Math 1 Exam

1)

a)

i) $\begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$

$= \begin{bmatrix} 3+3 & 2-1 \\ 2-1 & 5+5 \end{bmatrix}$

$= \begin{bmatrix} 6 & 1 \\ 1 & 10 \end{bmatrix}$

ii) Impossible, matrices different dimensions

iii) $\begin{bmatrix} 2 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 6 \end{bmatrix}$

$\begin{bmatrix} 2 & 0 & 12 \\ 4 & 0 & 24 \end{bmatrix}$

iV)

$A = \begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix}$

$A^{T} = \begin{bmatrix} 3 & -(2) \\ -(-1) & 5 \end{bmatrix} \frac{1}{(3\times5)-(\;\times -2)}$

$= \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix} \frac{1}{17}$

$= \begin{bmatrix} \frac{5}{17} & -\frac{2}{17} \\ \frac{1}{17} & \frac{3}{17} \end{bmatrix}$

V) Impossible

C's columns $\neq$ B's rows

b) $P = \langle 3, 2, -1 \rangle \qquad Q = \langle 1, 0, 2 \rangle$

i) $P \times Q = \langle 4, -(6-(-1)), -2 \rangle$

$$\begin{vmatrix} i & j & k \\ 3 & 2 & -1 \\ 1 & 0 & 2 \end{vmatrix}$$

$= \langle 4, -7, -2 \rangle$

$\therefore x = 2^2 - (1 \times 0) = 4$

$y = -((3 \times 2) - (-1)) = -7$

$z = (3 \times 0) - 2 = -2$

ii) $|Q| = \sqrt{1 + 2^2} = \sqrt{5}$

$\therefore \hat{Q} = \left\langle \frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}} \right\rangle$

iii) $\langle 5, 6, 10 \rangle + t \langle 3, 2, -1 \rangle$

$x = 3t + 5$
$y = 2t + 6$
$z = -t + 10$

iv) $P \cdot Q = 3 + (-2) = 1 \qquad |P| = \sqrt{3^2 + 2^2 + 1} = \sqrt{14} \qquad |Q| = \sqrt{5}$

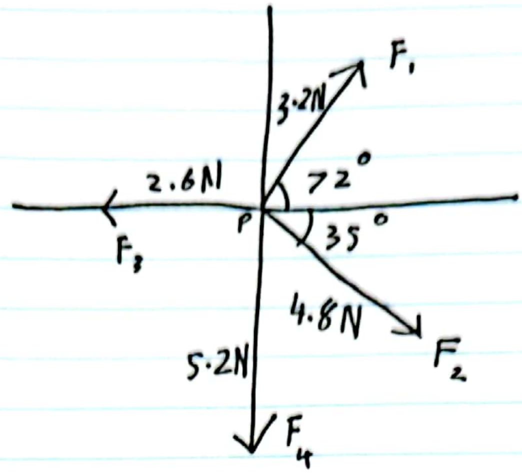
$\therefore \sqrt{14}\sqrt{5} \cos\theta = 1 \qquad \therefore \approx 83^\circ$

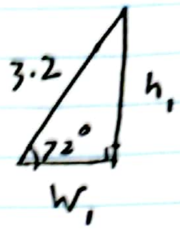
$\sqrt{70} \cos\theta = 1$

$\cos\theta = \frac{1}{\sqrt{70}}$

$\theta = \cos^{-1}\left(\frac{1}{\sqrt{70}}\right)$

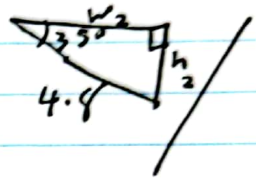
c)





$h_1 = 3.2\sin(72°)$

$W_1 = 3.2\cos(72°)$



$h_2 = -4.8\sin(35°)$

$w_2 = 4.8\cos(35°)$

$\therefore\ R = \sum_{n=1}^{4} F_n$

$F_3 = \langle -2.6, 0 \rangle$

$F_4 = \langle 0, -5.2 \rangle$

$\therefore\ F_1 = \langle 3.2\cos(72°), 3.2\sin(72°) \rangle$

$F_2 = \langle 4.8\cos(35°), -4.8\sin(35°) \rangle$

$\therefore\ R = \langle -2.6 + 3.2\cos(72°) + 4.8\cos(35°),\ -5.2 + 3.2\sin(72°) - 4.8\sin$

$\therefore\ R \approx \langle 2.32078, -4.90979 \rangle$

(Rounded to 5 decimals)

2)

a)

i) $y'' + 10y' + 25y = 0$

$$\frac{-10 \pm \sqrt{100-100}}{2} = \frac{-10}{2} = -5$$

$$\therefore y = (Ax+B)e^{-5x} \quad \text{or} \quad y = \frac{(Ax+B)}{e^{5x}}$$

ii)

$y'' = -64y \qquad y(0) = 36 \qquad y'(0) = 6$

$y'' + 64y = 0$

$$\frac{\pm\sqrt{-4(64)}}{2}$$

$$= \pm\frac{16i}{2} = \pm 8i$$

$$\therefore y = e^{0x}(A\cos(8x) + B\sin(8x))$$

$$y = A\cos(8x) + B\sin(8x)$$

$$y' = -8A\sin(8x) + 8B\cos(8x)$$

$6 = +8B \qquad 36 = A + \frac{3}{4}(0)$

$\therefore B = \frac{3}{4} \qquad \therefore A = 36$

$$\therefore y = 36\cos(8x) + \frac{3\sin(8x)}{4}$$

iii) $y'' + 8y' - 20y = 0$

$$\frac{-8 \pm \sqrt{64-4(-20)}}{2}$$

$$= \frac{-8 \pm \sqrt{144}}{2}$$

$$= \frac{-8 \pm 12}{2}$$

$$= 2, -20$$

$$\therefore \cancel{y} = Ae^{2x} + Be^{-20x}$$

iv) $y'' + 6y' + 13y = 0$

$$\frac{-6 \pm \sqrt{36-4(13)}}{2}$$

$$= \frac{-6 \pm 4i}{2}$$

$$= -3 \pm 2i$$

$$\therefore \boxed{y = e^{-3x}(A\cos(2x) + B\sin(2x))}$$

b) $2\frac{dy}{dx} + 8y = 5$

$$2\left(\frac{dy}{dx} + 4y\right) = 5$$

$$\frac{dy}{dx} + 4y = \frac{5}{2}$$

$$e^{\int 4dx} = I$$

$$I = e^{4x}$$

$$\left.y' e^{4x} + 4e^{4x} y = \frac{5}{2} e^{4x}\right\}$$

$$ye^{4x} = \int \frac{5}{2} e^{4x} dx$$

$$ye^{4x} = \frac{5e^{4x}}{8} + C$$

$$y = \frac{5}{8} + Ce^{-4x}$$

c)

i) $\frac{1}{(2x-1)(x+2)} \equiv \frac{A}{2x-1} + \frac{B}{x+2}$

$A(x+2)+B(2x-1)-1 \equiv 0$

$x=-2$	$x=\frac{1}{2}$
$B(2(-2)-1)-1 \equiv 0$	$A(\frac{1}{2}+2)-1 \equiv 0$
$-5B-1 \equiv 0$	$\frac{5}{2}A-1 \equiv 0$
$-5B \equiv 1$	$\frac{5}{2}A \equiv 1$
$B=-\frac{1}{5}$	$A=\frac{2}{5}$

ii) $\frac{dx}{dt} = k(1-x)(2-x)-3kx^2$ $x=0$ when $t=0$

$\frac{dx}{dt} = k(2-x-2x+x^2)-3kx^2$

$\frac{dx}{dt} = 2k-3kx+kx^2-3kx^2$

$\frac{dx}{dt} = k(-2x^2-3x+2)$

$\frac{dx}{dt} = -k(2x^2+3x-2)$

NEXT PAGE CONTINUES QUESTION

$$\frac{dx}{dt} = -k(2x-1)(x+2)$$

$$dx = -k(2x-1)(x+2)\,dt$$

$$\int \frac{dx}{(2x-1)(x+2)} = \int -k\,dt \quad \text{QED}$$

$$\int \frac{2}{5(2x-1)}\,dx + \int \frac{dx}{-5(x+2)} = \int -k\,dt$$

$$\frac{1}{5}\int \frac{10}{10x-5}\,dx + \frac{1}{-5}\int \frac{-5}{-5x+10}\,dx = \int -k\,dt$$

$$\frac{\ln|10x-5|}{5} + \frac{\ln|-5x+10|}{-5} = -kt$$

$$\cancel{\frac{\ln|-5|}{5} + \frac{\ln|10|}{-5} = c} \qquad + \frac{\ln\left|\frac{10x-5}{-5x+10}\right|}{5} = kt$$

when $x=0$

$$\frac{\ln(\frac{1}{2})}{5} = kt \qquad \ln(\tfrac{1}{2}) = 5kt$$

3)

a) $\int x^2 e^{5x}\,dx$

$u = x^2 \quad v = \frac{e^{5x}}{5}$

$u' = 2x \quad v' = e^{5x}$

$= \frac{x^2 e^{5x}}{5} - \int \frac{2x e^{5x}}{5}\,dx$

$u = 2x \quad v = \frac{e^{5x}}{25}$

$u' = 2 \quad v' = \frac{e^{5x}}{5}$

$= \frac{x^2 e^{5x}}{5} - \left(\frac{2x e^{5x}}{25} - \int \frac{2e^{5x}}{25}\,dx \right)$

$= \frac{x^2 e^{5x}}{5} - \frac{2x e^{5x}}{25} + \frac{2e^{5x}}{125} + C$

$= \frac{25x^2 e^{5x} - 10x e^{5x} + 2e^{5x}}{125} + C$

$= \frac{e^{5x}(25x^2 - 10x + 2)}{125} + C$

ii) $\displaystyle\int \frac{dx}{\sqrt{9+4x^2}}$

$$= \int \frac{dx}{\sqrt{3^2+4x^2}}$$

$$= \int \frac{dx}{\sqrt{3^2+(2x)^2}}$$

$$= \sinh^{-1}\left(\frac{2x}{3}\right)+C$$

iii) $\displaystyle\int e^x \sinh\left(\frac{x}{2}\right) dx$

$u = \sinh\left(\frac{x}{2}\right) \qquad v = e^x$

$u' = \frac{\cosh\left(\frac{x}{2}\right)}{2} \qquad v' = e^x$

$$= \sinh\left(\frac{x}{2}\right)e^x - \int \frac{\cosh\left(\frac{x}{2}\right)}{2} e^x \, dx$$

$$= \sinh\left(\frac{x}{2}\right)e^x - \left(\frac{\cosh\left(\frac{x}{2}\right)e^x}{2} - \int \frac{\sinh\left(\frac{x}{2}\right)}{4} e^x \, dx\right)$$

$$\int \sinh\left(\frac{x}{2}\right)e^x \, dx = \sinh\left(\frac{x}{2}\right)e^x - \frac{\cosh\left(\frac{x}{2}\right)e^x}{2} + \frac{1}{4}\int \sinh\left(\frac{x}{2}\right)e^x \, dx$$

$$\frac{3}{4}\int \sinh\left(\frac{x}{2}\right)e^x = \sinh\left(\frac{x}{2}\right)e^x - \frac{\cosh\left(\frac{x}{2}\right)e^x}{2}$$

$$= \frac{4\left(\sinh\left(\frac{x}{2}\right)e^x - \cosh\left(\frac{x}{2}\right)e^x\right)}{6} + C$$

b)

i) $\int_0^{2.1} \sin(x)\,dx$

$= \left[-\cos(x)\right]_0^{2.1}$

$= -\cos(2.1) + \cos(0)$

$= -\cos(2.1) + 1$

$\approx +1.50$

ii) $\frac{2.1-0}{3} = \frac{7}{10} = \Delta x$

$\therefore = \frac{\Delta x}{2}\left(\sin(0) + 2\sin\left(\frac{7}{10}\right) + 2\sin\left(\frac{7}{5}\right) + \sin\left(\frac{21}{10}\right)\right)$

$= \frac{7}{20}\left(\sin(0) + 2\sin\left(\frac{7}{10}\right) + 2\sin\left(\frac{7}{5}\right) + \sin\left(\frac{21}{10}\right)\right)$

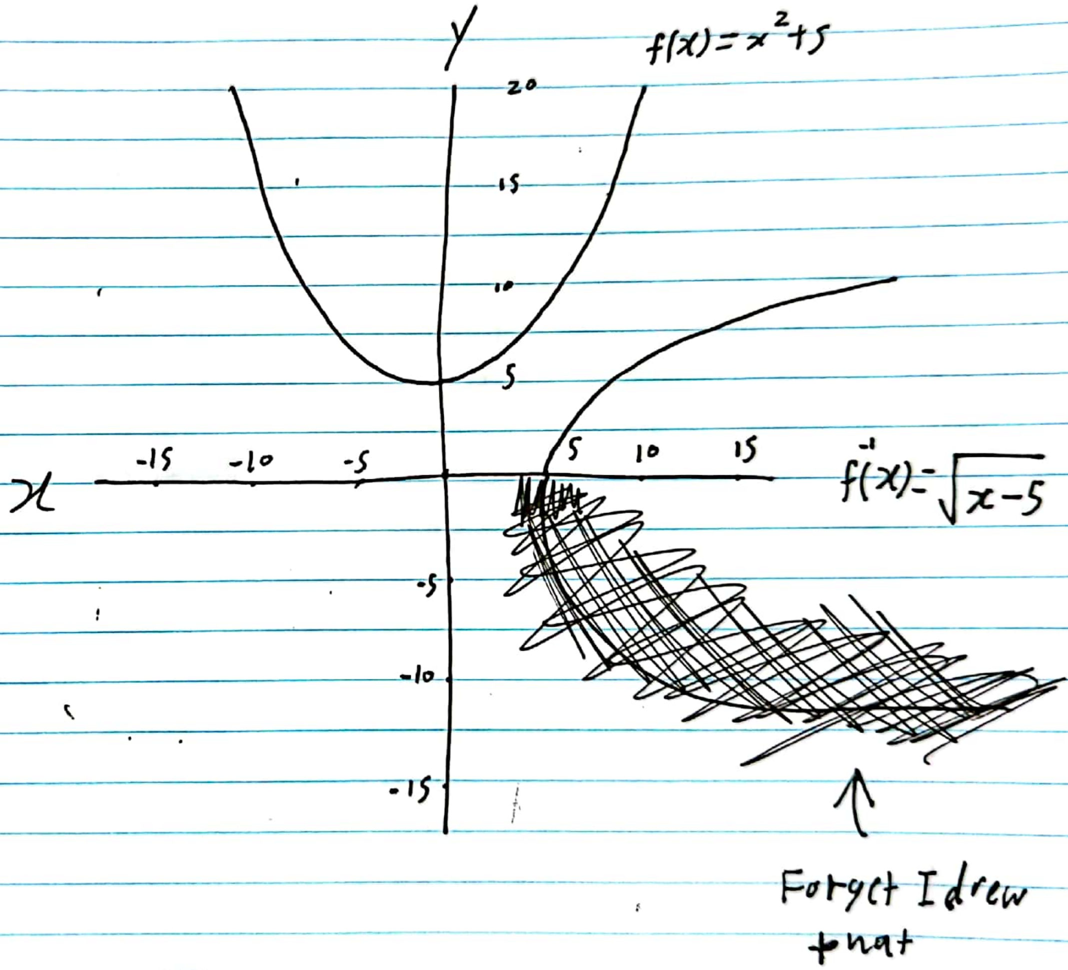
$\approx +1.44$

iii) In i, I used the integral to return a more precise value using calculus which works on an infinitesimal basis, where the trapezoidal rule or similarly riemann sums use shapes (or in this case, trapezoids) to estimate area under a curve by connecting lines to each value of a function. When the gradient is decreasing, the rule underestimates and when gradient increases, overestimates.

c) $f(x) = x^2 + 5$

i) When inverted, some functions don't pass the horizontal line test (and hence aren't functions I suppose). When the domain is restricted like in $\sin^{-1}(x)$ from $[-1, 1]$, a function can pass such a test

ii)



iii) $f^{-1}(x) = \sqrt{x-5}$ How I got this:

$y = x^2 + 5$

$x = y^2 + 5$

$x - 5 = y^2$ QED

$y = \sqrt{x-5}$

iv)

Domain : $[5, \infty)$

Range : $[0, \infty)$

d) $y = 2x^{3x}$

$\ln(y) = 3x\ln(2x)$

$e^{3x\ln(2x)} = y$

$y' = e^{3x\ln(2x)}\left(3 + 3\ln(2x)\right)$

$\frac{dy}{dx} = 3e^{3x\ln(2x)}\left(1 + \ln(2x)\right)$

4)

a) $z = 5 + 5i \qquad w = 3 + 3\sqrt{3}\, i$

i) $2(5-5i) + (3 + 3\sqrt{3}\, i)$

$= 10 - 10i + 3 + 3\sqrt{3}\, i$

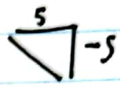
$= 13 - 10i + i\sqrt{27}$

$= 13 + (\sqrt{27} - 10)i$

$\therefore \mathcal{R}(2z + w) = 13$

$\therefore \mathcal{I}(2z + w) = \sqrt{27} - 10$

ii) $|z| = \sqrt{50}$



$z = \sqrt{50}\, e^{i \tan^{-1}(-1)}$

$z = \sqrt{50}\, e^{-i\frac{\pi}{4}}$

$\cancel{\text{or } \sqrt{50}\, e^{i\frac{7\pi}{4}}}$

iii) $z^{20} = \left(\sqrt{50}\, e^{+i\frac{7\pi}{4}}\right)^{20}$

$= 50^{10} e^{i\frac{140\pi}{4}}$

$= 50^{10} e^{i\pi}$

$\therefore -5^a 2^b + ci = -50^{10}$

$-5^a 2^b = -50^{10}$

$-5^a 2^b = (-5 \times 10)^{10}$

$-5^a 2^b = -5^{10} 10^{10}$

$-5^a 2^b = -5^{10} (2 \times 5)^{10}$

$-5^a 2^b = -5^{10} 2^{10} 5^{10}$

$-5^a 2^b = -5^{20} 2^{10}$

$\therefore c = 0$

$\therefore a = 20$

$\therefore b = 10$

CamScanner

b) $j^2 = -1$ $\qquad z = 1$

$$Z = \frac{(1+j)z - 2 + j4}{z + 2 + j}$$

$$Z = \frac{-(1+j) + 4j}{3+j} \cdot \frac{3-j}{3-j}$$

$$Z = \frac{-3(1+j) + 12j + j(1+j) - 4j^2}{9}$$

$$Z = \frac{-3 - 3j + 12j + j + j^2 + 4}{9}$$

$$Z = \frac{10j}{9}$$

$$\therefore Z = 0 + \frac{10j}{9}$$

$$\mathcal{R}(Z) = 0$$

$$\mathcal{I}(Z) = \frac{10}{9}$$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)!}{2^n}$

i)

$n_1 = \frac{2!}{2} = 1$

$n_2 = \frac{3!}{4} = \frac{3}{2}$

$n_3 = \frac{4!}{8} = 3$

ii) $\displaystyle\sum_{n=1}^{1} \frac{(n+1)!}{2^n} = 1$

$\displaystyle\sum_{n=1}^{2} \frac{(n+1)!}{2^n} = 1 + \frac{3}{2} = \frac{5}{2}$

$\displaystyle\sum_{n=1}^{3} \frac{(n+1)!}{2^n} = 1 + \frac{3}{2} + 3 = \frac{11}{2}$

CamScanner

iii)

$$\lim_{n\to\infty} \frac{(n+2)!}{2^n 2} \cdot \frac{2^n}{(n+1)!}$$

$$\lim_{n\to\infty} \frac{(n+2)!}{2(n+1)!}$$

$$\lim_{n\to\infty} \frac{(n+1)!(n+2)}{2(n+1)!}$$

$$\lim_{n\to\infty} \frac{n+2}{2}$$

approaches infinity, diverges

d)

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$\sin\left(\frac{\pi}{60}\right) \approx \frac{\pi}{60} - \frac{\left(\frac{\pi}{60}\right)^3}{3!} + \frac{\left(\frac{\pi}{60}\right)^5}{5!}$$

$$\sin\left(\frac{\pi}{60}\right) \approx \frac{\pi}{60} - \frac{\pi^3}{1296000} + \frac{\pi^5}{120 \times 60^5}$$

$$\approx 0.052$$