DISCRETE MATH 37181 TUTORIAL WORKSHEET 4

INSTRUCTIONS. Complete these problems in groups of 3-4 at the whiteboard. Partial solutions at the end of the PDF.

1. Prove (use induction) that for all $n \in \mathbb{N}_+$
$$2+4+6+8+\cdots+(2n) = n^2+n$$

2. Prove (use induction) that for all $n \in \mathbb{N}_+$
$$2+7+12+17+22+\cdots+(5n-3) = \frac{n(5n-1)}{2}$$

3. Prove that 6 divides n^3+5n for all $n \in \mathbb{N}$.

4. Let $P(n)$ be the statement that n^2+5n+1 is even.
   (a) Prove that $P(n)$ implies $P(n+1)$ for any $n > 0$.
   (b) For which values of n is $P(n)$ actually true?
   (c) What is the *moral* of this exercise?

5. Let $c \in \mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$. Prove that $\exists k \in \mathbb{N}$ such that for all $n \geqslant k$, $n! > c^n$. [1] [2]

6. Prove or disprove:
$$\forall n \in \mathbb{N}, \quad n^3+4n \equiv 0 \mod 5$$

7. Prove or disprove: For all $n \in \mathbb{N}_+$,
$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$$

8. Find an expression (which does not use $+\cdots+$ or $\sum$) for the function `unknown` computed by the following code:

```
int unknown(int n)
{
if n=1 return 1
else return (unknown(n-1)+(2*n-1)^2)
}
```

Date: Week 4 workshop (Wednesday 16, Thursday 17, Friday18 March).

[1]this means than *eventually* the function $n!$ will *dominate* or be bigger than c^n.

[2]Hint: If there are too many abstract things here, try doing the question for some *fixed* value of $c \in \mathbb{R}_+$ first. Eg $c = 5$. Once you understand how to do it for $c = 5$, try repeating it for a general positive real number c instead.

9. Prove that PMI implies WOP. Start with:

   Suppose (for contradiction) that WOP is false, so there is some non-empty set S which does not have a first element.

   Let $P(n)$ be the statement that "for all $i \in \mathbb{N}, i \leqslant n$, $i \notin S$".

10. Complete this definition: A *loop invariant* is ...

11. Consider this pseudocode.

```
    function mycode(int m, int n)
{
    while (m>= 0 and m<= 100)
       m:= m+1
       n:= n-1
    end while
    return n
 }
```

   (a) Show that "$m + n = 50$" is a loop invariant for the while loop.

   (b) Show that "$m + n$ is even" is a loop invariant for the while loop.

   (c) Show that "$m + n$ is odd" is a loop invariant for the while loop.

   (d) Why does it terminate? What is the output?

m	n	output
100	0	
-10	100	
50	0	
0	0	

   (You can use this table to try out the algorithm on different inputs, then go back and answer part (d).)

12. Recall problem 3 (vi) from worksheet 2. At that time you had just seen the WOP but not really used it. Now the following statement should be straightforward to prove using WOP [3].

   For all $n \in \mathbb{Z}$, if $n > 1$ then n is divisible by some prime number p.

   Hint: Given $n > 1$, let $D = \{s \in \mathbb{Z} \mid (s|n) \wedge (s > 1)\}$ be the set of integers that divide n. Explain why D is non-empty, and also why $D \subseteq \mathbb{N}$. Then show that the first element of D is a prime which divides n.

13. Prove that if $x > 0$ is any fixed real number, then

$$(1 + x)^n > 1 + nx$$

   for all $n \in \mathbb{N}, n \geqslant 2$.

14. Prove or disprove: for all $n \in \mathbb{N}$, $(3n + 1)7^n - 1$ is divisible by 9.

[3] or induction, which we did in the lecture, but for this question try to prove it using WOP.

Brief solutions:

1. Let $P(n)$ be the statement that
$$2+4+6+8+\cdots+(2n)=n^2+n.$$
Then $P(1)$ is true since LHS=2 and RHS$=1+1=2$.

Assume $P(k)$ is true. Then $P(k+1)$: LHS=
$$2+4+6+8+\cdots+(2k)+(2k+2)=k^2+k+(2k+2)$$
using the inductive assumption
$$\begin{aligned} &= k^2+3k+2=(k+1)(k+2)=(k+1)((k+1)+1)=(k+1)^2+(k+1)\\ &= \text{RHS}\end{aligned}$$
Then by PMI $P(n)$ is true for all $n\geqslant 1$.

2. Let $P(n)$ be the statement that
$$2+7+12+17+22+\cdots+(5n-3)=\frac{n(5n-1)}{2}.$$
Then $P(1)$ is true since LHS=2 and RHS$=\frac{1(4)}{2}=2$.

Assume $P(k)$ is true. Then $P(k+1)$:
$$\text{LHS}=2+7+12+17+22+\cdots+(5k-3)+(5k+5-3)=\frac{k(5k-1)}{2}+(5k+2)$$
using the inductive asssumption
$$=\frac{k(5k-1)}{2}+\frac{2(5k+2)}{2}=\frac{5k^2-k+10k+4}{2}=\frac{5k^2+9k+4}{2}$$
(secretly I will work out the RHS, then make them match up
(and write it nicely at the end))
$$\text{RHS}=\frac{(k+1)(5k+5-1)}{2}=\frac{(k+1)(5k+4)}{2}=\ \uparrow$$
Then by PMI $P(n)$ is true for all $n\geqslant 1$.

3. Let $P(n)$ be the statement that 6 divides n^3+5n, then $P(1)$ is true since $1^3+5.1=6$. Since we need to prove for all $n\in\mathbb{N}$, we must start at $n=0$: $P(0)$ is true since 6 divides $0=0^3+5.0$.

Assume for induction that $P(k)$ is true for $k\geqslant 0$, which means $k^3+5k=6s$ for some $s\in Z$. Then
$$(k+1)^3+5(k+1)=k^3+3k^2+3k+1+5k+5=k^3+3k^2+8k+6$$
Now use the inductive assumption
$$=k^3+5k+3k+3k^2+6=6s+3k(k+1)+6$$
so if we can show $3k(k+1)$ is a multiple of 6 we are done. It is because $k(k+1)$ is even, since either k is even or if not $k+1$ is even. Thus $P(k)$ implies $P(k+1)$.

Then by PMI $P(n)$ is true for all $n\geqslant 0$.

4. (a) Assuming $P(n)$ we have $n^2+5n+1=2s$ for some $s\in\mathbb{Z}$. Then
$P(n+1):(n+1)^2+5(n+1)+1=n^2+2n+1+5n+5+1=n^2+5n+1+2n+6=2s+2n+6$ is even, so $P(n)\to P(n+1)$.

   (b) None.

   (c) Moral: a proof by induction needs *both* $P(1)$ true (or some starting number) and $P(n)\to P(n+1)$ to work.

5. Let $P(n)$ be the statement that $n!>c^n$.

   Assume $P(k)$, then $P(k+1):(k+1)!=k!(k+1)>c^k(k+1)>c^k\cdot c$ if we assume that $k+1>c$ and $c^k\cdot c=c^{k+1}$, so $P(k)\to P(k+1)$ assuming we choose $k\geqslant c$.

   Base step: we need to find a value of k so that $P(k)$ is actually true (remember the moral learned from the last exercise!).

$$\begin{array}{lclllllllllll} k! & = & k & (k-1) & (k-2) & \dots & (\lceil c\rceil+1) & \lceil c\rceil & (\lceil c\rceil-1) & \dots & 3 & 2 & 1\\ c^k & = & c & c & c & \dots & c & c & c & \dots & c & c & c \end{array}$$

   Using this picture, I claim that if I choose $k=\lceil c\rceil^2$ (rounding up the value of $c\in\mathbb{R}^+$ to the nearest whole number) then $(\lceil c\rceil^2)!>c^{\lceil c\rceil^2}$.

   To prove this, I need another picture:

$$\begin{array}{lcllllllllll} k! & = & (\lceil c\rceil^2) & (\lceil c\rceil^2-1) & \dots & (\lceil c\rceil+1) & \lceil c\rceil & (\lceil c\rceil-1) & \dots & 3 & 2 & 1\\ c^k & = & c & c & \dots & c & c & c & \dots & c & c & c \end{array}$$

   (can you complete this proof?)

   Then by PMI $P(n)$ is true for all $n\geqslant\lceil c\rceil^2$.

6. False: true for 0,1 but for $n=2$ it is false.

7. This is true, and since its a statement about all n we use PMI.

   Let $P(n)$ be the statement that $\sum_{i=1}^{n}\frac{1}{i(i+1)}=\frac{n}{n+1}$. Then $P(1)$ is true since LHS$=\frac{1}{1.2}$ and RHS$=\frac{1}{1+1}=\frac{1}{2}$.

   Assume $P(k)$, then consider $P(k+1)$:

$$\begin{aligned} \textstyle\sum_{i=1}^{k+1}\frac{1}{i(i+1)} &= \textstyle\sum_{i=1}^{k}\frac{1}{i(i+1)}+\frac{1}{(k+1)(k+2)}\\ &= \frac{k}{k+1}+\frac{1}{(k+1)(k+2)}\\ &= \frac{k(k+2)}{(k+1)(k+2)}+\frac{1}{(k+1)(k+2)}\\ &= \frac{k^2+2k+1}{(k+1)(k+2)}\\ &= \frac{(k+1)(k+1)}{(k+1)(k+2)}\\ &= \frac{(k+1)}{(k+2)} \end{aligned}$$

   so $P(k+1)$ is true.

   So by PMI since $P(1)$ is true and $P(k)\to P(k+1)$ then $P(n)$ is true for all $n\in\mathbb{N}_+$.

8. $u(1) = 1, u(n) = u(n-1) + (2n-1)^2$, so
$$u(n) = 1 + 3^2 + 5^2 + 7^2 + \cdots + (2n-1)^2.$$
If we want a formula that doesn't involve a sum, guess a formula for $u(n)$ by computing small values, then use induction to prove it.
Solution:
$$1 + 3^2 + \cdots + (2n-1)^2 = \frac{n(2n+1)(2n-1)}{3}$$

9. Proof: Suppose (for contradiction) that WOP is false, so there is some non-empty set S which does not have a first element.

   Let $P(n)$ be the statement that "for all $i \in \mathbb{N}, i \leqslant n$, $i \notin S$".

   Then $P(0)$ is true because if 0 belongs to S, it would have to be the first element, and so it (and all $i \leqslant 0$, which is just $i = 0$) is not in S.

   If $P(k)$ is true then $0, 1, 2, \ldots, k \notin S$. If $k+1$ is in S then it would have to be the (recall we proved first elements, if they exist, must be unique) first element, but S does not have one, so $k+1$ is not in S, and so $P(k+1)$ is true.

   Then by PMI $P(n)$ is true for every $n \in \mathbb{N}$ which means that S is empty, which is a contradiction since we started by saying S is non-empty.

10. A *loop invariant* is a statement that if true before one iteration of the loop, remains true after one iteration.

11. (a) If $m + n = 50$, then in one iteration $m + n \to m + 1 + n - 1 = m + n = 50$ still.

    (b) It terminates because each iteration increases the value of m, and if m exceeds 100 then it will stop. If m starts of negative or greater than 100 then it will not enter the loop at all at stop straight away.
    Output: If $m < 0$ or $m > 100$ then it simply outputs the value n entered by the user.
    If $0 \leqslant m \leqslant 100$ then we subtract stuff from n. Let's use the loop invariant to help us. Call m', n' the new values of the variables at the end of the loop.
    At the end, $m' = 101$ (if $m = 100$ we perform one more iteration of the loop). We know that the sum $m + n$ remains invariant during the loop, so if originally $m + n = p$ then at the end $101 + n' = p$, so $n' = p - 101$. So I claim the output is $n + m - 101$.

m	n	output
100	0	-1
-10	100	100
50	0	-51
0	0	-101

13. Let $P(n)$ be the statement that $(1+x)^n > 1+nx$ where x is a fixed positive real number.

    Then $P(2)$ is true since $(1+x)^2 = 1+2x+x^2 > 1+2x$ since $x^2 > 0$.

    Assume $P(k)$ is true for $k \geqslant 2$. Then $P(k+1)$:

$$(1+x)^{k+1} = (1+x)^k(1+x) > (1+kx)(1+x) = 1+kx+x+kx^2 = 1+(k+1)x = kx^2 > 1+(k+1)x$$

    since $kx^2 > 0$ so $P(k) \to P(k+1)$.

    Then by PMI true for all $n \geqslant 2$.

14. Hint: You may need to prove another fact first: $7^m + 2$ is divisible by 3.