Solutions

1. a) (i) $\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) = \underset{\sim}{a} \cdot (-\hat{j})$

or

$= (0, 3, 4) \cdot (0, -1, 0)$

$= -3$

(ii) $\frac{7}{5}(0, 3, 4)$

or $\left(0, \frac{21}{5}, \frac{28}{5}\right)$

or $\frac{21}{5}\hat{\underset{\sim}{j}} + \frac{28}{5}\hat{\underset{\sim}{k}}$.

(iii) $(0, -5, 0)$

or $-5\hat{j}$.

b) $m = -\frac{7}{3}$

c) Cartesian equation is

$$x = 1$$

ie

$$x + 0y + 0z = 1$$

d) $\sqrt{\frac{35}{6}}$ units.

2. a) (i) $54 + 27i$

(ii) $\frac{-2+\sqrt{34}}{19} + i\,\frac{2+\sqrt{34}}{19}$

b) $-2\sqrt{3}+2i = 4e^{i\frac{5\pi}{6}}$

in polar form

$(-2\sqrt{3}+2i)^{17} = \frac{1}{2}4^{17}\sqrt{3} + i\,\frac{1}{2}4^{17}$.

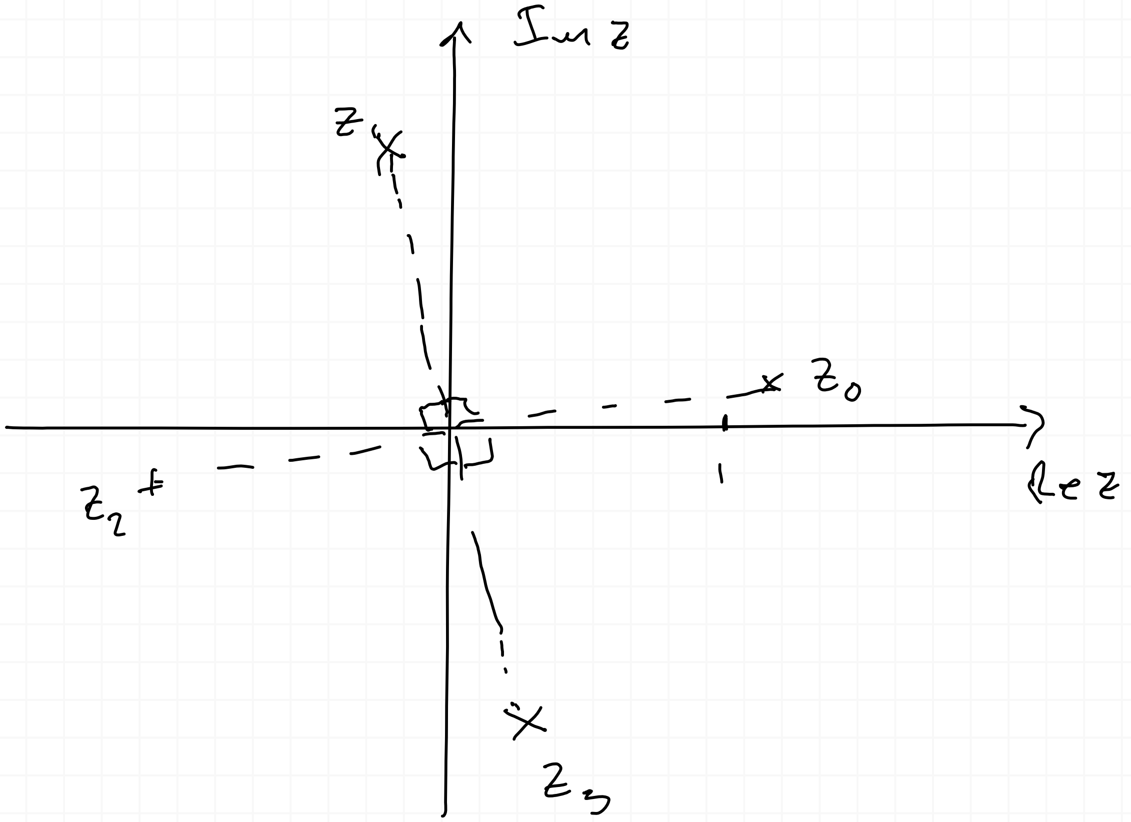
c) Roots are

$z_0 = 2^{\frac{1}{8}} e^{i\frac{\pi}{16}}$

$z_1 = 2^{\frac{1}{8}} e^{i\left(\frac{\pi}{16}+\frac{\pi}{2}\right)} = 2^{\frac{1}{8}} e^{i\frac{9\pi}{16}}$

$z_2 = 2^{\frac{1}{8}} e^{i\left(\frac{\pi}{16}+\pi\right)} = 2^{\frac{1}{8}} e^{i\frac{17\pi}{16}}$

$z_3 = 2^{\frac{1}{8}} e^{i\left(\frac{\pi}{16}+\frac{3\pi}{2}\right)} = 2^{\frac{1}{8}} e^{i\frac{25\pi}{16}}$



c) We would like to show that

$$\sin(A+B) = \sin A \cos B + \cos A \sin B.$$

Starting from the right-hand side,

$$\sin A \cos B + \cos A \sin B$$

$$= \frac{1}{2i}\left(e^{iA} - e^{-iA}\right)\frac{1}{2}\left(e^{iB} + e^{-iB}\right)$$

$$+ \frac{1}{2}\left(e^{iA} + e^{-iA}\right)\frac{1}{2i}\left(e^{iB} - e^{-iB}\right)$$

$$= \frac{1}{4i}\left(e^{i(A+B)} + e^{i(A-B)} - e^{i(B-A)} - e^{-i(A+B)}\right.$$

$$+\frac{1}{4i}\left(e^{i(A+B)} - e^{i(A-B)} + e^{-i(A-B)} - e^{-i(A+B)}\right)$$

$$= \frac{1}{4i}\left(2e^{i(A+B)} - 2e^{-i(A+B)}\right)$$

$$= \frac{1}{2i}\left(e^{i(A+B)} - e^{-i(A+B)}\right)$$

$$= \sin(A+B) \quad \text{as required.}$$

3 a) (i) $30(5x+3)^5$

(ii) $\dfrac{-2}{x} \dfrac{1}{(\ln x^2)^2}$

(iii) $\dfrac{1}{\sqrt{\left(\frac{1}{2}\right)^2 + x^2}}$ or $\dfrac{2}{\sqrt{1 + 4x^2}}$.

(iv)

$$6 \sin^2\left(e^{2\cos x}\right) \cos\left(e^{2\cos x}\right) \sin x . e^{2\cos x}$$

b) $-10 \, ms^{-1}$

c) Use the ratio test:

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \frac{(k+1)^2 e^{-2k-2}}{k^2 e^{-2k}}$$

$$= \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^2 e^{-2}$$

$$= \lim_{k\to\infty} \left(\frac{1+\frac{1}{k}}{1}\right)^2 e^{-2} = e^{-2} < 1$$

So the series converges.

d) First three nonzero terms:

- $f(x) \simeq 1 - x + x^2$

- $$\int_0^1 \frac{1}{1+x}\,dx \approx \frac{5}{6}$$

- The exact value of the integral is $\ln 2$

$$\text{Error} = \frac{5}{6} - \ln 2$$

$$\simeq 0.14$$

4. a)

i) $\frac{-1}{15}\left(1+5x\right)^{-3} + C$

ii) $\frac{1}{10}\cosh(5x) + C$

iii) $\frac{1}{2}e^{2x-1} + C$

iv) $\frac{-1}{4}e^{-x^4} + C$

v) $\ln\left|\frac{(x+3)^7}{(x+2)^4}\right| + C$

vi) $-\tan^{-1}(\cos x) + C$

vii) $2x\cos x + (x^2-2)\sin x + C$

viii) $\frac{1}{3}\sqrt{1+3x^2} + C$

c)

i) $\frac{\pi}{2}$

ii) $\frac{\pi}{2} - 2\tan^{-1}(2) + \frac{1}{2}\ln\frac{5}{2}$

iii) $\left(\sin\frac{\pi^2}{2}\right)^2$

iv) $\frac{(\ln 2)^2}{2}$

d) Let

$$I_n = \int \cos^n x \, dx$$

$$= \int \cos x \, . \cos^{n-1} x \, dx$$

Integrate by parts:

$$I_n = \cos^{n-1} x \, . \sin x - \int \sin x \, . (n-1)\cos^{n-2} x \, . (-\sin x) \, dx$$

$$= \cos^{n-1} x \, . \sin x + (n-1)\int \sin^2 \cos^{n-2} x \, dx$$

$$= \cos^{n-1}x \cdot \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2}x \, dx$$

$$= \cos^{n-1}x \cdot \sin x + (n-1)\int \cos^{n-2}x$$

$$- (n-1)\int \cos^n x \, dx$$

$$= \cos^{n-1}x \cdot \sin x + (n-1)I_{n-2} - (n-1)I_n.$$

Re-arranging,

$$I_n + (n-1)I_n = \cos^{n-1}x \sin x + (n-1)I_{n-2}$$

or

$$n I_n = \cos^{n-1}x \cdot \sin x + (n-1)I_{n-2}$$

$$\Rightarrow I_n = \frac{\cos^{n-1}x \cdot \sin x}{n}$$

$$+ \frac{n-1}{n} I_{n-2}$$

$$\int \cos^4 x \, dx = I_4$$

$$= \frac{\cos^3 x . \sin x}{4} - \frac{3}{4} I_2$$

$$= \frac{\cos^3 x \sin x}{4} - \frac{3}{4}\left[\frac{\cos x . \sin x}{2} - \frac{1}{2} I_0 \right]$$

$$= \frac{\cos^3 x \sin x}{4} - \frac{3}{8} \cos x \sin x - \frac{3x}{8} + C$$

5. a) The DE is linear:

The integrating factor is

$$I(x) = e^{\int \frac{1}{x}\,dx}$$

$$= e^{\ln x} = x.$$

Multiplying the DE by $I(x)$, we find

$$x\frac{dy}{dx} + y = x^3$$

$$\frac{d}{dx}(xy) = x^3$$

$$xy = \int x^3\,dx$$

$$= \frac{x^4}{4} + C$$

Finally the general solution is

$$y = \frac{x^3}{4} + \frac{C}{x}$$

When $x = 1$, $y = 2$, so

$$2 = \frac{1}{4} + C \Rightarrow C = \frac{7}{4}$$

so

$$y = \frac{x^3}{4} + \frac{7}{4}\frac{1}{x}.$$

b) (i) The auxilary equation is

$$m^2 - 5m - 6 = 0$$

or $$(m-6)(m+1) = 0 \Rightarrow m = 6, -1$$

So roots are real, distinct.

The general solution is

$$y(x) = Ae^{6x} + Be^{-x}$$

(ii) The aux. equation is

$$m^2 - 4m + 8 = 0$$

which has solutions

$$m = \frac{4 \pm \sqrt{16 - 4 \times 1 \times 8}}{}$$

$$= \frac{4 \pm \sqrt{16 - 32}}{2}$$

$$= 2 \pm \frac{1}{2}\sqrt{-16}$$

$$= 2 \pm \frac{1}{2} 4\sqrt{-1}$$

$$= 2 \pm 2i$$

Roots are complex,

$$y(x) = e^{2x}(A\cos 2x + B\sin 2x)$$

c) (i) The aux. eq$^{\underline{n}}$ is

$$m^2 + 2\frac{\gamma}{M}m + \frac{k}{M} = 0$$

This has solutions

$$m = \frac{1}{2}\left(-\frac{2\gamma}{M} \pm \sqrt{4\frac{\gamma^2}{M^2} - \frac{4k}{M}}\right)$$

$$= -\frac{\gamma}{M} \pm \frac{1}{2}\frac{2}{M}\sqrt{\gamma^2 - kM}$$

$$= -\frac{\gamma}{M} \pm \frac{1}{M}\sqrt{\gamma^2 - kM}.$$

If $\gamma^2 < kM$, $\gamma^2 - kM < 0$ so

$$m = -\frac{\gamma}{M} \pm \frac{1}{M}i\sqrt{kM - \gamma^2}$$

$$= \alpha \pm i\beta$$

where

$$\alpha = -\frac{\gamma}{M}$$

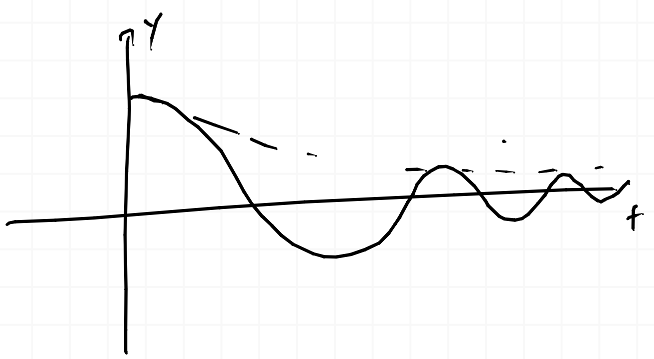
$$\beta = \frac{1}{M}\sqrt{kM - \gamma^2}$$

The roots are complex conjugates;

The general solution is the

$$y(t) = e^{-\alpha t}(A\cos\beta t + B\sin\beta t)$$

(ii)



(iii) The period is given by

$$T = \frac{2\pi}{\beta}$$

$$= \frac{2\pi}{\frac{1}{M}\sqrt{kM-\gamma^2}} = \frac{2\pi M}{\sqrt{kM-\gamma^2}}$$

If $\gamma = 0$, $T = \frac{2\pi M}{\sqrt{kM}} = 2\pi\sqrt{\frac{M}{k}}$

So if M is doubled,

T increases by a factor of $\sqrt{2}$.

6. (i)

$$AB = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 & 5 \\ 0 & -1 & 1 \end{pmatrix}$$

(ii)

$$AB^T = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 2 & 1 \end{pmatrix}$$ not possible

(iii) $A + B$ not possible

(iv)

$$A^{-1} = \frac{1}{-2-1} \begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= -\frac{1}{3} \begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix}$$

b)

$$\begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

or

$$A x = b$$

Now

$$A = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}$$

So

$$A^{-1} = \frac{1}{-1-8} \begin{pmatrix} -1 & -4 \\ -2 & 1 \end{pmatrix}$$

$$= -\frac{1}{9} \begin{pmatrix} -1 & -4 \\ -2 & 1 \end{pmatrix}$$

and

$$x = A^{-1} b$$

$$= -\frac{1}{9} \begin{pmatrix} -1 & -4 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$= \frac{-1}{9} \begin{pmatrix} -13 \\ -8 \end{pmatrix} \Rightarrow \begin{aligned} x_1 &= \frac{13}{9} \\ x_2 &= \frac{8}{9}. \end{aligned}$$

c)

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ -1 & 1 & -4 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{array}{cc} \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right) & \begin{array}{l} R_1 \\ R_1+R_2 \\ R_3 \end{array} \end{array}$$

$$\begin{array}{cc} \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 2 & -1 & -1 & 1 \end{array}\right) & \begin{array}{l} R_1 \\ \frac{1}{2}R_2 \\ R_3-R_2 \end{array} \end{array}$$

$$\begin{array}{cc} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{7}{2} & \frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right) & \begin{array}{l} R_1-R_2 \\ R_2 \\ \frac{1}{2}R_3 \end{array} \end{array}$$

$$\begin{array}{cc} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{9}{4} & \frac{5}{4} & -\frac{7}{4} \\ 0 & 1 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right) & \begin{array}{l} R_1-\frac{7}{2}R_3 \\ R_2+\frac{1}{2}R_3 \\ R_3 \end{array} \end{array}$$

So

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 9 & 5 & \\ 1 & 1 & 1 \\ -2 & -2 & 2 \end{pmatrix}$$

check:

$$A^{-1}A = \frac{1}{4}\begin{pmatrix} 9 & 5 & -7 \\ 1 & 1 & 1 \\ -2 & -2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 3 \\ -1 & 1 & -4 \\ 0 & 2 & 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = I \quad \checkmark$$

So

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 9 & 5 & -7 \\ 1 & 1 & 1 \\ -2 & -2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 7 \\ 3 \\ -2 \end{pmatrix} \Rightarrow \quad x_1 = \frac{7}{4}, \quad x_2 = \frac{3}{4}, \quad x_3 = \frac{-1}{2}.$$