Q1 a) $\int 3x\sqrt{x-2}\,dx = 3\int x\sqrt{x-2}\,dx.$

Let $u = x-2$ then $du = dx$, $u+2 = x$.

We get $3\int (u+2)u^{1/2}\,du$

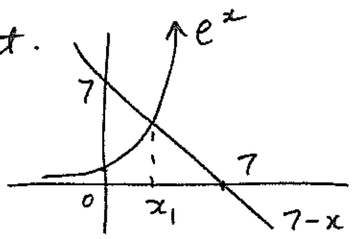
$= 3\int \left(u^{3/2} + 2u^{1/2}\right) du$

$= 3.\left[\frac{2}{5}u^{5/2} + 2.\frac{2}{3}.u^{3/2}\right] + C$

$= \frac{6}{5}\sqrt{(x-2)^5} + 4\sqrt{(x-2)^3} + C$

ie $\frac{6}{5}(x-2)^{5/2} + 4(x-2)^{3/2} + C.$

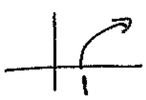
Q1 (b). Root(s) of $f(x) = e^x + x - 7$ occur at the intersection of $y_1 = e^x$ and $y_2 = 7-x$. Sketch these roughly to scale and see where they intersect.



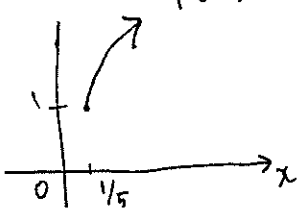
This is a good choice for x_1.

for example, $x_1 = 2$ is a very rough choice but would be suitable.

Q1 (c)(i) Since $\cosh x$ is , then

$\cosh^{-1} x$ is 

Now $f(x) = 1 + 4\cosh^{-1}(5x)$ would be:



Domain is $x \geq \frac{1}{5}$.

[Check by substituting some values].

Q1 (d)(i) From $\cos(A+B) = \cos A \cos B - \sin A \sin B$, put $B = A$ and we obtain

$$\cos(A+A) = \cos A \cos A - \sin A \sin A$$

so

$$\cos 2A = \cos^2 A - \sin^2 A.$$
$$= 1 - \sin^2 A - \sin^2 A$$
$$= 1 - 2\sin^2 A.$$

$$\therefore \quad \sin^2 A = \frac{1}{2}(1 - \cos 2A).$$

and

$$\int \sin^2 x \, dx = \frac{1}{2}\int (1 - \cos 2x)\, dx$$
$$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C$$
$$= \underline{\frac{x}{2} - \frac{1}{4}\sin 2x + C.}$$

Q1

(d)(ii). We will need to show that

$$\int_a^b f_1(x) \cdot f_1(x) = 1, \quad \text{and}$$

$$\int_a^b f_2(x) \cdot f_2(x) = 1, \quad \text{and}$$

$$\int_a^b f_1(x) \cdot f_2(x) = 0, \quad \text{where } a = 0, \; b = 2\pi.$$

For the first case,

$$\int_0^{2\pi} \left(\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi}}\right) dx = \left[\frac{x}{2\pi}\right]_0^{2\pi}$$

$$= \frac{2\pi}{2\pi} - 0$$

$$= 1.$$

(PTO).

Q1(d)(ii) For the second case,

$$\int_0^{2\pi} \left(\frac{1}{\sqrt{\pi}} \sin x \cdot \frac{1}{\sqrt{\pi}} \sin x\right) dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} \sin^2 x \quad dx$$

$$= \frac{1}{\pi} \left[\frac{x}{2} - \frac{1}{4} \sin 2x \right]_0^{2\pi} \qquad \text{from (i)}$$

$$= \frac{1}{\pi}\left(\frac{2\pi}{2} - \frac{1}{4}\sin 4\pi - (0 - 0)\right)$$

$$= \frac{1}{\pi}(\pi - 0)$$

$$= 1.$$

Then for the third case,

$$\int_0^{2\pi} \left(\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{\pi}} \sin x\right) dx$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{\pi} \int_0^{2\pi} \sin x \quad dx$$

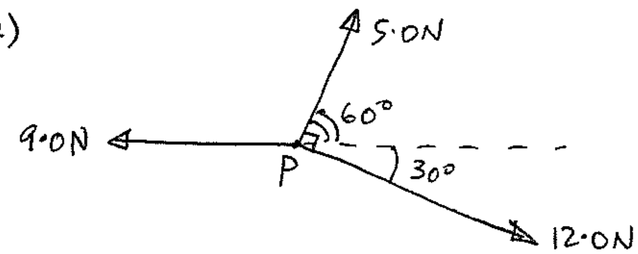
$$= \frac{1}{\sqrt{2} \cdot \pi} \cdot \left[\cos x\right]_0^{2\pi}$$

$$= \frac{1}{\sqrt{2}\pi} \cdot (\cos 2\pi - \cos 0)$$

$$= \frac{1}{\sqrt{2} \cdot \pi} \cdot (1 - 1)$$

$= 0.$ We have shown that $\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \sin x\right\}$ is an orthonormal set of functions.

Q2(a)

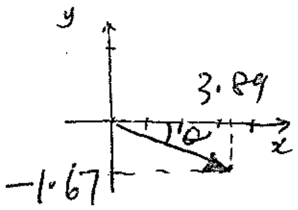


As a vector, the total force acting on point P is $(-9+12\cos 30^\circ+5\cos 60^\circ)\underline{i}+(5\sin 60^\circ-12\sin 30^\circ)\underline{j}$

$\doteqdot 3.8923048\ldots\underline{i} - 1.6698729\ldots\underline{j}$

The magnitude is $\sqrt{(3.89\ldots)^2+(1.66987\ldots)^2}$

$\doteqdot 4.235388\ldots \doteqdot \underline{4.2\text{ N}}$ (2 sig. figs.).



The direction is at an angle of θ ~~above~~ below the horizontal, where

$\theta = \tan^{-1}\left(\dfrac{1.66987\ldots}{3.8923\ldots}\right)$

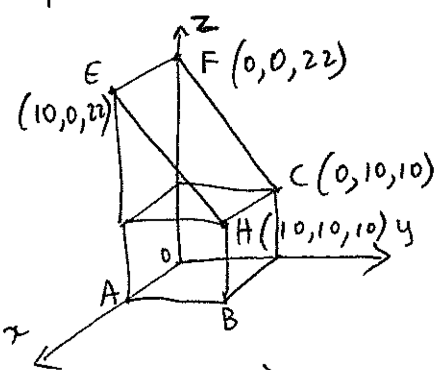
$\doteqdot 23.220\ldots$

$\doteqdot \underline{23^\circ}$ (to nearest degree)

OR: in usual orientation

337°

Q2(b). (i) To find the normal to the plane EHCF there are several ways :-



We could use the cross product of any two vectors in the plane.

e.g. $\overrightarrow{HE} = \langle 0,-10,12\rangle$

$\overrightarrow{HC} = \langle -10,0,0\rangle$

Then $\overrightarrow{HE}\times\overrightarrow{HC}$ is normal to the plane.

(PTO)

Q2 (b)(i) Continued.

$$\vec{HE} \times \vec{HC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & -10 & 12 \\ -10 & 0 & 0 \end{vmatrix}$$

$$= 0\underline{i} - \begin{vmatrix} 0 & 12 \\ -10 & 0 \end{vmatrix} \underline{j} + \begin{vmatrix} 0 & -10 \\ -10 & 0 \end{vmatrix} \underline{k}$$

$$= 0\underline{i} - 120\underline{j} - 100\underline{k}.$$

Note that any multiple of the vector $\langle 0, 6, 5 \rangle$ would be correct, as they are all normal to the plane.

(ii) The vector $\vec{FH} = \langle 10, 10, -12 \rangle$.

The line FH is given by the cartesian equations

$$x = 0 + 10t$$
$$y = 0 + 10t$$
$$z = 22 - 12t.$$

[This is not a unique expression for this line. eg $x = t_*$
$y = t_*$
$z = 22 - \left(\frac{12}{10} t_*\right)$
Would also generate all the points on the line FH.]

(iii) The line $x = 10t$, $y = 10t$, $z = 22 - 12t$ meets the xy plane where $z = 0$, ie $22 = 12t$, $t = \frac{22}{12} = \frac{11}{6}$. At this point on the line, $x = 10 \cdot \frac{11}{6} = \frac{55}{3}$, $y = \frac{55}{3}$, $z = 0$. The point is $\left(\frac{55}{3}, \frac{55}{3}, 0\right)$.

Q2(c). We need to solve

$$xy' - 4y = x^6 e^x.$$

This is equivalent to:-

$$y' - \frac{4}{x}y = x^5 e^x.$$

The integrating factor is

$$I = e^{\int -\frac{4}{x}dx} = e^{-4\ln x} = e^{\ln x^{-4}}$$

$$= x^{-4}.$$

Multiplying by I we get

$$\underbrace{x^{-4}y' - \frac{4}{x^5}y}_{\frac{d}{dx}\left(y.\,x^{-4}\right)} = xe^x$$

$$\frac{d}{dx}\left(y.\,x^{-4}\right) = xe^x$$

$$\therefore \quad y.\,x^{-4} = \int xe^x\,dx$$

We need integration by parts on the RHS.

$$y.\,x^{-4} = xe^x - \int e^x\,dx$$

$$y.\,x^{-4} = xe^x - e^x + c$$

So $\underline{y = x^5e^x - x^4e^x + cx^4}.$

Q3(a)(i) Solve $y''+25y = 3\sin(2t)$ —— ①

First solve $y''+25y=0$.

The Auxiliary Equation is $m^2+25=0$

so $m^2=-25$

$\underline{m = \pm 5i}$

The general solution of $y''+25y=0$ is

$y = C_1 \cos 5t + C_2 \sin 5t$.

We call this the complementary soln. of $y''+25y = 3\sin(2t)$, so

$y_c = C_1\cos(5t) + C_2\sin(5t)$.

Now choose $y_p = A\sin(2t) + B\cos(2t)$.

From the appearance of ①, $B=0$.

[But we can leave it in for now.]

So let $y_p = A\sin(2t) + B\cos(2t)$

$y_p' = 2A\cos(2t) - 2B\sin(2t)$

$y_p'' = -4A\sin(2t) - 4B\cos(2t)$.

Put these into ① to get

$-4A\sin(2t) - 4B\cos(2t) + 25A\sin(2t) + 25B\cos(2t)$
$= 3\sin(2t)$

Coefficients of $\cos(2t)$ on LHS and RHS :-

$-4B+25B=0 \quad \therefore \underline{B=0}.$

Coefficients of $\sin(2t)$ on LHS & RHS :-

$-4A+25A = 3 \quad \therefore \quad 21A=3, A=\frac{3}{21}=\underline{\frac{1}{7}}.$

Finally, $y = y_c + y_p$ so

$\underline{y = C_1\cos(5t) + C_2\sin(5t) + \frac{1}{7}\sin(2t)}$.

(general solution).

Q3(a)(ii) To find the particular solution:-

We have

$$y = C_1\cos(5t) + C_2\sin(5t) + \frac{1}{7}\sin(2t).$$

Now $y(0) = 0$ so

$$0 = C_1\cos(0) + C_2\sin(0) + \frac{1}{7}\sin(0)$$

$$0 = C_1 + 0 + 0 \quad \therefore \underline{\underline{C_1 = 0}}$$

So $y = C_2\sin(5t) + \frac{1}{7}\sin(2t)$, and

$$y' = 5C_2\cos(5t) + \frac{2}{7}\cos(2t).$$

Now $y'(0) = 1$, so

$$1 = 5C_2\cos(0) + \frac{2}{7}\cos(0)$$

$$1 = 5C_2 + \frac{2}{7}$$

$$\frac{5}{7} = 5C_2 \quad \therefore \quad \underline{\underline{C_2 = \frac{1}{7}}}.$$

The Particular Solution will be

$$\underline{y = \frac{1}{7}\sin(5t) + \frac{1}{7}\sin(2t).}$$

Q3(b).

$$Y = \frac{1}{Z}$$

$$= \frac{1}{65-20j} \times \frac{65+20j}{65+20j}$$

$$= \frac{65+20j}{4225+400}$$

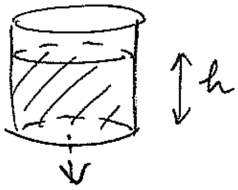
$$= \frac{65}{4625} + \frac{20}{4625}j$$

The Real Part is $\frac{13}{925}$.

The Imaginary Part is $\frac{4}{4625}$.

Q3 (c)

(i)



We have $\dfrac{dh}{dt} = -k\sqrt{h}$

so $\displaystyle\int h^{-1/2}\,dh = \int -k\,dt$

$$2h^{1/2} = -kt + c$$

$$h^{1/2} = \frac{c-kt}{2}$$

$$h = \frac{(c-kt)^2}{4}.$$

(ii) Now $k = 0.3$ and $h(0) = 1.6$ so

$$1.6 = \frac{(c - 0.3\times 0)^2}{4}.$$

$$6.4 = c^2 \qquad \therefore c \doteqdot 2.529822$$

ie $\quad c = \sqrt{\dfrac{64}{10}} = \dfrac{8}{\sqrt{10}} = \dfrac{8\sqrt{10}}{10}$

$$= \frac{4\sqrt{10}}{5}.$$

When the water has drained from the tank, $h = 0$ so

$$0 = \frac{\left(\frac{4\sqrt{10}}{5} - 0.3t\right)^2}{4}$$

So $\dfrac{4\sqrt{10}}{5} = \dfrac{3t}{10} \quad \therefore t = \dfrac{\overset{8}{\cancel{40}}\sqrt{10}}{\cancel{15}_3}$

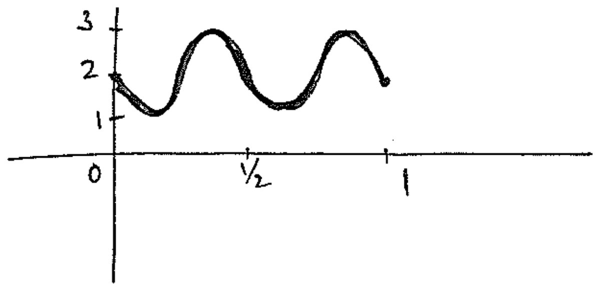
$$t = \frac{8\sqrt{10}}{3}\text{ h.} \doteqdot \underline{8.4327\text{ h.}}$$

Q3 (d).

The output is the function

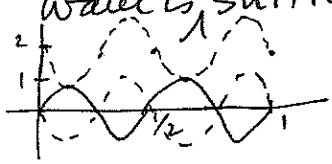
$$f(t) = \sin(4\pi t)$$

which is defined, then a graph is plotted of $2 - \sin(4\pi t)$ on $0 \leq t \leq 1$.



Wavelength $= \frac{2\pi}{4\pi} = \frac{1}{2}$.

Amplitude is 1, wave is inverted & shifted up 2.



Then the second derivative is given.

$$f(t) = \sin(4\pi t)$$

$$f'(t) = 4\pi \cos(4\pi t)$$

$$f''(t) = -16\pi^2 \sin(4\pi t)$$

Q4 (a) There are various ways to express the answer here.

eg. "For the series $\sum_{n=1}^{\infty} a_n$, the mth partial sum is $\sum_{n=1}^{m} a_n$."

---

another way to put this:—

"The partial sums of the series $a_1 + a_2 + a_3 + a_4 + \cdots$

are $S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

etc."

"If $S_k \to L$ as $k \to \infty$ then the series $\sum_{n=1}^{\infty} a_n$ converges to L."

another way to say this :—

"If $\lim_{k\to\infty} S_k$ exists and is finite then the ~~series~~ series $\sum_{n=1}^{\infty} a_n$ converges.".

Q4(b). Find open interval of convergence for $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+1}$.

For convergence we need

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1 \quad \text{i.e.}$$

$$\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+1)^2+1} \div \frac{(x-2)^n}{n^2+1} \right| < 1$$

$$\lim_{n\to\infty} \left| \frac{\cancel{(x-2)^n}(x-2)}{n^2+2n+2} \times \frac{n^2+1}{\cancel{(x-2)^n}} \right| < 1$$

$$|x-2| \cdot \underbrace{\lim_{n\to\infty} \left| \frac{n^2+1}{n^2+2n+2} \right|}_{1.} < 1$$

So $|x-2| < 1$

ie $\quad -1 < x-2 < 1$

$$\underline{1 < x < 3}.$$

Q4 (c). $\sum_{n=1}^{\infty} \frac{n+3}{2^n(n+1)!}$

We have $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$= \lim_{n\to\infty} \left| \frac{n+4}{2^{n+1}(n+2)!} \div \frac{n+3}{2^n(n+1)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n+4}{2^{n+1}(n+2)\cancel{(n+1)!}} \times \frac{2^n\cancel{(n+1)!}}{n+3} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n+4}{2(n+2)(n+3)} \right|$$

$$= 0$$

so the series $\sum_{n=1}^{\infty} \frac{n+3}{2^n(n+1)!}$ converges.

Q4 (d) (i) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} (-1)^n .$$

[Other answers are possible.].

(ii) For convergence, $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$

so $\lim_{n\to\infty} \left| \frac{x^{2(n+1)+1}}{(2(n+1)+1)!} \div \frac{x^{2n+1}}{(2n+1)!} \right| < 1.$

PTO

Q4(d)(ii) continued.

$$\lim_{n\to\infty}\left|\frac{x^{2n+3}}{(2n+3)(2n+2)(2n+1)!}\cdot\frac{(2n+1)!}{x^{2n+1}}\right| < 1$$

$$\underbrace{\lim_{n\to\infty}\left|\frac{1}{(2n+3)(2n+2)}\right|}_{0.}\cdot\left|x^2\right| < 1$$

Since LHS is zero for all values of x, the interval of convergence is $\underline{-\infty < x < \infty}$.

Q4(d)(iii).

$$\sin(0.1) \doteqdot 0.1 - \frac{0.1^3}{3!} + \frac{0.1^5}{5!}$$

$$\doteqdot 0.0998334166 7\ldots$$

Checking to see if A.S.E.T. applies:-

$|a_1| = 0.1$

$|a_2| = \frac{0.1^3}{3!} \doteqdot 0.0001666666\ldots$

$|a_3| = \frac{0.1^5}{5!} \doteqdot 0.00000008333\ldots$

The conditions $|a_n| \to 0$ and $|a_{n+1}| < |a_n|$ apply*, so the A.S.E.T. predicts that the error will be less than $|a_4|$ ie $\frac{0.1^7}{7!} \doteqdot 1.984\times 10^{-11}$, which is very small.

Note that calculator has $\sin(0.1) \doteqdot 0.0998334166$

[NB 0.1 RADIANS] (agreeing to 9 decpl.)

Note to Q4 (d)(iii)

(*) We could also use the Ratio Test to show that the series converges.