DISCRETE MATH 37181 TUTORIAL WORKSHEET 2

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INSTRUCTIONS. Complete these problems in groups of 3-4 on the whiteboard (real or virtual). Partial solutions at the end of the PDF. Your tutor will come around to each group throughout the tutorial and check your progress/ask questions as you work together on the worksheet.

Recall:

Definition 1. Let $a, b \in \mathbb{Z}$. We say a *divides* b if $\exists s \in \mathbb{Z}$ such that $b = as$.

For example, 3 divides -18 since there exists -6 such that $-18 = 3 \cdot (-6)$, and 3 does not divide 14 since for all $s \in \mathbb{Z}$ $14 \neq 3s$.

Notation: $a|b$.

Definition 2. $p \in \mathbb{N}, p > 1$ is called *prime* if $s \in \mathbb{N}$ divides p implies $s = 1$ or $s = p$.

1. Let the universe of discourse be all people, $C(x) =$"x likes cheese" and $D(x) =$"x cannot eat dairy foods".
   (a) The logical statement $\forall x[D(x) \rightarrow \neg C(x)]$ has the following meaning in English:

   A. All people either cannot eat dairy or cannot eat cheese.

   B. If a person cannot eat dairy then they cannot eat cheese.

   C. For all people, if the person eats cheese then they like cheese.

   D. If a person cannot eat dairy then they don't like cheese.

   E. None of the above.

   (b) What is the *negation* of the statement $\forall x[D(x) \rightarrow \neg C(x)]$? Express in symbols, simplify then write in English.

2. Using a *direct* proof, prove that if $k, l \in \mathbb{Z}$ have the same parity[1] then $k + l$ is even.

3. Consider the following statements (don't try to prove/disprove them yet!):
   (i) For all $x \in \mathbb{Z}$, if 3 divides x then 3 divides x^2.

   (ii) For all $x \in \mathbb{Z}$, if 3 divides x^2 then 3 divides x.

   (iii) $\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}\ [(xy > 0) \vee (x = 0) \vee (y = 0)]$

Date: Week 2 workshop (Wednesday 11, Thursday 12, Friday 13 August).

[1]parity means being even or odd. So your proof can be Case 1: both even. Case 2: both odd.

(iv) For all $n \in \mathbb{Z}$, n is divisible by some prime number p.

(v) For all $n \in \mathbb{N}$, $n^2 + 5n + 5$ is prime.

(vi) For all $n \in \mathbb{Z}$, if $n > 1$ then n is divisible by some prime number p.

(a) Read each of the statements (i)–(vi) together as a group, and discuss between the group if you think they are true or false. Write down your votes for *true*, *false* on the whiteboard.You don't have to agree, at the start. Don't try to prove them yet.

(b) Next to any votes for *true*, write *direct, contrapositive* or *contradiction* as the approach you think will be the best to prove it. Next to any votes for *false*, write down some ideas for a counterexample.

(c) Now, either prove using either a *direct, contrapositive* or *contradiction* proof, or show that it is false by giving a counterexample.

4. Write the output of the following (somewhat useless) code: [2]

```
int j = 9;
for(int i=0; i<10; i++)
   j--;
print j
```

5. Determine whether or not the following statements is true or false. If false, give a counterexample. Assume the universe of discourse to be $\mathbb{Z}$.

   (a) $\forall x \exists y \exists z \; [x = 7y + 5z]$ (b) $\forall x \exists y \exists z \; [x = 4y + 6z]$

6. Use the table on your formula sheet (last page of this worksheet) to *simplify* the expression $\neg(q \to (p \vee (q \wedge r)))$. Check your answer by drawing a truth table.

7. Is the formula $(p \vee q \vee r) \wedge (p \vee q \vee \neg r) \wedge (p \vee \neg q \vee r) \wedge (p \vee \neg q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (\neg p \vee q \vee \neg r) \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ satisfiable?

8. (Bonus question) Prove that $\sqrt{2}$ is not rational by following these steps.

   (a) Suppose (for contradiction) that $\sqrt{2}$ is rational. Then $\exists a, b \in \mathbb{Z}$ such that ...

   (b) If a, b have a common factor, we can divide top and bottom to get a smaller pair of numbers. So assume (here is the thing we will contradict later) that ...

   (c) Now multiply both sides of your answer to (a) by b then square both sides to get

   (d) It follows that a^2 is even, so by Lemma 4.3 of the lecture notes this implies a is even. Then $2b^2 = a^2 = (2s)^2 = 4s^2$ so divide both sides by 2 to get ...

   (e) What is the contradiction?

[2] `++` means increment by 1, `--` means decrement by 1

Sample formula sheet:

Some tautologies with names:

logic rule (tautology)	name
$\neg(\neg p) \leftrightarrow p$	double negative
$\neg(p \vee q) \leftrightarrow \neg p \wedge \neg q$	DeMorgan
$\neg(p \wedge q) \leftrightarrow \neg p \vee \neg q$	
$p \vee q \leftrightarrow q \vee p$	commutative
$p \wedge q \leftrightarrow q \wedge p$	
$p \vee (q \vee r) \leftrightarrow (p \vee q) \vee r$	associative
$p \wedge (q \wedge r) \leftrightarrow (p \wedge q) \wedge r$	
$p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r)$	distributive
$p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$	
$p \vee p \leftrightarrow p$	idempotent
$p \wedge p \leftrightarrow p$	
$p \vee F \leftrightarrow p$	identity
$p \wedge T \leftrightarrow p$	
$p \vee (p \wedge q) \leftrightarrow p$	absorption
$p \wedge (p \vee q) \leftrightarrow p$	
$p \to q \leftrightarrow \neg p \vee q$	useful one
$p \vee \neg p \leftrightarrow T$	inverse
$p \wedge \neg p \leftrightarrow F$	
$p \vee T \leftrightarrow T$	domination
$p \wedge F \leftrightarrow F$	
$p \to q \leftrightarrow \neg q \to \neg p$	contrapositive
$(\neg p \to F) \to p$	contradiction

Definition. Let $a, b \in \mathbb{Z}$. Then a *divides* b if $\exists s \in \mathbb{Z}$ such that $b = as$. Notation: $a|b$.

Definition. Let $p \in \mathbb{N}, p > 1$. Then p is called *prime* if $s \in \mathbb{N}$ divides p implies $s = 1$ or $s = p$.

For the Learning Progress Check (LPC), you are allowed to use a formula sheet that you create for yourself. You can use this as a start, and keep adding to it each week. A formula sheet is much more efficient during a test that scanning through lecture notes etc. LPC1 Question 1 asked you to prove (with truth tables) double negative, distributive and adsorption.

Brief solutions:

1. D. "If a person" in English means "if **any** person", so the $\forall$ is hidden here. Literally, for all people, if a person cannot eat dairy then that person does not like cheese.

2. *Proof.* We will prove two cases: k, l both even, and k, l both odd.

   If k, l both even then $\exists b, c \in \mathbb{Z}$ with $k = 2b, l = 2c$. Then $k + l = 2b + 2c = 2(b + c)$ is even since $b + c \in \mathbb{Z}$.

   If k, l both odd then $\exists b, c \in \mathbb{Z}$ with $k = 2b + 1, l = 2c + 1$. Then $k + l = 2b + 1 + 2c + 1 = 2(b + c) + 2 = 2(b + c + 1)$ is even since $b + c + 1 \in \mathbb{Z}$. $\square$

3. (i) *Proof.* Direct. By definition, if $3|x$ then $x = 3c$ for some $c \in \mathbb{Z}$ so $x^2 = (3c)^2 = 9c^2 = 3(3c^2)$ is divisible by 3. $\square$

   (ii) *Proof.* Contrapositive. Suppose 3 does not divide x. Then $x = 3c+i$ for $c \in Z$ and $i \in \{1, 2\}$ (that is, $i = 1$ or 2). Then $x^2 = (3c + i) = 9c^2 + 6ci + i^2 = 3(3c^2 + 2c) + i^2$.

   If $i = 1$ then $i^2 = 1$ and $x^2 = 3(3c^2 + 2c) + 1$ is not divisible by 3. If $i = 2$ then $i^2 = 4$ and $x^2 = 3(3c^2 + 2c + 1) + 1$is not divisible by 3. $\square$

   (iii) False, there exist $x = -1, y = 1$ such that $xy \leqslant 0$ and $x \neq 0$ and $y \neq 0$ (*i.e.* the negation of the statement is true).

   (iv) False, $n = 1$ is only divisible by ± 1. Also -1 is a counterexample.

   (v) False, if $n = 5$ then $n^2 + 5n + 5 = 55 = 5.10$ is not prime (has two divisors $\neq 1$).

   (vi) This is true. Here is a tricky proof using WOP. We will learn how to use WOP more in a future lectures, so this is an advanced proof for today. Some tutorial questions are designed to get your groups talking and arguing, so don't worry if you all decide you don't know how to prove it and move on. This is a key part of learning proofs.

   Proof: Given $n > 1$, let $D = \{s \in \mathbb{Z} \mid s|n \wedge s > 1\}$ be the set of integers that divide n. Since $n > 1$ and $n|n$, D is non-empty, and also $D \subseteq \mathbb{N}$. So I can use the WPO: D has a first element p. If p is not prime then $p = ab$ for two integers $a, b > 1$. Then $n = sp = sab$ so $a|n$ and $a > 1$ so $a \in D$, but a is smaller than p (since $b > 1$) contradicting that p was first. Thus p is prime. $\square$

4. It just prints once after it finishes the loop. Here is a table showing the value of i, j during the computation.

i	j
	9
0	8
1	7
2	6
3	5
4	4
5	3
6	2
7	1
8	0
9	-1

So the output is -1.

This question is here for the non-computer scientists in the class. Don't be afraid of the syntax, just ask someone or look it up. This is C or C++ code that I found somewhere. `i++` means decrease the value of `i` by 1 and `j--`, well, guess. Obviously this is not a programming course (you should do one, or several!) but later in this course we will apply mathematics to understand computer code, so that's why we want to get familiar with pseudocode here. For the computer programmers, we want you to be able to trace what the code does (draw a table like the one here, on paper) not just hack without thinking what steps are happening.

5. (a) If $x = 1$ we need $7y + 5z = 1$ so $y = \frac{1-5z}{7}$ top should be a multiple of 7, yes $z = 3$ gives $1 - 15 = -14$ so $y = -2, z = 3$. For any other value of x, we just multiply everything by x: $y = -2x, z = 3x$. So its true.

   (b) This time the right hand side is always even, since we have 4 and 6, so it is false for $x = 1$.

6. Simplify means try to get rid of nested brackets, and negations out the front of brackets, etc.

$$
\begin{array}{lll}
 & \neg(q \to (p \vee (q \wedge r))) & \\
\leftrightarrow & \neg(\neg q \vee (p \vee (q \wedge r))) & \text{using useful fact } a \to b \leftrightarrow \neg a \vee b \\
\leftrightarrow & \neg\neg q \wedge \neg(p \vee (q \wedge r)) & \text{using De Morgan} \\
\leftrightarrow & q \wedge (\neg p \wedge \neg(q \wedge r)) & \text{using double negative and De Morgan} \\
\leftrightarrow & q \wedge (\neg p \wedge (\neg q \vee \neg r)) & \text{using De Morgan} \\
\leftrightarrow & \neg p \wedge (q \wedge (\neg q \vee \neg r)) & \text{using commutative, associative, commutative} \\
\leftrightarrow & \neg p \wedge ((q \wedge \neg q) \vee (q \wedge \neg r)) & \text{distributive} \\
\leftrightarrow & \neg p \wedge (F \vee (q \wedge \neg r)) & \text{useful fact } a \wedge \neg a \text{ is always false} \\
\leftrightarrow & \neg p \wedge q \wedge \neg r & \text{identity}
\end{array}
$$

Check we didn't make a mistake by drawing the truth table for both statements:

p	q	r	$\neg(q \to (p \vee (q \wedge r)))$	$\neg p \wedge q \wedge \neg r$
1	1	1		
1	1	0		
1	0	1		
1	0	0		
0	1	1		
0	1	0		
0	0	1		
0	0	0		

or by saying "$\neg p \wedge q \wedge \neg r$ is true only when $p = 0, q = 1, r = 0$, and $\neg(q \to (p \vee (q \wedge r)))$ is true only when $q \to (p \vee (q \wedge r))$ is false, which happens only when $q = 1$ and $p \vee (q \wedge r) = 0$ (implication only false when $1 \to 0$) which happens only when $p = 0$ and $q \vee r = 0$ which, since $q = 1$ already, means $r = 0$". Hmm I think just drawing the truth table might be easier than that!

7. No. To prove it, must check every possible value assignment (2^3 of them).

8. (a) Suppose (for contradiction) that $\sqrt{2}$ is rational. Then $\exists a, b \in \mathbb{Z}$ such that $\sqrt{2} = \frac{a}{b}$.

   (b) If a, b have a common factor, we can divide top and bottom to get a smaller pair of numbers. So assume (here is the thing we will contradict later) that the *greatest common divisor* [3] of a and b is 1.

[3] $\gcd(a, b) = 1$

(c) Now multiply both sides of your answer to (a) by b then square both sides to get $b\sqrt{2} = a$, $2b^2 = a^2$.

(d) It follows that a^2 is even, so by Lemma 4.3 of the lecture notes this implies a is even. Then $2b^2 = a^2 = (2s)^2 = 4s^2$ so divide both sides by 2 to get $b^2 = 2s^2$, which means b^2 is even so by Lemma 4.3 b is even as well.

(e) What is the contradiction? We assumed that we had reduced any common factors in a, b at the start, that is, $\gcd(a, b) = 1$ but now 2 divides both a and b, which is a contradicition. Therefore $\sqrt{2}$ is NOT rational.