DISCRETE MATH 37181 TUTORIAL WORKSHEET 5

INSTRUCTIONS. Complete these problems in groups of 3-4 at the whiteboard. Partial solutions at the end of the PDF.

1. What does the following pseudocode algorithm do?

```
input: a real number x
if x>=0
    set i=0
    while x>0
        x--
        i++
    return i
else if x< 0
    set i=-1
    set y=-x
    while y>0
        y--
        i++
    return -i
```

2. (a) Give an example of a set A with 3 elements and a set B with 4 elements.
   (b) Give an example of a one-to-one function from A to B.
   (c) Givean example of an onto function from A to B.
   (d) How many different functions are there from A to B?
   (e) Give an example of a relation from A to B that is not a function.

3. Let $\mathscr{U}$ be a universal set. Let "$\perp$" be the relation on $\mathscr{U}$ defined by $A \perp B$ if $A \cap B = \emptyset$.
   (a) Is $\perp$ reflexive
   (b) Is $\perp$ symmetric?
   (c) Is $\perp$ antisymmetric?
   (d) Is $\perp$ transitive?

4. Prove that the set of all positive even integers is in bijection with $\mathbb{Z}$.
   (Hint: a guess for a function $f : \mathbb{Z} \to 2\mathbb{N}$ is given in the solutions. Use that if you want.)

Date: Week 5 workshop (Wednesday 23, Thursday 24, Friday 25 March).

5. If A is a set the notation $|A|$ means the number of elements in A. Let $|A| = m$ and $|B| = n$ with $m, n \in \mathbb{N}$.

   (a) What is $|A \times B|$?

   (b) How many functions are there from A to B?

   (c) How many relations are there from A to B? [1]

   (d) How many one-to-one functions are there from A to B?

6. Define a function $A : \mathbb{N}^2 \to \mathbb{N}$ using the following *recursive* definition.

$$\begin{array}{lcll} A(0,n) & = & n+1 & n \geqslant 0, \\ A(m,0) & = & A(m-1,1) & m > 0, \\ A(m,n) & = & A(m-1, A(m,n-1)) & m,n > 0. \end{array}$$

   (a) Compute $A(1,3)$.

   (b) Compute $A(2,3)$.

   (c) Prove that $A(1,n) = n+2$ for all $n \in \mathbb{N}$. [2]

   (d) Prove that $A(2,n) = 2n+3$ for all $n \in \mathbb{N}$.

   (e) Prove that $A(3,n) = 2^{n+3} - 3$ for all $n \in \mathbb{N}$.

   (f) Find (guess then prove) a formula for $A(4,n)$. [3]

7. Recall the definition of *equivalence relation*: reflexive, symmetric, transitive. If $\mathscr{R}$ is an equivalence relation on a set A, define the *equivalence class* of $a \in A$ to be a set $[a]_{\mathscr{R}} = \{b \in A \mid a\mathscr{R}b\}$.

   (a) Show that if $a \in A$ then $[a]_{\mathscr{R}}$ is not empty.

   (b) If $\mathscr{R}_6$ is the equivalence relation defined by "$a\mathscr{R}_6 b$ if 6 divides $(b-a)$", write down all the different equivalence classes of elements in $\mathbb{Z}$. [4]

   (c) Prove that for any equivalence class $\mathscr{R}$ on a set A, if $a, b \in A$ then either $[a]_{\mathscr{R}} = [b]_{\mathscr{R}}$ or $[a]_{\mathscr{R}} \cap [b]_{\mathscr{R}} = \emptyset$.

8. Let A be a set. Define a *partition* of A to be a collection of subsets $A_1, A_2, \ldots$ of A which satisfy the following two rules:

   - the union of all of the A_i is equal to A. That is, $\bigcup_i A_i = A$.
   - $A_i \cap A_j = \emptyset$ if $i \neq j$.

   Prove that if $\mathscr{R}$ is an equivalence relation on A then the set of equivalence classes of $\mathscr{R}$,

$$\{[a]_{\mathscr{R}} \mid a \in A\},$$

   is a partition of A.

[1] Recall: by induction, if A has size n then $\mathscr{P}(A)$ has size 2^n.

[2] obviously you will use PMI for (c)-(e).

[3] This part is for those students looking for challenge questions on the worksheets.

[4] Hint: start by writing $[0]_{\mathscr{R}_6}, [1]_{\mathscr{R}_6}, \ldots$.

Brief solutions:

1. Computes $\lceil x \rceil$. To prove it, show 1. it terminates 2. use a loop invariant

2. (a) $A = \{1, 2, 3\}, B = \{a, b, c, d\}$.
   (b) $f = \{(1, a), (2, b), (3, c)\}$
   (c) Does not exist.
   (d) 4^3
   (e) $\mathscr{R} = \emptyset$. Many other choices.

3. (a) Not reflexive since if $\mathscr{U}$ contains a set $A = \{1\}$ then $A \cap A \neq \emptyset$ so $A \not\perp A$.
   (b) Yes. Reason: rule from Worksheet 3 table: $A \cap B = B \cap A$ commutative law
   (c) Not antisymmetric, if $A = \{1\}$ and $B = \{2\}$ are in $\mathscr{U}$ then $A \perp B$ and $B \perp A$ but $A \neq B$.
   (d) Not transitive. $A = \{1\}$, $B = \{2\}$,$C = \{1\}$

4. Define the function $f : \mathbb{Z} \to \{2, 4, 6, 8, \ldots\}$ by
$$f(n) = \begin{cases} 4n + 2 & n \geqslant 0 \\ -4n & n < 0 \end{cases}$$
Hard part is to guess the right function. Easy part (if the guess is correct) is to prove it is 1-1 and onto.

5. (a) $|A \times B| = mn$
   (b) Each $f(a)$ can be one of $|B|$ things, so $|B|^{|A|} = n^m$.
   (c) A relation is any subset of $A \times B$ so we have $m.n$ pairs (elements) and we can choose any subset of them. A subset corresponds to choosing a black mark for "in" and a red mark for "out", so we need to choose from 2 options mn times, so this is 2^{mn}.
   (d) We can choose anything for $f(a_1)$, so $|B| - n$ choices, but then for $f(a_2)$ we cannot choose the same so we have $|B| - 1 = n - 1$ choices. Then $f(a_3)$ we have $n - 2$ choices, and so on:
$$n(n-1)(n-2)\ldots(n-m+1)$$

   Note: we don't ask how many onto functions there are, because that is pretty difficult to work out. Fun to try though.

6. (a)
$$\begin{aligned} A(1,3) &= A(0, A(1,2)) \\ &= A(1,2) + 1 \\ &= A(0, A(1,1)) + 1 \\ &= A(1,1) + 1 + 1 \\ &= A(0, A(1,0)) + 2 \\ &= A(1,0) + 3 \\ &= A(0,1) + 3 \\ &= 1 + 1 + 3 \\ &= 5 \end{aligned}$$

(b)

$$\begin{array}{rcll} A(2,3) & = & A(1,A(2,2)) & \\ & = & A(2,2)+2 & \text{using part (c)} \\ & = & A(1,A(2,1))+2 & \\ & = & A(2,1)+4 & \text{using part (c)} \\ & = & A(1,A(2,0))+2 & \\ & = & A(2,0)+6 & \text{using part (c)} \\ & = & A(1,1)+6 & \\ & = & 3+6=9 & \end{array}$$

(c) Induction. Let $P(n)$ be the statement that $A(1,n) = n+2$. Then $P(0)$ is true since $A(1,0) = A(0,1) = 1+1 = 2$ from the definition of A.

Assume $P(k)$ is true for some $k \in \mathbb{N}$. Then

$$A(1,k+1) = A(0,A(1,k)) = A(1,k)+1$$

by the definition of A

$$= (k+2)+1$$

using the fact that $P(k)$ is true

$$= k+3 = (k+1)+2$$

so $P(k+1)$ is true.

So by PMI $P(n)$ is true for all $n \in \mathbb{N}$.

(d) Induction. Let $P(n)$ be the statement that $A(2,n) = 3+2n$. Then $P(0)$ is true since

$$A(2,0) = A(1,1) = 1+2$$

using part (c)

$$= 3 = 3+2.0$$

Assume $P(k)$ is true for some $k \in \mathbb{N}$. Then

$$A(2,k+1) = A(1,A(2,k)) = A(2,k)+2$$

using part (c)

$$= (3+2k)+2$$

using the fact that $P(k)$ is true

$$= 3+2(k+1)$$

so $P(k+1)$ is true.

So by PMI $P(n)$ is true for all $n \in \mathbb{N}$.

(e) Induction. Let $P(n)$ be the statement that $A(3,n) = 2^{n+3}-3$. Then $P(0)$ is true since LHS$= A(3,0) = A(2,1) = 3+2 = 5$ using part (d) and RHS$= 2^{0+3}-3 = 2^3-3 = 8-3 = 5$.

Assume $P(k)$ is true for some $k \in \mathbb{N}$. Then

$$A(3,k+1) = A(2,A(3,k)) = 3+2A(3,k)$$

using part (d)

$$= 3+2(2^{k+3}-3)$$

using the fact that $P(k)$ is true

$$= 3+2^{k+4}-6 = 2^{(k+1)+3}-3$$

so $P(k+1)$ is true.

So by PMI $P(n)$ is true for all $n \in \mathbb{N}$.

(f) Exercise for students with too much time on their hards. In general as we increase m, n this function becomes massive, and will break your computer.

7. (a) Since $\mathscr{R}$ is reflexive, $a\mathscr{R}a$ so $a \in [a]_{\mathscr{R}}$.

   (b) $[0]_{\mathscr{R}_6} = \{\dots, -12, -6, 0, 6, 12, 18, \dots\}, [1]_{\mathscr{R}_6} = \{\dots, -11, -5, 1, 7, 13, 19, \dots\}, \dots$.

   (c) Let $a, b \in A$. Either $[a]_{\mathscr{R}} \cap [b]_{\mathscr{R}} = \emptyset$ or not, so suppose $[a]_{\mathscr{R}} \cap [b]_{\mathscr{R}} \neq \emptyset$. Then $\exists x \in [a]_{\mathscr{R}} \cap [b]_{\mathscr{R}}$ so $x\mathscr{R}a$ and $x\mathscr{R}b$ so by transitive property of equivalence relations $a\mathscr{R}b$. Now we must show the two sets $[a]_{\mathscr{R}}$ and $[b]_{\mathscr{R}}$ are equal. We do a formal set equality proof:
      (i) let $p \in LHS$ then $p \in RHS$,
      (ii) let $p \in RHS$ then $p \in LHS$.

8. We have the definition of partition above, so we just have to satisfy it.

$$\{[a]_{\mathscr{R}} \mid a \in A\}$$

is a collection of subsets of A.

The union of all of them is A since for each $a \in A, a \in [a]_{\mathscr{R}}$ so $A \subseteq \bigcup_{a \in A}[a]_{\mathscr{R}}$, and if $b \in [a]_{\mathscr{R}}$ then by definition $b \in A$ so $\bigcup_{a \in A}[a]_{\mathscr{R}} \subseteq A$ so the two sets are equal.

When we write $[a]_{\mathscr{R}}$ we mean the set $\{a, b, c, \dots\}$ where each $b, c, \dots$ in the set are $\mathscr{R}a$. That is, we don't write the same set twice, if a then we don't write both $[a]_{\mathscr{R}}$ and $[b]_{\mathscr{R}}$. Thus the set of all sets of the form $[a]_{\mathscr{R}}$ for all $a \in A$ includes a copy of each distinct set exactly once. By the previous question, when two of these equivalence classes are not equal, then they are disjoint. So we satisfy the second rule, and we have a partition.

Note that the *converse* to this question is also true: any time you have a partition of a set (as defined here) you can define a corresponding equivalence relation: x if x, y both lie in the same set of the partition A_i.