2011 Exam - Solutions

1. a)

(i) $$\underset{\sim}{b} \times \underset{\sim}{c} = \begin{vmatrix} \hat{\underset{\sim}{i}} & \hat{\underset{\sim}{j}} & \hat{\underset{\sim}{k}} \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{vmatrix}$$

$$= \hat{\underset{\sim}{i}}\left(+1-3\right) - \hat{\underset{\sim}{j}}\left(-2-1\right) + \hat{\underset{\sim}{k}}\left(6+1\right)$$

$$= -2\hat{\underset{\sim}{i}} + 3\hat{\underset{\sim}{j}} + 7\hat{\underset{\sim}{k}}.$$

$$= \left(-2, 3, 7\right)$$

So $$\underset{\sim}{a} \cdot \left(\underset{\sim}{b} \times \underset{\sim}{c}\right) = \left(1, -1, 0\right) \cdot \left(-2, 3, 7\right)$$

$$= -2 - 3 + 0$$

$$= -5$$

(ii) Such a vector is

$$2\hat{\underset{\sim}{a}} = 2\frac{\underset{\sim}{a}}{|\underset{\sim}{a}|} = \ldots$$

$$= 2\frac{\left(1, -1, 0\right)}{\sqrt{1^2 + 1^2 + 0^2}}$$

$$= \frac{2}{\sqrt{2}}\left(1, -1, 0\right)$$

$$= \sqrt{2}\left(1, -1, 0\right)$$

$$= \left(\sqrt{2}, -\sqrt{2}, 0\right)$$

(iii) The cross product $\underset{\sim}{b} \times \underset{\sim}{c}$ is $\perp$ to both $\underset{\sim}{b}$ and $\underset{\sim}{c}$,

So

$$5\frac{\underset{\sim}{b} \times \underset{\sim}{c}}{|\underset{\sim}{b} \times \underset{\sim}{c}|}$$

$$= 5\frac{\left(-2, 3, 7\right)}{\sqrt{4 + 9 + 49}}$$

$$= \frac{5}{\sqrt{62}}\left(-2, 3, 7\right).$$

b)

$$|(3,1,0)| = |(m,-1,0)|$$

or

$$\sqrt{9+1} = \sqrt{m^2+1}$$

$$\Rightarrow \quad 9+1 = m^2+1$$

$$m^2 = 9$$

So $\quad m = \pm 3.$

c) Vector eq$\underline{\text{n}}$ is

$$\underline{r}(u,v) = a + u\underline{p} + v\underline{q}$$

here

$$\underline{a} = (0,0,-1)$$

$$\underline{p} = (1,2,1) - (0,0,-1)$$

$$= (1,2,2)$$

$$\underline{q} = (1,-1,0) - (0,0,-1)$$

$$= (1,-1,1)$$

So

$$\underline{r}(u,v) = (0,0,-1) + u(1,2,2)$$

$$+ v(1,-1,1)$$

$$= (u+v,\ 2u-v,\ -1+2u+v)$$

$$\left.\begin{aligned} x &= u+v \\ y &= 2u-v \\ z &= -1+2u+v \end{aligned}\right\}$$

$$u = x - v$$

$$y = 2(x-v) - v$$

$$= 2x - 3v$$

$$v = \frac{1}{3}(2x-y).$$

$$u = x - \frac{1}{3}(2x-y)$$

$$= \frac{1}{3}x + \frac{1}{3}y$$

Then

$$z = -1 + \frac{2}{3}x + \frac{2}{3}y$$

$$+ \frac{1}{3}2x - \frac{1}{3}y$$

$$= -1 + \frac{4}{3}x + \frac{1}{3}y$$

So $$\frac{4}{3}x + \frac{1}{3}y - z = 1$$

or $$4x + y - 3z = 3.$$

f) We require

$$(\underline{p} + t\underline{q})\cdot(\underline{p} - t\underline{q}) = 0$$

or $$\underline{p}\cdot\underline{p} + \cancel{t\underline{p}\cdot\underline{q}} - \cancel{t\underline{q}\cdot\underline{p}} - t^2\underline{q}\cdot\underline{q} = 0$$

so $$\underline{p}\cdot\underline{p} - t^2\underline{q}\cdot\underline{q} = 0$$

$$t^2 = \frac{\underline{p}\cdot\underline{p}}{\underline{q}\cdot\underline{q}} = \frac{|\underline{p}|^2}{|\underline{q}|^2}$$

but $|\underline{p}| = 2|\underline{q}|$, so

$$t^2 = \frac{4|\underline{q}|^2}{|\underline{q}|^2} = 4 \implies t = \pm 2$$

2. (i)

$$(z_1+1)(z_2-i) = ((1+3i)+1)(1-i)$$

$$= (2+3i)(1-i)$$

$$= 2 - 2i + 3i + 3$$

$$= 5 + i$$

(ii) $$\frac{|z_1|}{z_2 - 2i} = \frac{\sqrt{1^2+3^2}}{1-i-2i}$$

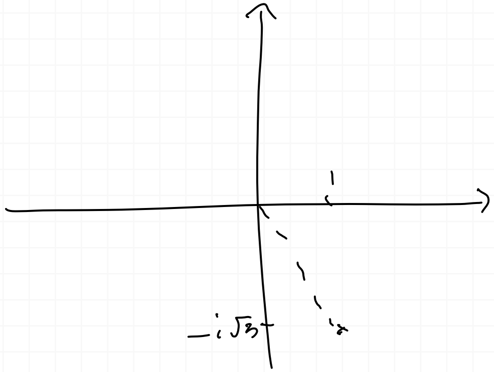
$$= \frac{\sqrt{10}}{1-3i} \times \frac{(1+3i)}{1+3i}$$

$$= \frac{(1+3i)}{1+9}\sqrt{10} = \frac{\sqrt{10}}{10}(1+3i)$$

$$= \frac{1}{\sqrt{10}}(1+3i) = \frac{1}{\sqrt{10}} + \frac{3}{\sqrt{10}}i$$

b)

$$r = \sqrt{1^2 + (\sqrt{3})^2}$$



$$= 2$$

$\theta = -\frac{\pi}{3}$ by inspection

So

$$1 - i\sqrt{3} = 2e^{-i\frac{\pi}{3}}.$$

Then

$$(1-i\sqrt{3})^{10} = \left(2e^{-i\frac{\pi}{3}}\right)^{10}$$

$$= 2^{10} e^{-i\frac{10\pi}{3}}$$

$$= 2^{10} e^{-i\left(\frac{12-2}{3}\right)\pi}$$

$$= 2^{10} e^{+i\frac{2\pi}{3}}$$

$$= 2^{10}\cos\left(\frac{2\pi}{3}\right) + i2^{10}\sin\left(\frac{2\pi}{3}\right)$$

$$= 2^{10}\left(-\frac{1}{2}\right) + i2^{10}\frac{\sqrt{3}}{2}$$

$$= -2^9 + i2^9\sqrt{3}.$$

c) Solve

$$z^3 = i$$

$$= e^{i\left(\frac{\pi}{2} + m2\pi\right)} \qquad m = 0, 1, 2, \dots$$

$$z_m = \left(e^{i\left(\frac{\pi}{2}+2m\pi\right)}\right)^{\frac{1}{3}}$$

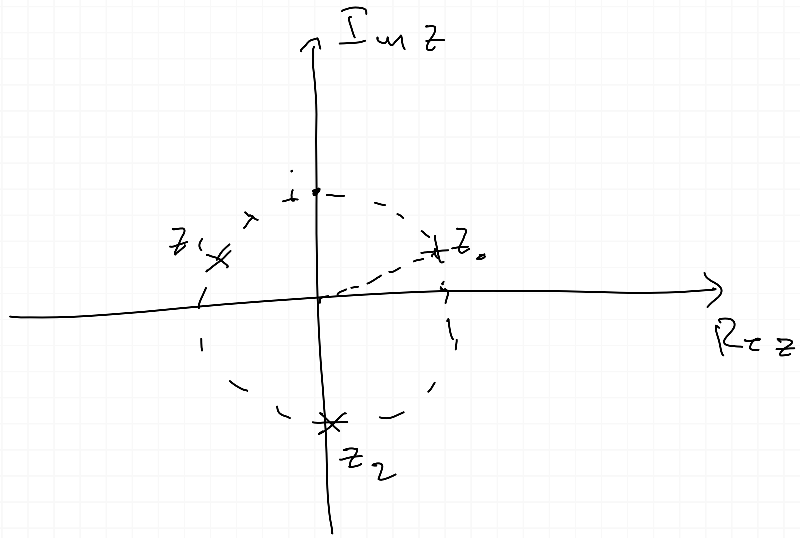
$$= e^{i\left(\frac{\pi}{6}+\frac{2m\pi}{3}\right)}$$

So

$$z_0 = e^{i\frac{\pi}{6}}$$

$$z_1 = e^{i\left(\frac{\pi}{6}+\frac{2\pi}{3}\right)} = e^{i\frac{5\pi}{6}}$$

$$z_2 = e^{i\left(\frac{\pi}{6}+\frac{4\pi}{3}\right)} = -i$$



d) Substitute:

$$\frac{1}{1-\frac{1}{2}e^{i\theta}} = 1 + \frac{1}{2}e^{i\theta} + \left(\frac{1}{2}e^{i\theta}\right)^2 + \dots$$

$$= 1 + \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{2i\theta} + \dots$$

Take real parts of both sides:

$$\frac{1}{1-\frac{1}{2}e^{i\theta}} = \frac{1}{1-\frac{1}{2}\cos\theta - i\frac{1}{2}\sin\theta}$$

$$= \frac{1}{1-\frac{1}{2}\cos\theta - i\frac{1}{2}\sin\theta} \times \frac{1-\frac{1}{2}\cos\theta + i\frac{1}{2}\sin\theta}{1-\frac{1}{2}\cos\theta - i\frac{1}{2}\sin\theta}$$

$$= \frac{1-\frac{1}{2}\cos\theta + i\frac{1}{2}\sin\theta}{\left(1-\frac{1}{2}\cos\theta\right)^2 + \left(\frac{1}{2}\sin\theta\right)^2}$$

$$= \frac{1-\frac{1}{2}\cos\theta + i\frac{1}{2}\sin\theta}{1-\cos\theta+\frac{1}{4}\cos^2\theta+\frac{1}{4}\sin^2\theta}$$

$$= \frac{1-\frac{1}{2}\cos\theta + i\frac{1}{2}\sin\theta}{1+\frac{1}{4}-\cos\theta}$$

So $$\mathrm{Re}\left[\frac{1}{1-z}\right] = \frac{1-\frac{1}{2}\cos\theta}{\frac{5}{4}-\cos\theta}$$

$$= \frac{4-2\cos\theta}{5-4\cos\theta}$$

Also,

$$\operatorname{Re}\left[1 + e^{i\theta} + \frac{1}{2}e^{2i\theta} + \dots\right]$$

$$= 1 + \cos\theta + \frac{1}{2}\cos(2\theta) + \dots$$

So

$$\frac{4 - 2\cos\theta}{5 - 4\cos\theta} = 1 + \cos\theta + \frac{1}{2}\cos(2\theta) + \dots$$

as required.

3.

(i) $f'(x) = 25(1+5x)^4$

(ii) Let $y = \cos^{-1}(2x)$

then $\cos y = 2x$

$$-\sin y \frac{dy}{dx} = 2$$

$$\frac{dy}{dx} = \frac{-2}{\sin y} = \frac{-2}{\sqrt{1-\cos^2 y}}$$

$$= \frac{-2}{\sqrt{1-4x^2}}.$$

(iii)

$$f(x) = x\cos\left(\frac{1}{x^3}\right)$$

$$= x\cos(x^{-3})$$

$$f'(x) = \cos\left(\frac{1}{x^3}\right) + x\sin\left(\frac{1}{x^3}\right) \times (-3x^{-4})$$

$$= \cos\left(\frac{1}{x^3}\right) + \frac{3}{x^3}\sin\left(\frac{1}{x^3}\right)$$

b)

$$y = -\sqrt{4-x^2} = -(4-x^2)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = -\frac{1}{2}(4-x^2)^{-\frac{1}{2}} \times (-2x)$$

$$= x(4-x^2)^{-\frac{1}{2}}$$

$$= \frac{x}{\sqrt{4-x^2}}$$

Now

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

So

$$\frac{dx}{dt} = \frac{1}{dy/dx} \cdot \frac{dy}{dt}$$

$$= \frac{\sqrt{4-x^2}}{x} \times \frac{dy}{dt}$$

$$= \frac{\sqrt{4-1}}{1} \times 1\times10^3 \ \text{ms}^{-1}$$

$$= \sqrt{3} \times 1000 \ \text{ms}^{-1}$$

$$\simeq 1.7 \times 10^3 \ \text{ms}^{-1}.$$

c) Use the ratio test:

$$R = \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right|$$

$$= \lim_{k \to \infty} \left| \frac{k+1}{3^{k+1}} \times \frac{3^k}{k} \right|$$

$$= \lim_{k \to \infty} \left| \frac{k+1}{k} \times \frac{1}{3} \right|$$

$$= \frac{1}{3} \lim_{k \to \infty} \left| \frac{k+1}{k} \right|$$

$$= \frac{1}{3} \lim_{k \to \infty} \left| \frac{1 + \frac{1}{k}}{1} \right| = \frac{1}{3}$$

This is less than 1, so the series converges.

d)

$$f(x) = \ln x \implies f(1) = 0$$

$$f'(x) = \frac{1}{x} \implies f'(1) = 1$$

$$f''(x) = \frac{-1}{x^2} \implies f''(1) = -1$$

So

$$f(x) \simeq f(1) + f'(1)(x-1)$$

$$+ \frac{1}{2} f''(1)(x-1)^2$$

$$\simeq 0 + (x-1) + \frac{1}{2}(-1)(x-1)^2$$

$$\simeq (x-1) - \frac{1}{2}(x-1)^2$$

Then

$$\ln 1.1 \simeq (0.1) - \frac{1}{2}(0.1)^2$$

$$\simeq 0.1 - 0.5 \times 0.01$$

$$= 0.1 - 0.005$$

$$= 0.995$$

4. (i) $\dfrac{1}{-14}\left(1-2x\right)^{8} + C$

(ii) $\dfrac{1}{3}\sin(3x) + C$

(iii) $\dfrac{1}{5}e^{5x+4} + C$

(iv) $\displaystyle\int \sin 3x \, e^{\cos 3x} \, dx$

$$= -\frac{1}{3}\int \left(-3\sin 3x\right) e^{\cos 3x} \, dx$$

$$= -\frac{1}{3}e^{\cos 3x} + C$$

v)

$$\int x e^{-3x}\,dx = -x\,\frac{1}{3}e^{-3x} - \int 1.\left(-\frac{1}{3}e^{-3x}\right)dx$$

$$= -\frac{1}{3}x e^{-3x} + \frac{1}{3}\int e^{-3x}\,dx$$

$$= -\frac{1}{3}x e^{-3x} - \frac{1}{9}e^{-3x} + C$$

vi)

$$\frac{2x+3}{x^2-9x+20} = \frac{2x+3}{(x-4)(x-5)}$$

$$= \frac{A}{x-4} + \frac{B}{x-5}$$

So

$$2x+3 = A(x-5) + B(x-4)$$

$x=4$: $\quad 11 = A(-1) \implies A = -11$

$x=5$: $\quad 13 = B\times 1 \implies B = 13.$

So

$$\int \frac{2x+3}{x^2-9x+20}\,dx = -11\int \frac{1}{x-4}\,dx + 13\int \frac{1}{x-5}\,dx$$

$$= -11\ln|x-4| + 13\ln|x-5| + C$$

$$= \ln\left|\frac{(x-4)^{11}}{(x-5)^{13}}\right| + C$$

b)

(i) $$\int_0^{\frac{\pi}{2}} x \sin(2x^2)\,dx$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} 4x \sin(2x^2)\,dx$$

$$= \frac{1}{4}\left[-\cos(2x^2)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{-1}{4}\left(\cos\left(2\left(\frac{\pi}{2}\right)^2\right) - \cos(0^2)\right)$$

$$= \frac{-1}{4}\left(\cos\left(\frac{\pi^2}{2}\right) - 1\right)$$

(ii) $$\int_{-1}^{1} \frac{dx}{x^2+10x+25} = \int_{-1}^{1} \frac{dx}{(x+5)^2}$$

$$= \int_{-1}^{1} (x+5)^{-2}\,dx$$

$$= \frac{1}{-1}\left[(x+5)^{-1}\right]_{-1}^{1}$$

$$= -\left(\frac{1}{1+5} - \frac{1}{-1+5}\right)$$

$$= -\left(\frac{1}{6} - \frac{1}{4}\right) = \frac{1}{4} - \frac{1}{6}$$

$$= \frac{1}{12}.$$

(iii) Now

$$2^{3x-1} = e^{\ln 2^{3x-1}}$$

$$= e^{(3x-1)\ln 2}$$

So $$\int_0^1 2^{3x-1}\,dx = \int_0^1 e^{(3x-1)\ln 2}\,dx$$

$$= \frac{1}{3\ln 2}\left[e^{(3x-1)\ln 2}\right]_0^1$$

$$= \frac{1}{3\ln 2}\left(e^{2\ln 2} - e^{-\ln 2}\right)$$

$$= \frac{1}{3\ln 2}\left(4 - \frac{1}{2}\right)$$

$$= \frac{1}{3\ln 2}\cdot\frac{7}{2} = \frac{7}{6\ln 2}.$$

c) Integrating by parts,

$$\int e^x p(x)\,dx = e^x p(x) - \int e^x p'(x)\,dx$$

$$= e^x p(x) - e^x p'(x) + \int e^x p''(x)\,dx$$

$$= e^x p(x) - e^x p'(x) + e^x p''(x) - \int e^x p'''(x)\,dx$$

$$= e^x\left(p(x) - p'(x) + p''(x) + \cdots\right)$$

Hence

$$\int e^x(x^2+2x+7)\,dx$$

$$= e^x\left(x^2+2x+7 - (2x+2) + 2\right) + C$$

$$= e^x\left(x^2+7\right) + C.$$

5.

a) The DE is linear;

The integrating factor is

$$I(x) = e^{\int(-3)\,dx}$$

$$= e^{-3x}$$

Then

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}dx = e^{-3x}e^{2x}$$

or

$$\frac{d}{dx}\left(e^{-3x}y\right) = e^{-x}$$

Integrating,

$$e^{-3x}y = \int e^{-x}\,dx$$

$$= -e^{-x} + C$$

So

$$y(x) = -e^{2x} + Ce^{3x}.$$

when $x=0$, $y=0$, so

$$0 = -1 + C$$

$$\Longrightarrow \quad C = 1$$

Then $y = -e^{2x} + e^{3x}$.

b) (i) The aux. eqⁿ is

$$m^2 - 7m + 10 = 0$$

$$(m-5)(m-2) = 0$$

$$m = 5,\ 2$$

distinct real roots

So $y(x) = Ae^{5x} + Be^{2x}$

(ii) The aux. eq^{n} is

$$m^2 - 8m + 16 = 0$$

$$(m-4)(m-4) = 0$$

$\Rightarrow$ double root at $m = 4$

So $$y(x) = (Ax + B)e^{4x}.$$

c)

(i) Aux. eq^{n} is

$$m^2 + k^2 = 0 \quad \Rightarrow \quad m = \pm ik$$

complex roots, so gen. sol^{n} is

$$y(t) = A\cos kt + B\sin kt.$$

The period of motion is

$$P = \frac{2\pi}{k} = \frac{2\pi}{\frac{\pi}{L}\sqrt{\frac{T}{\rho}}}$$

$$= 2L\sqrt{\frac{\rho}{T}}.$$

If T is doubled, P is reduced by a factor of $\sqrt{2}$.

ii) The ansatz is

$$y_p(t) = Ct\cos(kt).$$

$$y_p'(t) = C\left(\cos(kt) - tk\sin(kt)\right)$$

$$y_p''(t) = C\left[-k\sin(kt) - k\sin(kt) - tk^2\cos(kt)\right]$$

Subbing into the DE, we find

$$-2kC\sin(kt) - \cancel{tk^2C\cos(kt)} + \cancel{Ck^2t\cos(kt)} = a\sin(kt)$$

So

$$-2kC\sin(kt) = a\sin(kt).$$

$$C = -\frac{a}{2k}.$$

The gen. sol^{n} is

$$y(t) = y_c(t) + y_p(t)$$

$$= A\cos kt + B\sin kt$$

$$-\frac{a}{2k} t\cos(kt).$$

Initial conditions

$y(0) = 0$:

$$0 = A + B \times 0$$

$$-\frac{a}{2k} \times 0 \implies A = 0.$$

$y'(0) = 1$:

$$y'(t) = Bk\cos kt$$

$$-\frac{a}{2k}\cos(kt)$$

$$+\frac{a}{2} t\sin(kt)$$

So

$$1 = Bk - \frac{a}{2k} + 0$$

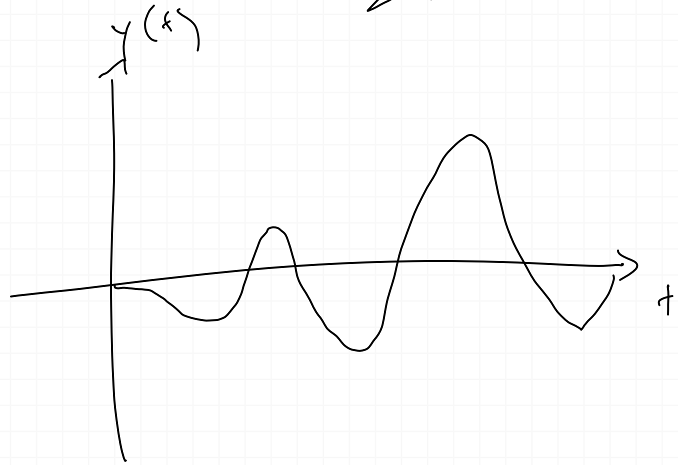
Then

$$B = \frac{1}{k}\left(1 + \frac{a}{2k}\right)$$

So

$$y(t) = \frac{1}{k}\left(1 + \frac{a}{2k}\right)\sin kt$$

$$-\frac{a}{2k} t\cos(kt)$$



6 a) (i)

$$AB = \begin{pmatrix} 0 & 2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 5 & 0 & 2 \\ 1 & 3 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 6 & -2 \\ -3 & 6 & -4 \end{pmatrix}$$

(ii)

$$A^T = \begin{pmatrix} 0 & -1 \\ 2 & 2 \end{pmatrix}$$

(iii) not possible

(iv) not possible.

b)

$$\begin{pmatrix} 2 & 2 \\ 7 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 7 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \frac{1}{6-14}\begin{pmatrix} 3 & -2 \\ -7 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \frac{1}{-8}\begin{pmatrix} -1 \\ -3 \end{pmatrix} = \begin{pmatrix} \frac{1}{8} \\ \frac{3}{8} \end{pmatrix}$$

So $x_1 = \frac{1}{8}$

$x_2 = \frac{3}{8}$.

c) Use row-ops:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 \\ R_2 - R_1 \\ R_3 + R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 2 & -1 & 1 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 - R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -\frac{1}{2} & \frac{1}{2} \end{array}\right) \begin{array}{l} R_1 - R_2 \\ R_2 \\ \frac{1}{2} R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & -\frac{1}{2} & \frac{1}{2} \end{array}\right) \begin{array}{l} R_1 - R_3 \\ R_2 + R_3 \\ \end{array}$$

So

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 2 & -1 & -1 \\ 0 & 1 & 1 \\ 2 & -1 & 1 \end{pmatrix}$$

Then

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & -1 & -1 \\ 0 & 1 & 1 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -3 \\ 5 \\ 3 \end{pmatrix}$$

$$x_1 = -\frac{3}{2}$$

$$x_2 = \frac{5}{2}$$

$$x_3 = \frac{3}{2}.$$