# Final exam
# 33130 Mathematics 1 - Autumn 2022

Please submit your solutions as a single PDF

- **Please name the PDF with your name and your student number.**
- **By submitting this exam, you agree to the following conditions:**
- **I have completed the exam on my own. I have not discussed the solutions with anyone.**
- **I have neither sought nor obtained help from anyone for the duration of this exam.**
- **I agree that I will not post the questions or my solutions online, nor will I show them to anyone.**

**Question 1 (20 marks)**

**(a)** Evaluate the integral

$$\int x\cos(7x)dx$$

**(4 marks)**

**(b)** Find the first derivative of

$$f(x) = \arctan\left(\pi\sqrt{x}\right)$$

**(4 marks)**

**(c)** Find the first derivative of

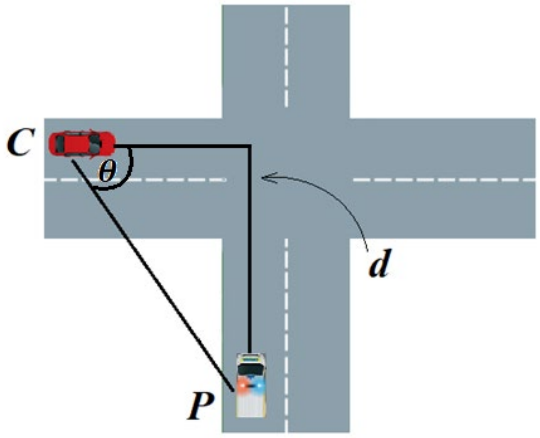
$$y = x^{\ln(x)}$$

**(5 marks)**

(Hint: use logarithmic differentiation)

**(d)** The stationary police car ***P*** (see the figure below), which is located at a distance d=100m from the intersection, monitors the speed of cars at the intersection. The speed limit in all directions at this intersection is 60km/h (this is approximately 17m/s). The car ***C*** moves towards the intersection with a constant velocity. The rate of change of the angle $\theta$ at the moment when $\theta = \pi/6$ is 0.05rad/s. What is the velocity of car ***C***?

**(7 marks)**



**Question 2 (20 marks)**

**(a)** Calculate the angle between the vectors $\mathbf{a} = (1,1,1)$ and $\mathbf{b} = (1,-1,0)$

**(4 marks)**

**(b)** Calculate $\mathbf{c} \times \mathbf{d}$, where $\mathbf{c} = (2,1,1)$ and $\mathbf{d} = (1,-1,2)$

**(4 marks)**

**(c)** Prove the identity $|\mathbf{e} \times \mathbf{f}|^2 = |\mathbf{e}|^2 |\mathbf{f}|^2 - (\mathbf{e} \cdot \mathbf{f})^2$, where **e** and **f** are vectors in $R^3$

**(4 marks)**

(Hint: use the definitions of dot and cross products)

**(d)** Find all solutions of the equation. Leave your solutions in the exponential form.

$$64z^3 = -1.$$

**(4 marks)**

**(e)** According to one model of rabbit population growth, the rate of change in the number of rabbits is proportional to the cubic root of the number of rabbits present. If there are 8 rabbits present initially and 4 months later there are 27 rabbits, how many rabbits are there after 4 years?

**(4 marks)**

**Question 3 (20 marks)**

**(a)** Find all solutions of the differential equation

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = e^x$$

**(5 marks)**

**(b)** Evaluate the integrals

i) $$\int \frac{x^3 dx}{(1+x^4)^{1/3}},$$

**(5 marks)**

ii) $$\int \frac{2x+7}{(x+3)(x+4)} dx.$$

**(5 marks)**

**(c)** Determine whether or not the series converges

$$\sum_{k=1}^{\infty} \frac{2^k e^{-k}}{k+5}.$$

**(5 marks)**

**Question 4 (20 marks)**

**(a)** Solve the following differential equation using the integrating factor method:

$$xy' - 2y = x^4$$ **(5 marks)**

**(b)** The concentration $c(t)$ of a particular toxin at time $t$ is given by

$$c(t) = \frac{3}{2}\sqrt{t}e^{-t^{3/2}}$$

in units of *m*g/L. The *average concentration C*, which is related to the deadliness of the toxin, is given by

$$C = \frac{1}{T}\int_0^T c(t)dt,$$

where $T$ is the total time during which the toxin circulates. Compute the value of $C$ if $T$ = 1min.

**(5 marks)**

**(c)** Find the value of the following determinant

$$\begin{vmatrix} -1 & -2 & -3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{vmatrix}$$

**(4 marks)**

**(d)** Find the derivative of the function y(x) given implicitly

$$xy^4 + \sin(xy) = 1$$ **(6 marks)**