THIS PAPER **MUST NOT BE REMOVED**: TO BE RETURNED AT THE END OF THE EXAMINATION

Cover Page - Type B

NAME: ....

STUDENT NO: ....

COURSE: ....

AUTUMN SEMESTER 2012 PRACTICE EXAMINATION

33190

Mathematical Modelling for Science

Bachelor of Science in Mathematics
Bachelor of Mathematics and Finance

Date: Practise Examination

Time allowed: Three hours plus ten minutes reading time

All questions are to be attempted
All questions are of equal value
Only non-progammable calculators permitted
Answer each question in a separate booklet

Examiner: C. G. Poulton

Assessor: M. Coupland

Question 1. - *Begin a new booklet for this question.*

(a) Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be the vectors $\mathbf{a} = (0, 3, 4)$, $\mathbf{b} = (1, 0, -2)$, and $\mathbf{c} = (-1, 0, 3)$.

(i) Evaluate $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

(ii) Find a vector parallel to $\mathbf{a}$ with length equal to 7.

(iii) Find a vector perpendicular to both $\mathbf{b}$ and $\mathbf{c}$ and with the same length as vector $\mathbf{a}$.

(6 marks)

(b) Find a scalar number m such that the vector $\mathbf{p} = (5, 1, 3)$ is perpendicular to the vector $\mathbf{q} = (1, 2, m)$. (4 marks)

(c) Find the Cartesian equation of the plane passing through the three points $(1, 0, -1)$, $(1, 2, 1)$ and $(1, -1, 0)$. (5 marks)

(d) A line goes through the point $(-1, -2, 3)$ and is parallel to the vector $(2, 1, -1)$. How close to the origin does the line come? *[Hint: At the closest point the direction of the line is perpendicular to the vector from the origin to the line.]* (5 marks)

Question 2. - *Begin a new booklet for this question.*

(a) For the complex numbers $z_1 = 2+2i$ and $z_2 = 3-5i$, express each of the following complex numbers in the form $a+ib$, with a and b real:

(i) $(z_1^2 + i)(z_2 - i)$

(ii) $\dfrac{z_1}{|z_2| - 2i}$ (5 marks)

(b) Express the complex number $-2\sqrt{3}+2i$ in exponential polar form (i.e. in the form $re^{i\theta}$). Hence or otherwise, express

$$(-2\sqrt{3}+2i)^{17}$$

in Cartesian form (i.e. in the form $a+ib$). (5 marks)

(c) Find all fourth roots of $1+i$, and plot the solutions in the complex plane. (5 marks)

(c) Starting from the complex identities for sine and cosine

$$\begin{aligned} \sin\theta &= \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) \\ \cos\theta &= \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \quad , \end{aligned}$$

prove the trigonometric identity

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

(5 marks)

Question 3. - *Begin a new booklet for this question.*

(a) Find the first derivatives of the following functions:

(i) $f(x) = (5x+3)^6$

(ii) $f(x) = \dfrac{1}{\ln x^2}$

(iii) $f(x) = \sinh^{-1}(2x)$

(iv) $f(x) = \sin^3(e^{2\cos x})$ (4 marks)

(b) To test zero-gravity devices, NASA uses aeroplanes that trace parabolic arcs, given by the equation

$$y(x) = h - \frac{1}{a}x^2$$

where $h = 8000$ m is the initial height of the run and $a = 2000\text{m}^2$ is a measure of the curvature. At the point $x = 100$m, it is observed that the aeroplane's horizontal velocity is 100 m s^{-1}. Compute the vertical velocity at this point. (6 marks)

(c) Show that the series

$$\sum_{k=0}^{\infty} k^2 e^{-2k}$$

converges. (5 marks)

(d) Find the first three non-zero terms of the Taylor series of

$$f(x) = \frac{1}{1+x}$$

expanded about $x = 0$. Using the *first three* terms of this series, estimate the value of the integral

$$\int_0^1 \frac{1}{1+x}\mathrm{dx}\ .$$

Then evaluate this integral directly and obtain a value for the error of your estimate. (5 marks)

Question 4. - *Begin a new booklet for this question.*

(a) Evaluate the following indefinite integrals

(i)
$$\int \frac{1}{(1+5x)^4}dx$$

(ii)
$$\int \frac{1}{2}\sinh 5x dx$$

(iii)
$$\int e^{2x-1}dx$$

(iv)
$$\int x^3 e^{-x^4}dx$$

(v)
$$\int \frac{3x+2}{x^2+5x+6}dx$$

(vi)
$$\int \frac{\sin x}{1+\cos^2 x}dx$$

(vii)
$$\int x^2 \cos x dx$$

(viii)
$$\int \frac{x}{\sqrt{1+3x^2}}dx$$

(8 marks)

(c) Evaluate the following definite integrals

(i)
$$\int_0^{\pi} \sin^2 x dx$$

(ii)
$$\int_0^1 \frac{x-1}{x^2+2x+2} dx$$

(iii)
$$\int_0^{\pi} x \sin x^2 dx$$

(iv)
$$\int_1^2 \frac{\ln x}{x} dx$$

(8 marks)

(d) Given the definition
$$I_n = \int \cos^n x dx$$
then prove that, for $n > 1$,
$$I_n = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2} \ .$$
Hence evaluate
$$\int \cos^4 x dx \ .$$

(4 marks)

Question 5. - *Begin a new booklet for this question.*

(a) Find the general solution to the differential equation

$$\frac{dy}{dx} + \frac{1}{x}y = x^2$$

Hence find the complete solution that satisifies the initial condition $y = 2$ when $x = 1$. (8 marks)

(b) Find the general solution to the following differential equations:

(i)

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 6y = 0\ .$$

(ii)

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 8y = 0\ .$$

(6 marks)

(c) A physical mass-spring system is modelled by the second-order differential equation

$$M\frac{d^2y}{dt^2} + 2\gamma\frac{dy}{dt} + ky = 0$$

where k is the spring constant, γ is the damping coefficient, and M is the mass.

(i) Show that, if $\gamma^2 < kM$, the general solution to this system is

$$y(t) = e^{-\alpha t}\left(A\cos(\beta t) + B\sin(\beta t)\right)$$

and find expressions for α and β in terms of γ, k, and M.

(ii) Graph the motion of the system for the initial conditions $y(0) = 1$, $y'(0) = 0$.

(iii) What is the period of the oscillation? In the limit that $\gamma \to 0$, what happens to the period if the mass M is doubled?

(6 marks)

Question 6. - *Begin a new booklet for this question.*

(a) Given the matrices $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$ determine, where possible:

(i) $\mathbf{AB}$

(ii) $\mathbf{AB}^T$

(iii) $\mathbf{A} + \mathbf{B}$

(iv) $\mathbf{A}^{-1}$. (8 marks)

(b) Write the system of equations

$$\begin{array}{rcrcl} x_1 & + & 4x_2 & = & 5 \\ 2x_1 & - & x_2 & = & 2 \end{array}$$

in matrix form. By finding the inverse, solve for x_1 and x_2. (8 marks)

(c) Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 1 & 3 \\ -1 & 1 & -4 \\ 0 & 2 & 1 \end{pmatrix} .$$

Hence solve the system of equations

$$\begin{array}{rrrcl} x_1 & + x_2 & + \ 3x_3 & = & 1 \\ -x_1 & + x_2 & - \ 4x_3 & = & 1 \\ & 2x_2 & + x_3 & = & 1 \ . \end{array}$$

(4 marks)

Table of integrals

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} + C$$
$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\frac{x}{a} + C_1 = \log\left(x + \sqrt{a^2 + x^2}\right) + C_2$$
$$\int \frac{dx}{1 - x^2} = \tanh^{-1}x + C_1 = \frac{1}{2}\log\left|\frac{1 + x}{1 - x}\right| + C_2$$
$$\int \cosh x \, dx = \sinh x + C$$
$$\int \sinh x \, dx = \cosh x + C$$
$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$$
$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$