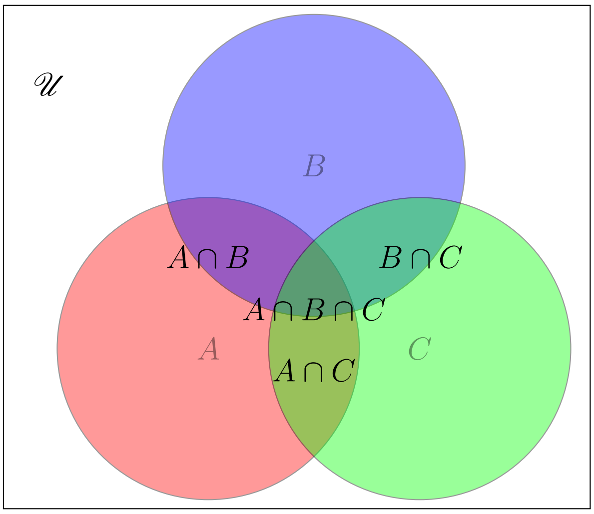
DISCRETE MATH 37181 TUTORIAL WORKSHEET 3

Instructions. Complete these problems in groups of 3-4 on the whiteboard (real or virtual). Partial solutions at the end of the PDF. Your tutor will come around to each group throughout the tutorial and check your progress/ask questions as you work together on the worksheet.

1. Use the Euclidean algorithm to find

   (a) $\gcd(29, 287)$ (b) $\gcd(28, 240)$ (c) $\gcd(233, 377)$

2. A *Venn diagram* is a picture (usually 2-D) of how some sets interrelate. Draw $\mathscr{U}$ as a box and sets $A, B, C, \ldots$ as circles with overlaps. If $a \in A$ then we draw it inside the circle for A, and if $b \notin A$ we draw it outside the circle.



   (a) Use a Venn diagram to decide if you believe the statement
   $$A \cap (B \cup C) = (A \cap B) \cup C.$$

   (b) If your picture in part (a) convinced you the statement is wrong, give an example of (small, easy) sets A, B, C for which it is false. If your picture convinced you it was correct, prove with a LHS,RHS argument.

   (c) Use a Venn diagram to decide if you believe the statement
   $$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$
   Prove or disprove it.

   (d) How many different regions are there in a Venn diagram for 3 sets? How many for 4 sets? How many for n sets?

   (e) How many rows are there in a truth table for 3 statements p, q, r? How many for 4 statements? How many for n statements?

Date: Week 3 workshop (Wednesday 9, Thursday 10, Friday 11 March).

3. Consider Table 1 below.
   (a) Fill in the missing parts of the rules.
   (b) Choose **one** of the statements about sets in column 1 of the table and prove it (use the method: let $x \in$ LHS, then $x \in$ RHS, then vice versa).
   (c) Use the rules in the table (column 1) to simplify
   $$\overline{\overline{(A \cup B)} \cap C \cup \overline{B}}.$$
   Write the name of each rule used next to each step.
   (d) Use the rules in the table (column 2) to simplify
   $$\neg(\neg((p \vee q) \wedge r) \vee \neg q)$$

4. Write out the *power set* $\mathscr{P}(A)$ for the following sets. What is the *size* of each set $\mathscr{P}(A)$?
   (a) $A = \emptyset$
   (b) $A = \{1\}$
   (c) $A = \{1, 2\}$
   (d) $A = \{1, 2, 3\}$
   (e) $A = \{1, 2, 3, 4\}$
   (f) $A = \{1, 2, 3, 4, 5\}$

5. Let A, B, C be sets in some arbitrary universal set $\mathscr{U}$. Show that
   $$(A \cup C) \cup (B \cap C) = (A \cap B) \cup C$$
   is false in general by giving an example using $\mathscr{U} = \{1, 2, 3, 4, 5\}$.

6. (1 mark) Define $P(S)$ to be the statement about sets: "$S \notin S$". Then define
   $$\mathscr{T} = \{S \mid \neg P(S)\}.$$
   In English, which of the following best describes $\mathscr{T}$?
   A. $\mathscr{T}$ is the set of all sets which are non-empty.
   B. $\mathscr{T}$ is the set of all sets which include themselves as elements.
   C. $\mathscr{T}$ is the set of all sets which do not include themselves as elements.
   D. $\mathscr{T}$ is the set of all sets which have a first element.
   E. $\mathscr{T}$ is the set of all sets which do not have a first element.

Set law	logic rule	name
$\overline{\overline{A}} = A$	$\neg(\neg p) \leftrightarrow p$	double negative/ double complement
$\overline{A \cup B} = \overline{A} \cap \overline{B}$		DeMorgan
$\overline{A \cap B} = \overline{A} \cup \overline{B}$		
$A \cup B = B \cup A$		commutative
$A \cap B = B \cap A$		
$A \cup (B \cup C) = (A \cup B) \cup C$		associative
$A \cap (B \cap C) =$		
$A \cup (B \cap C) =$	$p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r)$	distributive
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$		
$A \cup A = A$	$p \vee p \leftrightarrow p$	idempotent
$A \cap A =$	$p \wedge p \leftrightarrow$	
$A \cup \emptyset = A$	$p \vee F \leftrightarrow p$	identity
$A \cap \mathscr{U} =$	$p \wedge T \leftrightarrow$	
$A \cup \overline{A} = \mathscr{U}$	$p \vee \neg p \leftrightarrow$	
$A \cap \overline{A} = \emptyset$	$p \wedge \neg p \leftrightarrow$	
$A \cup (A \cap B) = A$		absorption
$A \cap (A \cup B) = A$		

TABLE 1. Sets versus logic. In this table p, q, r are statements in some universe of discourse, and A, B, C are sets in some universal set $\mathscr{U}$.

Brief solutions:

1. (a)

$$287 = 9 \times 29 + 26$$
$$29 = 1 \times 26 + 3$$
$$26 = 8 \times 3 + 2$$
$$3 = 1 \times 2 + 1$$
$$2 = 1.2 + 0$$ [1]

so $\gcd(29, 287) = 1$

(b)

$$240 = 8 \times 28 + 16$$
$$28 = 1 \times 16 + 12$$
$$16 = 1 \times 12 + 4$$
$$12 = 3 \times 4 + 0$$

so $\gcd(28, 240) = 4$

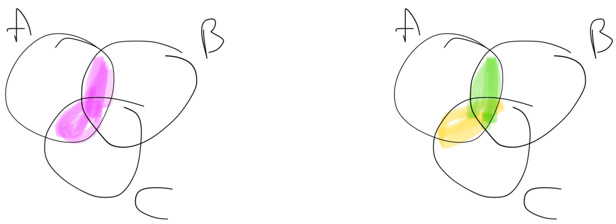
(c) These are consecutive *Fibonacci numbers*, so you will always have $q_i = 1$ and r_i the next smallest Fibonacci number (it can be shown that this maximises the most number of steps on any input of the algorithm. More to say on this later).

2. (a)

Not same.

(b) Let $A = \emptyset, B = \emptyset, C = \{1\}$. Then $A \cap (B \cup C) = \emptyset$ and $(A \cap B) \cup C = \{1\}$, so in general the statement is false because there exist sets for which it fails.

(c)



Yes both regions are the same, so I believe it. To prove it, needs a "let $x \in$ LHS, let $x \in$ RHS" proof.

[1](but you don't really need this)

Proof. Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. Either $x \in B$ or not. If $x \in B$ then $x \in (A \cap B)$, else $x \notin B$ so $x \in C$ since $x \in B \cup C$, so $x \in A \cap C$. Thus $x \in A \cap B$ or $x \in A \cap C$, so $x \in (A \cap B) \cup (A \cap C)$ and

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C).$$

Now suppose $x \in (A \cap B) \cup (A \cap C)$. So either $x \in A \cap B$ or $x \in A \cap C$. In the first case, $x \in A$ and $x \in B$ so $x \in B \cup C$ so $x \in A \cap (B \cup C)$. In the other case, $x \in A$ and $x \in C$ so $x \in C \cup B$ so $x \in A \cap (B \cup C)$. Thus

$$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C).$$

Thus since each set contains the other, they are equal [2]. □

(d) 3 sets: 8 or 2^3. 4 sets: for each set you are either in (1) or out (0). So there are 2^4 possible ways.

Guess that in general there are 2^n regions in a Venn diagram for n sets. (We need a different proof method to prove this – next week).

(e) Same answer as (d).

3. (b) I will do $A \cap B = B \cap A$. Proof: Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. Then $x \in B$ and $x \in A$ is also true. So $x \in B \cap A$. So $A \cap B \subseteq B \cap A$.

   Now suppose $x \in B \cap A$. Then $x \in B$ and $x \in A$. Then $x \in A$ and $x \in B$ is also true. So $x \in A \cap B$. So $B \cap A \subseteq A \cap B$.

   It follows that since both sets contain each other, they are equal.

   (c) Part (b) shows us that to write out proofs for the whole table would be really tedious, but assuming someone has already written out proofs for all of them, we can use them to prove other statements as follows:

$$\begin{array}{rcll}
\overline{\overline{(A \cup B) \cap C} \cup \overline{B}} & = & \overline{\overline{(A \cup B) \cap C}} \cap \overline{\overline{B}} & \text{DeMorgan} \\
 & = & \overline{\overline{(A \cup B) \cap C}} \cap B & \text{double negation (double complement)} \\
 & = & ((A \cup B) \cap C) \cap B & \text{double negation} \\
 & = & (A \cup B) \cap (C \cap B) & \text{associative} \\
 & = & (A \cup B) \cap (B \cap C) & \text{commutative} \\
 & = & ((A \cup B) \cap B) \cap C & \text{associative} \\
 & = & (B) \cap C & \text{absorption} \\
 & = & B \cap C &
\end{array}$$

   Check with a Venn diagram.

[2] $X \subseteq Y \wedge Y \subseteq X \leftrightarrow X = Y$ by definition of set.

(d) Same steps

$$\begin{array}{rcll} \neg[\neg[(p\vee q)\wedge r]\vee\neg q] & \leftrightarrow & \neg\neg[(p\vee q)\wedge r]\wedge\neg\neg q & \text{DeMorgan} \\ & \leftrightarrow & [(p\vee q)\wedge r]\wedge q & \text{double negation} \\ & \leftrightarrow & (p\vee q)\wedge(r\wedge q) & \text{associative} \\ & \leftrightarrow & (p\vee q)\wedge(q\wedge r) & \text{commutative} \\ & \leftrightarrow & ((p\vee q)\wedge q)\wedge r & \text{associative} \\ & \leftrightarrow & q\wedge r & \text{absorption} \end{array}$$

Check with a truth table.

4. (a) $\mathscr{P}(\emptyset)=\{\emptyset\}$ contains 1 thing.
   (b) $\mathscr{P}(\{1\})=\{\emptyset,\{1\}\}$ contains 2 things.
   (c) $\mathscr{P}(\{1,2\})=\{\emptyset,\{1\},\{2\},\{1,2\}\}$ contains 4 things.
   (d) $\mathscr{P}(\{1,2,3\})$ contains 8 things.
   (e) $\mathscr{P}(\{1,2,3,4\})$ contains $16=2^4$ things.

6. B: $\neg(S\notin S)$ is the same as $S\in S$.