

# DISCRETE MATH 37181 TUTORIAL WORKSHEET 6

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INSTRUCTIONS. Complete these problems in groups of 3-4 at the whiteboard. Partial solutions at the end of the PDF. Spend about 40-50 minutes on Big O and 30-40 minutes on PHP, so that you cover both topics with your group.

Big-O form	Name
$O(1)$	constant
$O(\log_2 n)$	logarithmic
$O(n)$	linear
$O(n \log_2 n)$	$n \log n$ (sometimes called quasilinear)
$O(n^2)$	quadratic
$O(n^3)$	cubic
$O(n^p), p \in \mathbb{N}$	polynomial
$O(c^n), c > 1$	exponential
$O(n!)$	factorial

TABLE 1. Some standard functions for comparison.

Recall:

**Axiom 1** (Pigeonhole principle). *If  $m$  pigeons occupy  $n$  pigeonholes and  $m > n$  then some pigeonhole has at least two pigeons in it.*

To give a proof using it, you need to specify what are the  $m$  pigeons, what are the  $n$  pigeonholes, and some rule for placing pigeon  $x$  in pigeonhole  $y$ .

**Axiom 2** (Generalised Pigeonhole principle). *If  $m$  pigeons occupy  $n$  pigeonholes and  $m > kn$  then some pigeonhole has more than  $k$  pigeons in it.*

To give a proof using it, you need to specify what are the  $m$  pigeons, the  $n$  pigeonholes, some rule for placing pigeon  $x$  in pigeonhole  $y$ , and what is the  $k$ .

- For each of the following functions, guess a Big-O form from one of those in Table 1, then prove your guess.

(a)  $f(n) = 4n + 7$

(e)  $f(n) = n + \frac{1}{n}$

(b)  $f(n) = 3 + \sin(n)$

(f)  $f(n) = \log_4 n + 7$

(c)  $f(n) = 5n^2 + 3n \log_2 n$

(g)  $f(n) = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{n(n+1)}$

(d)  $f(n) = 2 + 4 + 6 + \cdots + 2n$

(h)  $f(n) = 1 + 2^3 + 3^3 + \cdots + n^3$

2. Let  $f, g : \mathbb{N} \rightarrow \mathbb{N}$  defined by  $f(n) = \begin{cases} n & \text{for } n \text{ odd} \\ 1 & \text{for } n \text{ even} \end{cases}$  and  $g(n) = \begin{cases} 1 & \text{for } n \text{ odd} \\ n & \text{for } n \text{ even} \end{cases}$

(a) Draw plots of  $f(n)$  and  $g(n)$ .

(b) Is  $f \in O(g)$  or  $g \in O(f)$ ?

3. Prove that the relation of *domination* of the set of all functions from  $\mathbb{N}_+$  to  $\mathbb{R}$  to is reflexive and transitive<sup>1</sup>. Is it symmetric and/or antisymmetric?

4. Here is a pseudocode program. (In this and the next question code arrays are numbered starting at 1).

```

procedure Foo (n int; array=[a_1, a_2, ..., a_n] )
  begin
    num := array[1]
    for i := 2 to n do
      if array[i] > num
        num := array[i]
    print num
  end

```

Work out what it does, try it out on input (5; [3, 1, 4, 2, 0]), then compute a Big-O complexity for how many steps it takes in worst-case. A *step* can be considered as how many times the comparison

array[i] > num  
is performed.

5. Here is some more pseudocode.

```

procedure Bar (n int; array=[a_1,...,a_n] array of integers)
  begin
    for i := 2 to n do
      for j := 1 to n-i+1 do
        if array[j] > array[j+1]
          temp := array[j]
          array[j] := array[j+1]
          array[j+1] := temp
      print array
    end

```

Work out what it does, try it out on input (5; [3, 1, 4, 2, 0]), then compute a Big-O complexity for how many steps it takes in worst-case. A *step* can be considered as how many times the comparison

array[j] > array[j+1]  
is performed.<sup>2</sup>

<sup>1</sup>i.e. for all  $f, g, h : \mathbb{N}_+ \rightarrow \mathbb{R}$ ,  $f \in O(f)$ , and if  $f \in O(g)$  and  $g \in O(h)$  then  $f \in O(h)$ .

<sup>2</sup>see <http://www.geeksforgeeks.org/bubble-sort/> if you prefer to read real C, Java or Python code.

6. Consider the following problem: input is a sorted array of distinct integers, eg:

1	3	5	8	12	13	15	16	18	20	22	30	40	50	55	67
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

↑

Devise an efficient algorithm to locate a particular entry of the array, say the number 50. How many steps (Big O) roughly does your algorithm need? Hint: don't start at the start of the array and move left!

<sup>3</sup>

Also, will your algorithm be able to return No if the entry does not appear?

7. Use PHP to show that if  $S \subseteq \mathbb{N}_+$  and

- (a)  $|S| \geq 3$  then  $S$  contains two distinct elements  $x, y$  such that  $x + y$  is even.
- (b)  $|S| > 6$  then  $S$  contains three distinct elements  $x, y, z$  such that  $x + y + z$  is a multiple of 3.

8. There are 51 students taking a course, and the available grades are  $Z, P, C, D, HD$ . Show that at least 11 students will get the same grade.

9. Let  $p$  be an odd number and let  $S$  be any subset of  $\mathbb{Z}_{p+1} = \{0, 1, \dots, p\}$  which contains at least  $2 + \frac{p-1}{2}$  elements. Show that there are at least two elements of  $S$  with sum equal to  $p$ .

10. Let  $\Delta$  be an equilateral triangle with side length 1. Show that if we select 10 points in the interior of  $\Delta$  then at least two of them lie distance  $< \frac{1}{3}$  of each other. <sup>4</sup>

11. A factory produces robots. It needs to fill an order of 44 robots in 30 days, and in order to maintain the machinery it must produce at least one robot a day. Prove that there must be a string of some number of consecutive days where the factory produces exactly 15 robots. <sup>5</sup>

12. Question 2 (c) on LPC5 was easy because the statement was false (eg.  $B(0, 2) = 4 \neq 1$ ). However, if the question instead asked: for all  $x, y \in \mathbb{N}, x \geq 1, B(x, y) = 1$  then the statement would be true. How would you prove this (induction on what?  $x?, y?, x + y?, \dots, x$  first, then for  $P(0)$  and  $P(k + 1)$  you need to do a second induction using  $y?, \dots$ ).

<sup>3</sup>Hint here: <https://hackernoon.com/what-does-the-time-complexity-o-log-n-actually-mean-45f94bb5bfbf>

<sup>4</sup>hint: divide the triangle up into 9 smaller triangles

<sup>5</sup>hint: Let  $x_i$  be the number of robots produced between day 1 and day  $i$ . Let  $y_i = x_i + 15$ . Use PHP to show that some  $x_i = y_j$ .

Brief solutions:

1. (a) Linear.  $m = 5, k = 7$  then for all  $n \geq 7$  we have  $5n = 4n + n \geq 4n + 7$  so  $mg(n) \geq f(n)$  holds.

- (b) Constant.  $g(n) = 1$ , since  $\sin(n)$  bounces around between -1 and 1. Proof: for  $m = 4, k = 1$  we have

$$f(n) = 3 + \sin(n) \leq 3 + 1 = 4 = 4.g(n)$$

for all  $n \geq 1$ .

- (c) Quadratic.  $\log_2 n \leq n$  for  $n \geq 1$ , so for  $k = 1$  and  $m = 8$  we have

$$f(n) = 5n^2 + 3\log_2 n \leq 5n^2 + (3n).n = 8n^2 = 8g(n)$$

for all  $n \geq 1$ .

- (d) Induction we know that this is  $2 \sum_{i=1}^n i = n(n+1)$  so: quadratic.  $f(n) = n(n+1) = n^2 + n$ ,  $g(n) = n^2$ , then  $m = 2, k = 1$  we have

$$f(n) = n^2 + n \leq n^2 + n.n = 2n^2 = 2g(n)$$

for all  $n \geq 1$ .

- (e) Linear.  $m = 2, k = 1$ . For  $n \geq 1$  we have

$$f(n) = n + \frac{1}{n} \leq n + 1$$

since  $\frac{1}{n} \leq 1$

$$\leq n + n = 2n = 2g(n).$$

- (f) Logarithmic.

$$f(n) = \log_4 n + 7 = \frac{\log_2 n}{\log_2 4} + 7$$

using  $\log_b y = \frac{\log_a y}{\log_a b}$

$$= \frac{\log_2 n}{2} + 7 \leq \frac{\log_2 n}{2} + \frac{\log_2 n}{2}$$

since  $\log_2 n \geq 14$  for  $n$  sufficiently big, so  $k = 2^{15}$  would make this true, and  $m = 1$ .

- (g) Induction we know this is  $\frac{n}{n+1}$  which is always less than 1, so  $g(n) = 1$ , Constant,  $m = k = 1$ .

- (h) Induction we proved a formula for this in Quiz 4:

**Lemma 3.** For all  $n \in \mathbb{N}_+$

$$1^3 + 2^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$$

So  $O(n^4)$ .

2. (b) Neither is true.

3. Let  $f, g, h : \mathbb{N}_+ \rightarrow \mathbb{R}$  be three functions from positive integers to real numbers.

Then  $f \in O(f)$  since there exists  $m = 1, k = 1$  so that  $f(n) \leq 1 \cdot f(n)$  for all  $n \geq 1$ , so domination is reflexive.

By definition there exists  $m_1, m_2 \in \mathbb{R}_+$  and  $k_1, k_2 \in \mathbb{N}_+$  with  $|f(n)| \leq m_1|g(n)|$  for  $n \geq k_1$  and  $|g(n)| \leq m_2|h(n)|$  for  $n \geq k_2$

so letting  $m = m_1m_2$  and  $k = \max\{k_1, k_2\}$  we have for all  $n \geq k$

$$|f(n)| \leq m_1|g(n)| \leq m_1m_2|h(n)| = m|h(n)|$$

so  $f \in O(h)$ . This shows domination is transitive.

Since  $6n^2 + 4n + 3$  and  $n^2$  are both Big O of each other, domination is not antisymmetric, and since  $n \in O(n^2)$  but  $n^2 \notin O(n)$  it is not symmetric.

4. Finds the maximum entry in the array. Takes linear ie.  $O(n)$  steps.

5. On input  $(5; [3, 1, 4, 2, 0])$ ,  $n = 5$  so we do  $i = 2 : j = 1, 2, 3, 4$ ;  $i = 2, j = 1, 2, 3$ ;  $i = 3, j = 1, 2$ ;  $i = 4, j = 1$ , and rearranged the array:

```

3, 1, 4, 2, 0
i = 2 :
    1, 3, 4, 2, 0
    1, 3, 4, 2, 0
    1, 3, 2, 4, 0
    1, 3, 2, 0, 4
i = 3
    1, 3, 2, 0, 4
    1, 2, 3, 0, 4
    1, 2, 0, 3, 4
i = 4
    1, 2, 0, 3, 4
    1, 0, 2, 3, 4
i = 5
    0, 1, 2, 3, 4

```

Sorts the array into ascending order. Takes quadratic time (compares each pair of numbers in the array, so takes  $(n-1) + (n-2) + \dots = \sum_{k=1}^{n-1} k = \dots$  (use formula proved by induction Week 5) steps which is  $O(n^2)$ .

6. The best (prove it!) method is binary search: start at the middle of the array, and compare that entry with your target value. If less, move to the middle of the first half (and throw the right half away), and if greater, move to the middle of the right half. Repeat until you either locate the target value, or if the length of the remaining segment of the array has size 1. Count the number of times you make a “step”. <sup>6</sup>
7. (a) Let  $S = \text{pigeons}$ ,  $m = |S| \geq 3$ , and holes are labeled 0, 1, send  $x \in S$  to hole  $[x]_2$ . Since  $\geq 3$  pigeons in 2 holes, by PHP some hole has two pigeons, call them  $x, y$ . If  $[x]_2 = [y]_2 = 0$  then  $x + y$  is even, and if  $[x]_2 = [y]_2 = 1$  then  $x + y \equiv 1 + 1 \equiv 0$  so again  $x + y$  is even.
- (b) Let  $S = \text{pigeons}$ ,  $m = |S| \geq 7$ , and holes are labeled 0, 1, 2, send  $x \in S$  to hole  $[x]_3$ , and put  $k = 2$ . Then since  $nk = 2 \cdot 3 = 6$  and  $m > 6$  by GPHP some hole has more than  $k = 2$  pigeons, so there are  $x, y, z \in S$  with  $[x]_3 = [y]_3 = [z]_3 = i$ . Then  $x + y + z \equiv 3i \equiv 0 \pmod{3}$  so  $x + y + z$  is a multiple of 3.

<sup>6</sup>Does a step include moving the pointer between cells, or just whenever you make a comparison of two integers?

8.  $m = 51$  students=pigeons,  $n = 5$  grades=holes,  $k = 10$ . Then by GPHP  $m > kn = 50$  so some hole has more than 10 pigeons in it, so at least 11 students will get the same grade.

9. Pigeons=elements of  $S$ , pigeonholes:

$$\{0, p\}, \{1, p-1\}, \{2, p-2\}, \dots, \left\{\frac{p-1}{2}, \frac{p+1}{2}\right\}.$$

Place each element of  $S$  in the subset containing it. There are  $n = 1 + \frac{p-1}{2}$  holes and  $m = 2 + \frac{p-1}{2}$  pigeons so some subset contains two elements. The sum of two elements in a subset is  $p$ .

10. Divide the triangle up into 9 equilateral triangles of side length  $\frac{1}{3}$ , these are the pigeonholes.



Pigeons are the 10 points, which are placed in the mini-triangle in which they sit. If a point lies on a border between two, either always place the point into the triangle *above* (say), or you could make two points and place one in each. In the second case, you have  $m \geq 10$  pigeons, and in the first case  $m = 10$  to put into 9 holes, so by PHP some triangle contains two points. To finish, show that any two points in an equilateral triangle of side length  $\frac{1}{3}$  lie within distance  $\frac{1}{3}$  of each other.

11. Let  $x_i$  be the number of robots produced between day 1 and day  $i$ . Let  $y_i = x_i + 15$ . Then  $1 \leq x_1 < x_2 < \dots < x_{30} = 44$  and  $16 \leq y_1 < y_2 < \dots < y_{30} = 59$ . Note that the sequence of  $x_i$ 's is strictly increasing since each day at least one robot is produced, and same for the  $y_i$ 's.

Let the pigeonholes be the numbers from 1 to 59, and the pigeons the values for  $x_1, \dots, x_{30}, y_1, \dots, y_{30}$ , with the rule that  $x_i$  and  $y_i$  are placed in the hole corresponding to their value. So there are 60 pigeons and only 59 holes, so PHP says some hole has two values in it. But the  $x_i$  are all distinct, and the  $y_i$  are all distinct, so this hole must contain exactly one  $x_i$  and one  $y_j$ . So we have  $x_i = y_j = x_j + 15$  and then we see  $j < i$  and so on day  $j$   $x_j$  robots have been produced, and then after  $i - j$  days an additional 15 are produced.

From

<https://math.stackexchange.com/questions/2413861/how-to-prove-that-there-is-a-string->

12. For example, let  $P(n)$  be the statement that " $n = x + y$  and if  $x \geq 1$  then  $B(x, y) = 1$ ". Then  $P(1)$  is true? Then assume  $P(k)$  is true for  $k \geq 1$ , etc.

Or, let  $P(x)$  be the statement that " $B(x, y) = 1$ ". Then  $P(1)$ : let  $Q(y)$  be the statement that " $B(1, y) = 1$ ". Then  $Q(0)$  is true? Then assume  $Q(k)$  is true for  $k \geq 0$ , etc.