Final exam
33130 Mathematics 1 - Autumn 2022

Please submit your solutions as a single PDF

- **Please name the PDF with your name and your student number.**
- **By submitting this exam, you agree to the following conditions:**
- **I have completed the exam on my own. I have not discussed the solutions with anyone.**
- **I have neither sought nor obtained help from anyone for the duration of this exam.**
- **I agree that I will not post the questions or my solutions online, nor will I show them to anyone.**

Question 1 (20 marks)

(a) Evaluate the integral

$$\int x\sin(10x)dx$$

(4 marks)

(b) Find the first derivative of

$$f(x) = \arcsin\left(\pi\sqrt{x^3}\right)$$

(4 marks)

(c) Find the first derivative of

$$y = [\ln x]^{-x}$$

(5 marks)

(Hint: use logarithmic differentiation)

(d) There was an outbreak of a virus in a community. When the number of infections *N* reached 100 the government introduced social distancing to stop the spread. Scientists predicted that the number of cases *N(t)* after the introduction of social distancing should obey the equation

$$\frac{dN}{dt} = \frac{N^2}{1000}e^{-t/9},$$

where *t* is measured in days.

(i) Find *N(t)* and deduce the maximum possible number of infections *N*

(5 marks)

(ii) What is the rate of spread after 70 days? **(2 marks)**

Question 2 (20 marks)

(a) Calculate the angle between the vectors $\mathbf{a} = (2, 2, 2)$ and $\mathbf{b} = (1, 0, -1)$

(4 marks)

(b) Calculate $\mathbf{c} \times \mathbf{d}$, where $\mathbf{c} = (1, 1, 2)$ and $\mathbf{d} = (-1, 1, -2)$

(4 marks)

(c) Prove the identity $(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b}) = -2(\mathbf{a} \times \mathbf{b})$, where **a** and **b** are vectors in R^3

(4 marks)

(Hint: use the definitions of dot and cross products

(d) Find all solutions of the equation. Leave your solutions in the exponential form.

$$32z^5 = 1$$

(4 marks)

(e) According to one model of rabbit population growth, the rate of change in the number of rabbits is proportional to the fifth root of the number of rabbits present. There are 32rabbits present initially and 60 days later there are 1024 rabbits. Will the number of rabbits exceed 100000 in 6 years?

(4 marks)

Question 3 (20 marks)

(a) Find all solutions of the differential equations

$$\frac{d^2y}{dx^2} + 4y = \sin x,$$

(5 marks)

(b) Evaluate the integrals

i) $$\int \frac{x^4 dx}{(1+x^5)^{2/3}},$$

(5 marks)

ii) $$\int \frac{2x-5}{x^2-5x+6} dx.$$

(5 marks)

(c) Determine whether or not the series converges

$$\sum_{k=1}^{\infty} \frac{3^{-k}\pi^k}{k+3}.$$

(5 marks)

Question 4 (20 marks)

(a) Solve the following differential equation using the integrating factor method:

$$xy' - 5y = x^6$$

(5 marks)

(b) The concentration $c(t)$ of a particular toxin at time t is given by

$$c(t) = \frac{5}{2} t^{3/2} e^{-t^{5/2}}$$

in units of *m*g/L. The *average concentration C*, which is related to the deadliness of the toxin, is given by

$$C = \frac{1}{T} \int_0^T c(t)\,dt,$$

where T is the total time during which the toxin circulates. Compute the value of C if T = 2min.

(5 marks)

(c) Find the value of the following determinant

$$\begin{vmatrix} -1 & 2 & 3 \\ -7 & 8 & 9 \\ -4 & 5 & 6 \end{vmatrix}$$

(4 marks)

(d) Find the derivative of the function y(x) given implicitly

$$x^3 y^4 + \cos(xy^3) = x^2$$

(6 marks)