Week 10

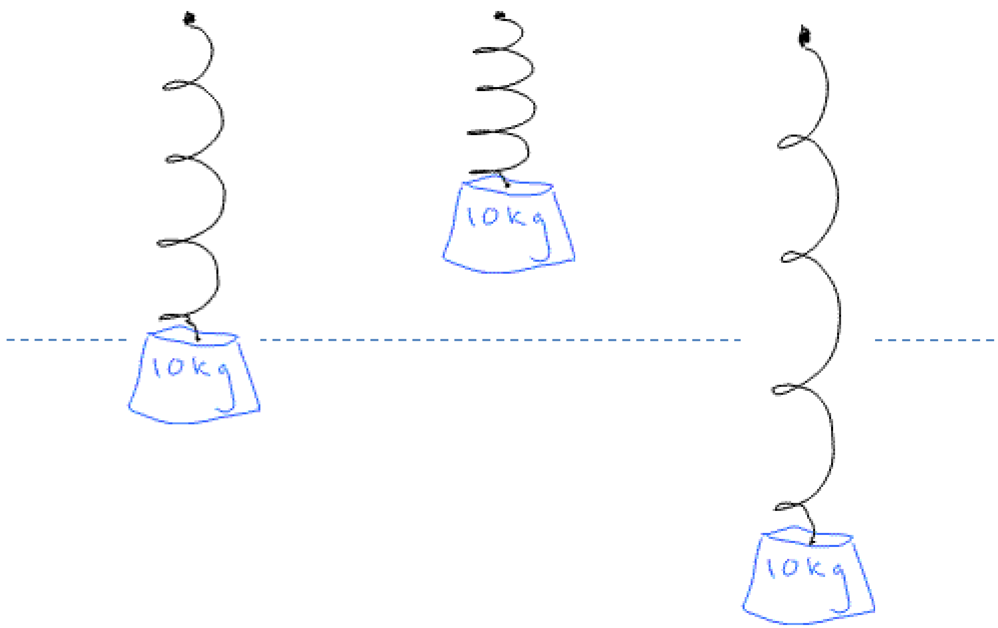
Second order differential equations [Textbook: 7.4]

Second order differential equations

We consider equations of the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

Example: The force on a mass on a spring is proportional to the extension of the spring.



We consider two types of 2nd order differential equations:

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

homogeneous

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

inhomogeneous

Homogeneous 2nd-order DEs

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

Strategy for solving: guess a solution, then substitute and see if it works.

So the ansatz $y = e^{mx}$ works for all x, provided

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This equation has solutions m_1 and m_2, so both

$$y = e^{m_1 x} \qquad \text{and} \qquad y = e^{m_2 x}$$

are solutions to the DE. We can multiply both of these by any constant and still end up with a solution:

$$y = Ae^{m_1 x} \qquad y = Be^{m_2 x}$$

So the general solution is $y = Ae^{m_1 x} + Be^{m_2 x}$

Result:

The general solution to the homogeneous equation is

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy \ =0$$

$$y(x)=Ae^{m_1x}+Be^{m_2x}$$

where m_1 and m_2 are the distinct solutions of the *auxiliary equation*

$$am^2+bm+c=0$$

Example:

Find the general solution to $$2\frac{d^2y}{dx^2}+5\frac{dy}{dx}-3y=0$$

Example:

Find the general solution to $$\frac{d^2y}{dx^2}+6\frac{dy}{dx}+8y=0$$

Example:

Find the general solution to $$\frac{d^2y}{dx^2}+7\frac{dy}{dx}=0$$

Complex roots

Sometimes the roots of the auxiliary equation may be *complex*.

e.g. Find the general solution to

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0$$

Find the general solution to $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 13y = 0$

Differential equation

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

Auxiliary equation

$$am^2+bm+c=0$$

Two distinct roots

m_1 and m_2

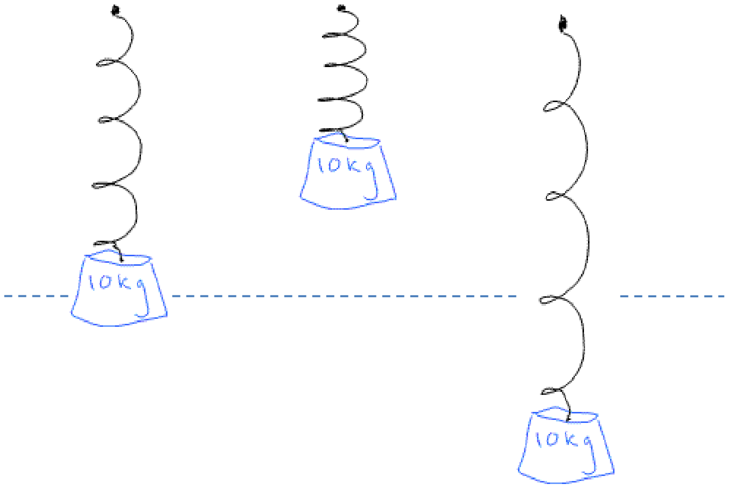
$$y(x)=Ae^{m_1x}+Be^{m_2x}$$

Complex roots

$m = \alpha \pm i\omega$

$$y(x)=e^{\alpha x}(A\cos\omega x+B\sin\omega x)$$

Example: The equation of motion for a mass on a spring is $M\dfrac{d^2y}{dt^2} + ky = 0$
Find the general solution for y(t).



Find the general solution to $$\frac{d^2y}{dx^2}+25y=0$$

What happens when the auxiliary equation has a double/repeated root?

e.g:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

When the auxiliary equation has a double root at m_1 the general solution is

$$y(x) = (Ax + B)e^{m_1 x}$$

Example:

$$\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

Differential equation

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

Auxiliary equation

$$am^2+bm+c=0$$

Two distinct roots
m_1 and m_2

$$y(x)=Ae^{m_1x}+Be^{m_2x}$$

Complex roots
$m = \alpha + i\omega$

$$y(x)=e^{\alpha x}(A\cos\omega x+B\sin\omega x)$$

Double root at
$m = m_1$

$$y(x)=(Ax+B)e^{m_1x}$$

Two distinct roots m_1 and m_2

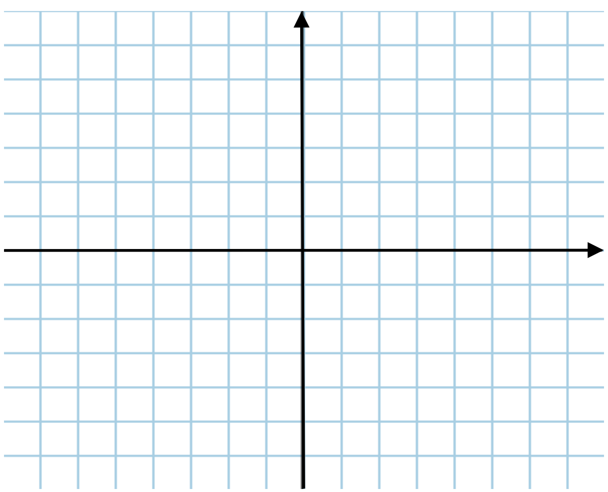
$$y(x) = Ae^{m_1 x} + Be^{m_2 x}$$

Complex roots $m = m = \alpha + i\beta$

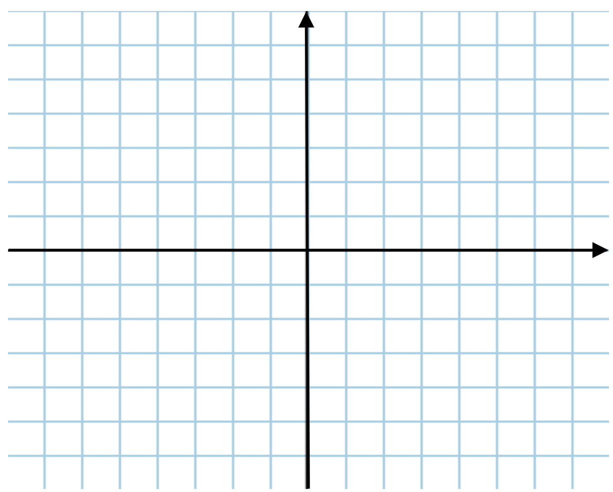
$$y(x) = e^{\alpha x}\left(A\cos(\beta x) + B\sin(\beta x)\right)$$

Double root at $m = m_1$

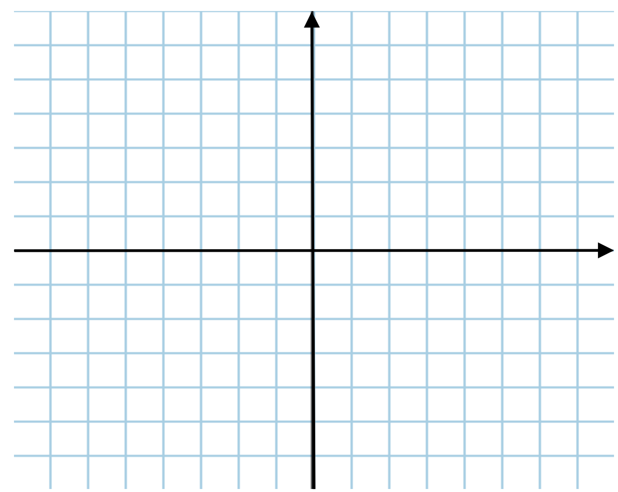
$$y(x) = (Ax + B)e^{m_1 x}$$



"overdamped"



"Damped oscillation"



"critically damped"

Examples:

$$2\frac{d^2y}{dx^2}+5\frac{dy}{dx}-3y=0$$

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$$

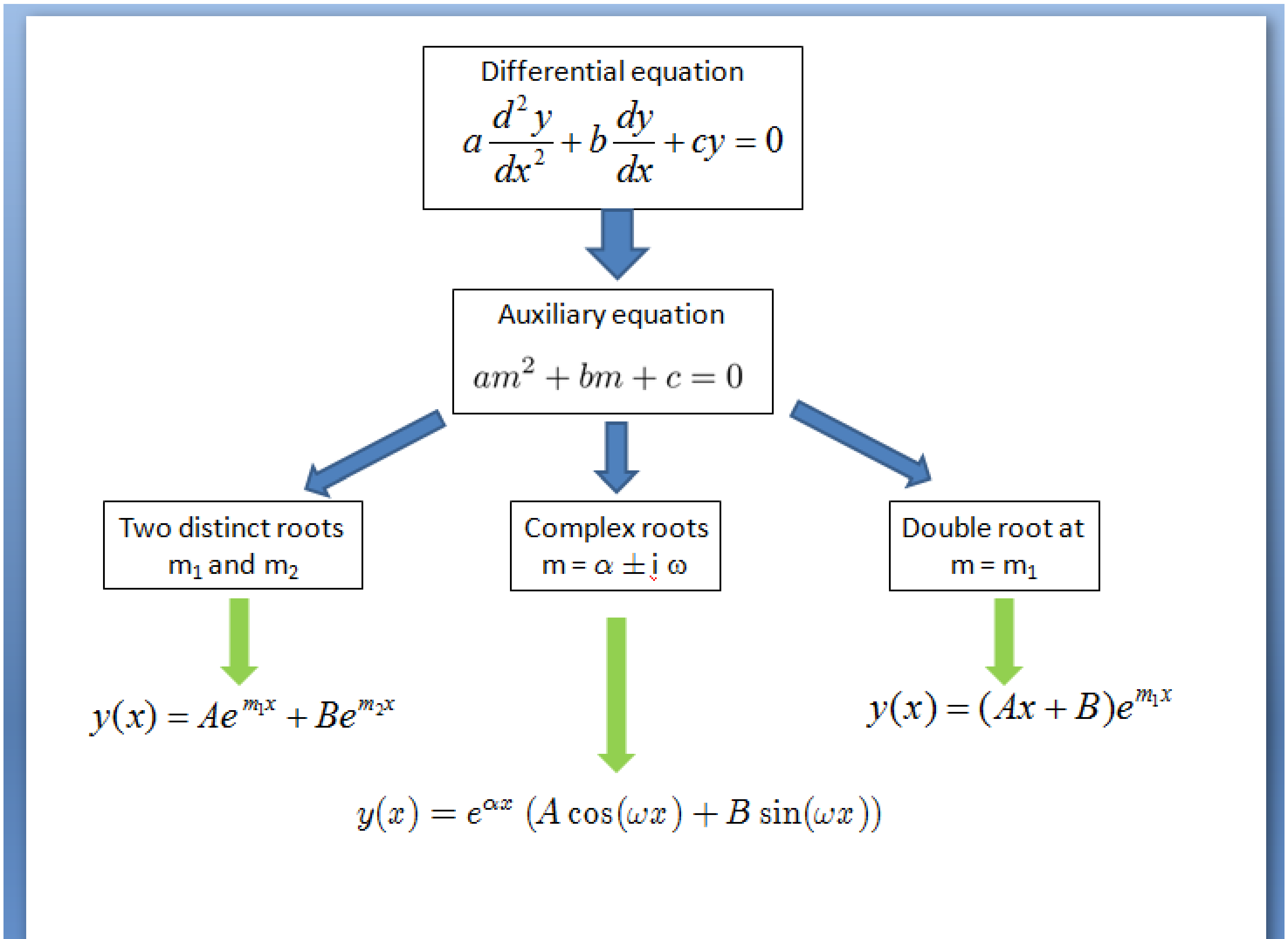
$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0$$

Example: Find the general solution to the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 0$$

Find the solution that satisfies the boundary condition y(0) = 6, y'(0) = -2.

Summary: we have seen how to find the general solution of *homogeneous* 2^{nd} order differential equations

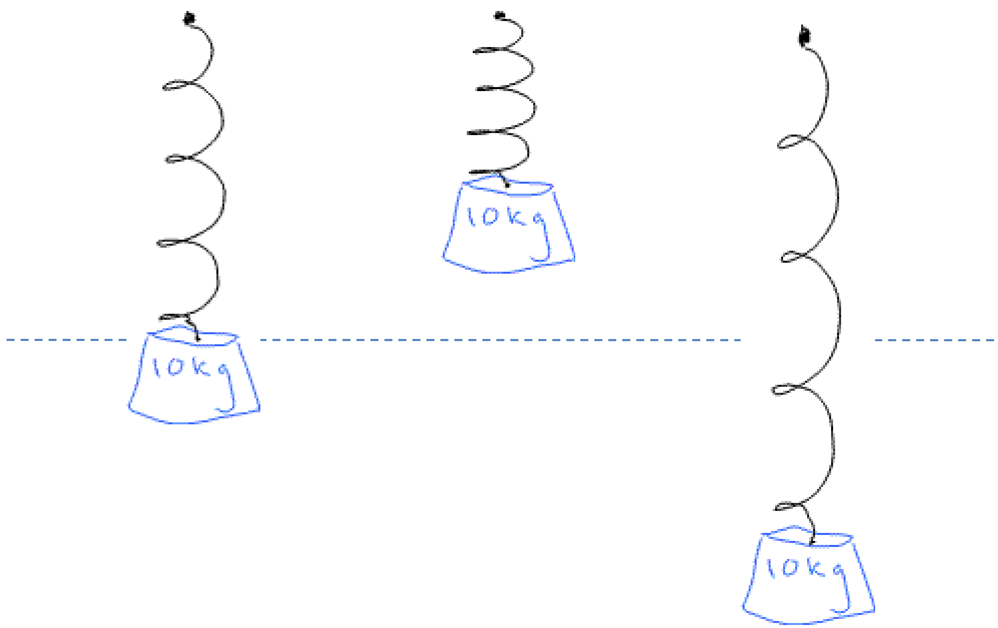


Second order inhomogeneous differential equations

We consider equations of the form

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

<u>Example:</u> The force on a mass on a spring is proportional to the extension of the spring.



Uniqueness

Recall: The general solution to a 2nd order linear DE has *2* arbitrary constants.

For all linear first and second order DEs,
there exists a uniqueness theorem, which says:

There is *only one* solution for inhomogeneous DE

If we can guess *any* solution, then this must be *the* solution

We now consider *inhomogeneous equations*, which have a *driving term* on the right-hand side

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

homogeneous

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

inhomogeneous

Inhomogeneous 2nd order DEs

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

The general solution is given by $y = y_C + y_P$

Where y_c is *the complementary function,* given by the general solution of

$$a\frac{d^2y_C}{dx^2}+b\frac{dy_C}{dx}+cy_C=0$$

And y_p is *the particular* solution to

$$a\frac{d^2y_P}{dx^2}+b\frac{dy_P}{dx}+cy_P=f(x)$$

1. Find general solution to this equation
2. Find particular solution to this equation
3. Add them together

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$$a\frac{d^2}{dx^2}(y_C+y_P)+b\frac{d}{dx}(y_C+y_P)+c(y_C+y_P)=f(x)$$

$$a\frac{d^2 y_P}{dx^2} + b\frac{dy_P}{dx} + cy_P = f(x)$$

1. Find general solution to this equation
2. Find particular solution to this equation
3. Add them together

Rule of thumb: *guess something that looks like $f(x)$, together with the derivatives of $f(x)$.*

$$\frac{d^2 y_P}{dx^2} + 3\frac{dy_P}{dx} + 2y_P = 6x + 10$$

To find the general solution to the inhomogeneous equation,
we add the particular integral to the complementary function:

$$\frac{d^2 y_P}{dx^2} + 3\frac{dy_P}{dx} + 2y_P = 3e^{4x}$$

Find the general solution to the inhomogeneous equation:

$$\frac{d^2 y_P}{dx^2} + 2\frac{dy_P}{dx} + y_P = 3e^{2x}$$

Find the general solution to the inhomogeneous equation:

$$\frac{d^2 y_P}{dx^2} - 3\frac{dy_P}{dx} + 2y_P = e^{2x}$$

$$y_P = Axe^{2x}$$

To find the general solution to the inhomogeneous equation,
we add the particular integral to the complementary function:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 6e^x$$

Summary: We consider two types of 2nd order differential equations:

homogeneous

Differential equation

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

inhomogeneous

$$y(x_0)=y_0$$
$$y'(x_0)=v_0$$

Boundary conditions

The *general solution* is the most general solution that doesn't take the boundary conditions into account.

Without boundary conditions

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

General solution

$$y(x)=(\textit{general}\ \text{homogeneous})+(\text{Inhomogeneous})$$

The *full solution* satisfies both the differential equation and the boundary conditions

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

$$y(x_0)=y_0$$
$$y'(x_0)=v_0$$

Problem specified completely

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

$$y(x_0)=y_0$$
$$y'(x_0)=v_0$$

Full solution:

$$y(x)=\text{Some function of } x$$

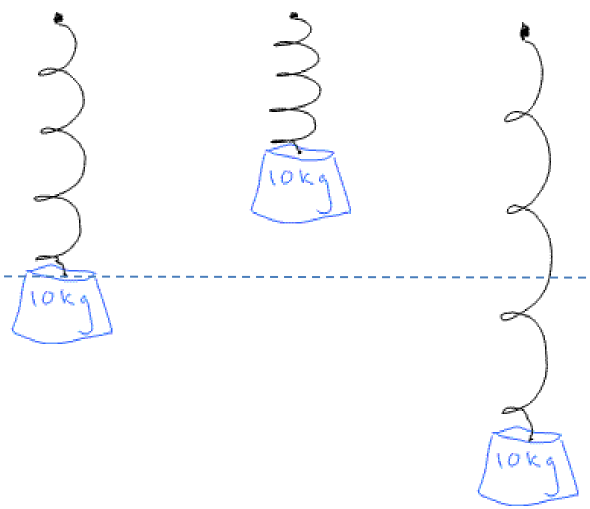
Example: Find the general solution to

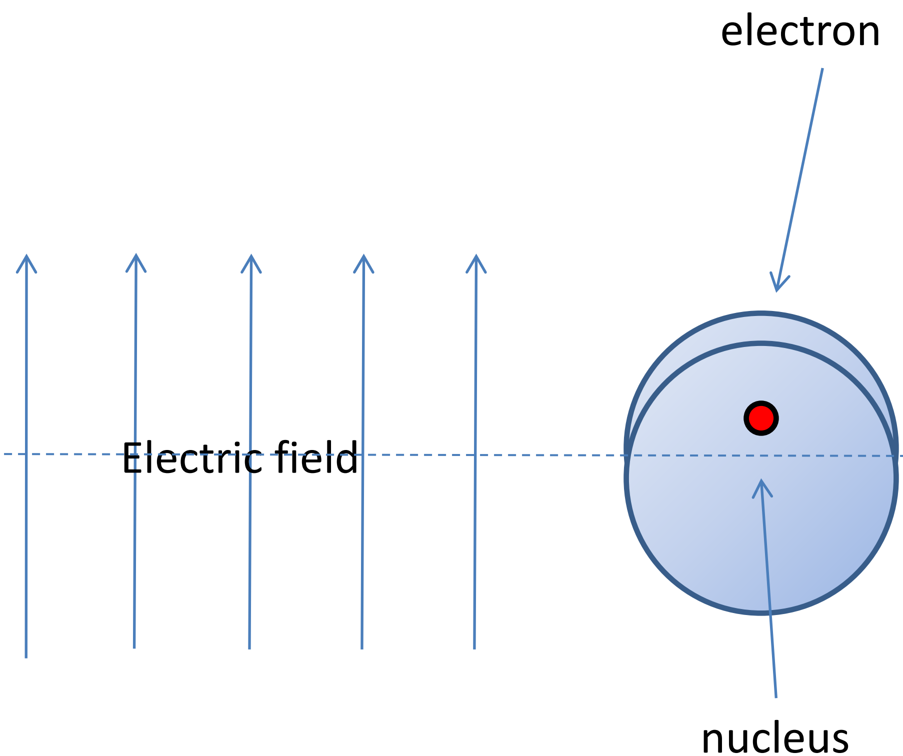
$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 6e^x$$

Hence find the full solution that obeys the boundary conditions y(0) = 1, y'(0) = -5.

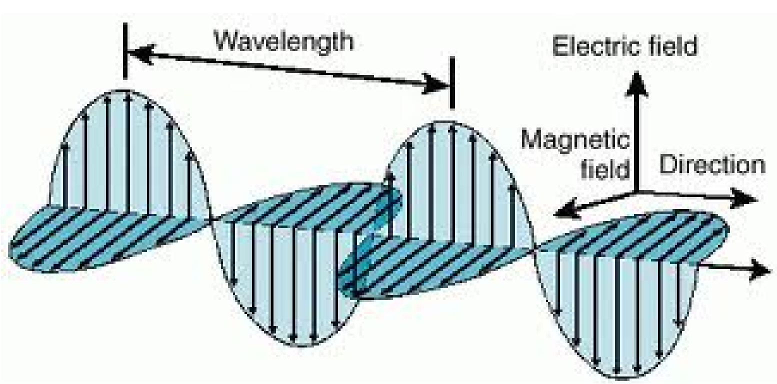
Second order differential equations are important in the theory of vibrations and resonances

E.g. Masses on springs:





Light interacting with matter:



Example: Find the general solution for the following DE

$$\frac{d^2y}{dx^2} + 9y = 10\sin 2x$$

What happens if the driving frequency ω_f is equal to the resonance frequency ω?

To see this, we have to find a new particular solution to

$$\frac{d^2 y}{dt^2} + \omega_f^2 y = f \sin \omega_f t$$

$$y_p = At \cos \omega_f t$$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