Question 1. - *Begin a new booklet for this question.*

(a) Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be the vectors $\mathbf{a} = (1, -1, 0)$, $\mathbf{b} = (2, -1, 1)$, and $\mathbf{c} = (1, 3, -1)$.

(i) Compute $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

(ii) Find a vector twice the length of $\mathbf{a}$ and pointing in the same direction.

(iii) Find a vector perpendicular to both $\mathbf{b}$ and $\mathbf{c}$ and with length equal to 5.

(6 marks)

(b) Find a scalar m such that the vectors $(3, 1, 0)$ and $(m, -1, 0)$ have the same length.

(4 marks)

(c) Find the Cartesian equation of the plane passing through the three points $(0, 0, -1)$, $(1, 2, 1)$ and $(1, -1, 0)$.

(5 marks)

(d) Consider two vectors $\mathbf{p}$ and $\mathbf{q}$ such that $\mathbf{p}$ is twice as long as $\mathbf{q}$. Find a scalar quantity t such that $\mathbf{p} + t\mathbf{q}$ is perpendicular to $\mathbf{p} - t\mathbf{q}$.

(5 marks)

Question 2. - *Begin a new booklet for this question.*

(a) For the complex numbers $z_1 = 1 + 3i$ and $z_2 = 1 - i$, express each of the following complex numbers in the form $a + ib$, with a and b real:

(i) $(z_1 + 1)(z_2 - i)$

(ii) $\dfrac{|z_1|}{z_2 - 2i}$ (5 marks)

(b) Express the complex number $1 - i\sqrt{3}$ in exponential polar form (i.e. in the form $re^{i\theta}$). Hence or otherwise, express

$$(1 - i\sqrt{3})^{10}$$

in Cartesian form (i.e. in the form $a + ib$). (5 marks)

(c) Find all cube roots of i, and plot the solutions in the complex plane. (5 marks)

(d) Using the series expansion

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$

(which converges for $|z| < 1$), substitute $z = \frac{1}{2}e^{i\theta}$ to show that

$$1 + \frac{\cos\theta}{2} + \frac{\cos 2\theta}{4} + \frac{\cos 3\theta}{8} + \cdots = \frac{4 - 2\cos\theta}{5 - 4\cos\theta} \quad .$$

(5 marks)

Question 3. - *Begin a new booklet for this question.*

(a) Find the first derivatives of the following functions:

(i) $f(x) = (1 + 5x)^5$

(ii) $f(x) = \cos^{-1}(2x)$

(iii) $f(x) = x\cos(\frac{1}{x^3})$ (5 marks)

(b) A particle released from a cyclotron moves along the path

$$y(x) = -\sqrt{4 - x^2}$$

At the point $x = 1$m, it is observed that the particle is moving at 1000 ms^{-1} in the y direction. How fast is the particle moving in the x-direction at this point? (5 marks)

(c) Show that the series

$$\sum_{k=0}^{\infty} \frac{k}{3^k}$$

converges. (5 marks)

(d) Find the first two non-zero terms of the Taylor series of

$$f(x) = \ln x$$

expanded about $x = 1$. Use this series to estimate the value of

$$\ln 1.1$$

without using a calculator. (5 marks)

Question 4. - *Begin a new booklet for this question.*

(a) Evaluate the following indefinite integrals

(i)

$$\int (1-2x)^7 dx$$

(ii)

$$\int \cos(3x) dx$$

(iii)

$$\int e^{5x+4} dx$$

(iv)

$$\int \sin(3x) e^{\cos(3x)} dx$$

(v)

$$\int x e^{-3x} dx$$

(vi)

$$\int \frac{2x+3}{x^2-9x+20} dx$$

(8 marks)

(b) Evaluate the following integrals

(i)

$$\int_0^{\pi/2} x \sin(2x^2) dx$$

(ii)

$$\int_{-1}^{1} \frac{dx}{x^2+10x+25}$$

(iii)

$$\int_0^1 2^{3x-1} dx$$

(9 marks)

NOTE: QUESTION CONTINUES OVER PAGE.

(c) Show that, for any polynomial $p(x)$,

$$\int e^x p(x)dx = e^x\left(p(x) - p'(x) + p''(x) - \cdots\right) + C$$

Hence or otherwise evaluate the indefinite integral

$$\int e^x (x^2 + 2x + 7)dx \ .$$

(5 marks)

Question 5. - *Begin a new booklet for this question.*

(a) Find the general solution to the differential equation

$$\frac{dy}{dx} - 3y = e^{2x}$$

Hence find the complete solution that satisifies the initial condition $y = 0$ when $x = 0$. (8 marks)

(b) Find the general solution to the following differential equations:

(i)

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 10y = 0 \ .$$

(ii)

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0 \ .$$

(6 marks)

(c) The motion of a point $y(t)$ on a guitar string can be modelled by the second-order differential equation

$$\frac{d^2y}{dt^2} + k^2y = 0$$

where $k = \frac{\pi}{L}\sqrt{\frac{T}{\mu}}$, in which L is the string's length, T is the tension on the string, and μ is the mass per unit length.

(i) Find the general solution to the above equation. How does the period of the motion change if the tension on the string is doubled?

(ii) The guitar string is then subjected to a driving vibration at resonance, so that the equation of motion is given by

$$\frac{d^2y}{dt^2} + k^2y = a\sin(kt)$$

Show that

$$y_P(t) = Ct\cos(kt)$$

can be a particular solution of this equation, and find C in temrs of k and a. Hence find the complete solution $y(t)$ satisfying the initial condition $y(0) = 0$, $y'(0) = 1$. How does the vibration change as time t increases?

(6 marks)

Question 6. - *Begin a new booklet for this question.*

(a) Given the matrices

$$\mathbf{A} = \begin{pmatrix} 0 & 2 \\ -1 & 2 \end{pmatrix}$$

and

$$\mathbf{B} = \begin{pmatrix} 5 & 0 & 2 \\ 1 & 3 & -1 \end{pmatrix}$$

determine, where possible:

(i) $\mathbf{AB}$

(ii) $\mathbf{A}^T$

(iii) $\mathbf{A} + \mathbf{B}$

(iv) $\mathbf{BA}$. (7 marks)

(b) Write the system of equations

$$\begin{array}{rcrcl} 2x_1 & + & 2x_2 & = & 1 \\ 7x_1 & + & 3x_2 & = & 2 \end{array}$$

in matrix form. By finding an appropriate inverse, solve for x_1 and x_2. (7 marks)

(c) Find the inverse of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

(6 marks)

Table of integrals

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\frac{x}{a} + C_1 = \log\left(x + \sqrt{a^2 + x^2}\right) + C_2$$

$$\int \frac{dx}{1 - x^2} = \tanh^{-1}x + C_1 = \frac{1}{2}\log\left|\frac{1 + x}{1 - x}\right| + C_2$$

$$\int \cosh x \, dx = \sinh x + C$$

$$\int \sinh x \, dx = \cosh x + C$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$