a) K_n for $n \geqslant 3$ yes: say edges are labeled $\{i, j\}$ for $1 \leqslant i, j \leqslant n$, then the path

$$\{1, 2\}, \{2, 3\}, \dots, \{n, 1\}$$

is a Hamilton cycle. For $n = 1, 2$, what did your group/classroom decide? Is the empty path a cycle that visits every vertex in K_1? Can a cycle cross the same edge twice for K_2?

(b) *Proof sketch.* For contradiction, suppose the theorem is false and there is some graph satisfying the conditions which does not have a Hamilton cycle. Let G be such an example which has the *smallest* number of vertices. (WOP used here). Say this G has n vertices. (We know $n \geqslant 3$).

Now out of all the graphs that have n vertices and satisfy the conditions which does not have a Hamilton cycle, choose one that has the *maximum* number of edges.

What we have done is narrowed down (using WOP) to having a fairly concrete example to play with. Call this graph $G_{\max}$.

We know that if we were to *add* an edge to this graph, it would have more edges than our maximal example, so it would not be a counterexample to the theorem.
See where you can go from here. (Note, $G_{\max}$ is not K_n since K_n has a Hamilton cycle by part (a).) □