Sequences and Series

[Textbook: 8.1, 8.2]

A <u>sequence</u> is an ordered set of numbers

$$a_0 \; , a_1 \; , a_2 \; , \ldots\ldots , a_k \; , \ldots$$

Often the sequence is written in abstract form by specifying the k^{th} term of the set $\{a_k\}$.

e.g. The sequence 0, 2, 4, 6, 8, ...

e.g. The Harmonic sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

Given a set of n numbers $a_1, a_2, a_3, \ldots a_n$, the expression

$$a_1 + a_2 + a_3 + \ldots\ldots + a_n$$

Karl Friedrich Gauss

is called a *finite series.* In shorthand,

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \ldots\ldots + a_n$$

What happens when we have an infinite set of numbers?




$\frac{1}{2}$ $\frac{1}{4}$ $\frac{1}{8}$ $\frac{1}{16}$ $\frac{1}{32}$

The expression

$$a_1 + a_2 + a_3 + \ldots\ldots + a_k + \ldots\ldots$$

is called an <u>infinite series</u>, written

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + :::$$

If the sum of the series is equal to a finite number, then we say that the series <u>converges.</u>

If not, the series <u>diverges.</u>

For an infinite series to converge, it is necessary that

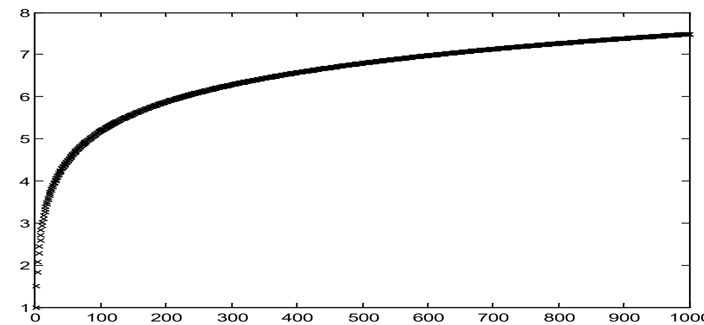
$$\lim_{k \to \infty} a_k = 0$$

However this condition is not sufficient.

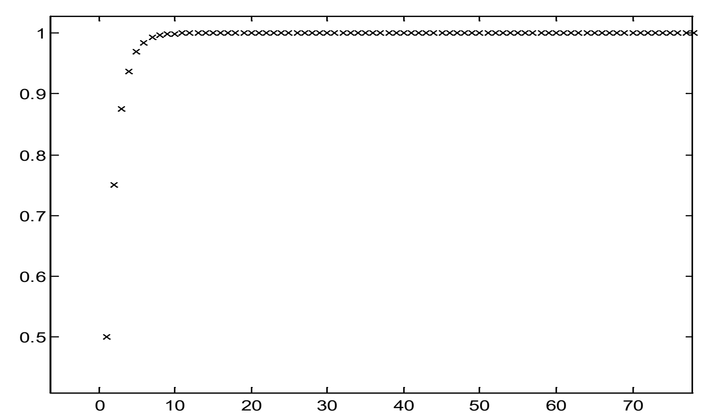
Examples of series:

$$\sum_{k=1}^{\infty} k = 1 + 2 + 3 + :::$$

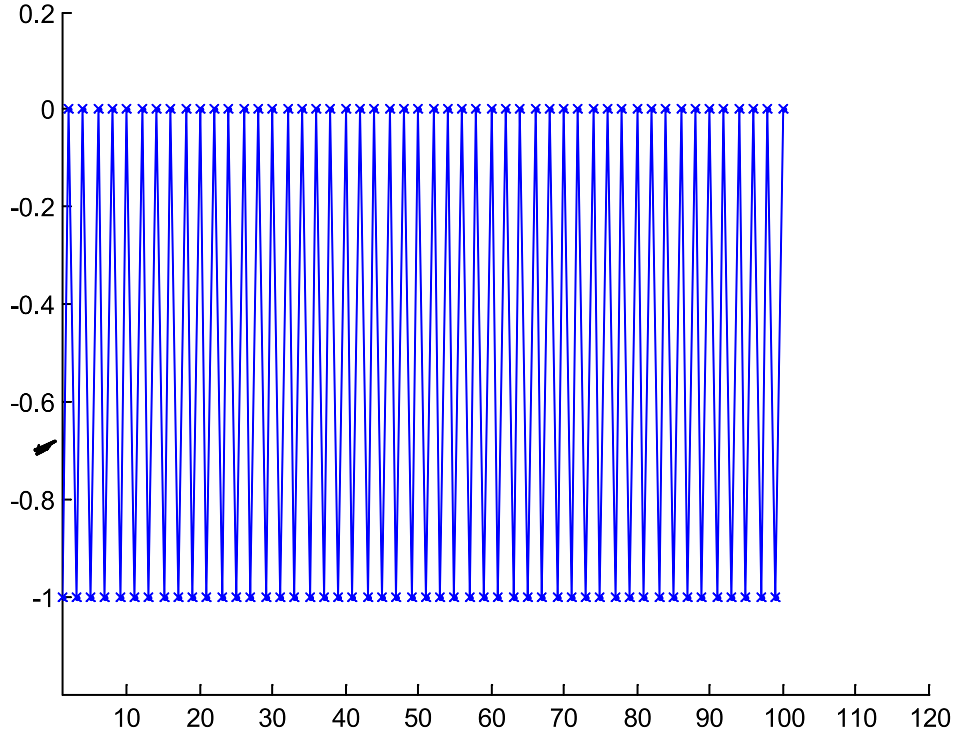
$$\sum_{k=1}^{\infty} \frac{1}{k} =$$



$$\sum_{k=1}^{\infty} \frac{1}{2^k} =$$



$$\sum_{k=1}^{\infty}(-1)^k =$$



Graph of $\sum_{k=1}^{n}(-1)^k$ for n values up to 100

A <u>Geometric series</u> has the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots$$

We can evaluate this series in the following way:
Define the *Nth partial sum* as

$$s_N = \sum_{k=0}^{N-1} ar^k = a + ar + ar^2 + \cdots + ar^{N-1}$$

Tests for convergence

1. The ratio test

2. Leibnitz test

3. The comparison test

The ratio test

Theorem: Let $S = \sum_{k=1}^{\infty} a_k$ be a series with $a_k > 0$, and suppose that

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = R$$

Then:

a) The series *converges* if $R<1$

b) The series *diverges* if $R>1$

c) If $R=1$ the series can either converge or diverge

The *interval of convergence* is the set of points for which the power series converges.

Example:

Determine whether the series

$$\sum_{k=1}^{\infty} k^2 2^{-k}$$

is convergent or divergent.

Example:

Determine whether the series

$$\sum_{k=1}^{\infty} \frac{1}{2^k}$$

is convergent or divergent.

Example:

Determine whether the series

$$\sum_{k=1}^{\infty} k!e^{-k}$$

is convergent or divergent.

Intervals of convergence

For each value of x in a series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{5^n (n+1)}$$

The Leibnitz Test

Theorem: The series

$$S = \sum_{k=1}^{\infty} (-1)^{k+1} b_k$$

is convergent if:

a) $b_k \geq 0$ for all k

b) $b_{k+1} \leq b_k$ for all k

c) $b_k \to 0$ as $k \to \infty$

Example:

Determine whether the series

$$\sum_{k=1}^{\infty} (-2)^k 3^{-k}$$

is convergent or divergent.

The comparison test

Theorem: The series

$$S = \sum_{k=1}^{\infty} a_k$$

is convergent if there is *another series* $\sum_{k=1}^{\infty} c_k$ such that

$$0 \leq a_k \leq c_k$$

for all k.

Example of applying the comparison test:

We know that the series $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges.

Now we also know that

$$0 < \frac{1}{3^k} < \frac{1}{2^k}$$

Therefore, using the comparison test, the series $\sum_{k=1}^{\infty} \frac{1}{3^k}$ converges.

Example:

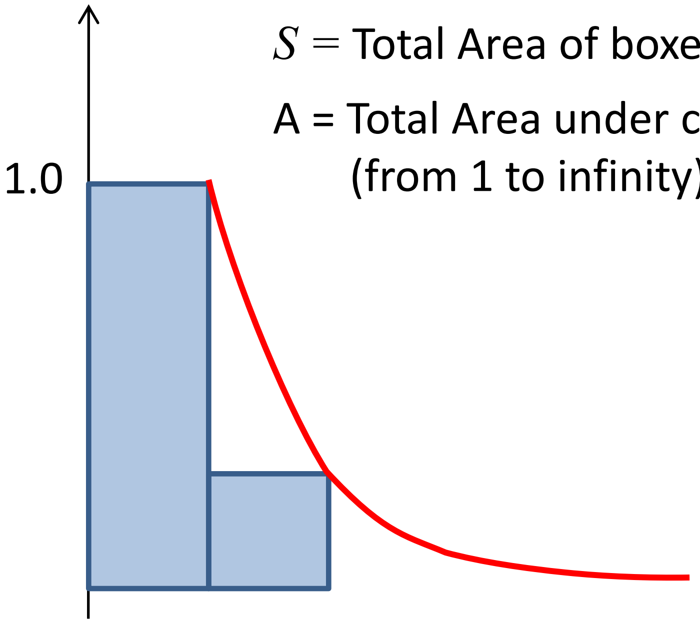
$$S = \sum_{k=1}^{\infty} \frac{\sin^2 k}{2^k}$$

Comparison with integrals

We can often show convergence of a series by comparing the series with an integral.

Example:

$$S=\sum_{k=1}^{\infty}\frac{1}{k^2}=1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$$

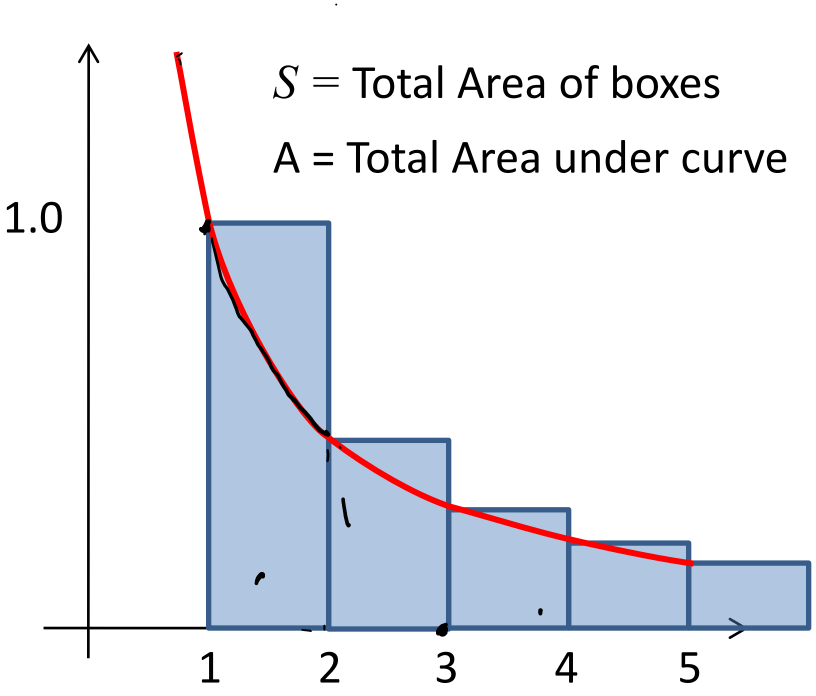


S = Total Area of boxes

A = Total Area under curve (from 1 to infinity)

The harmonic series:

$$S = \sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$



Taylor and MacLaurin Series [Textbook: 8.6,8.7]

Power Series: A different way of thinking of functions

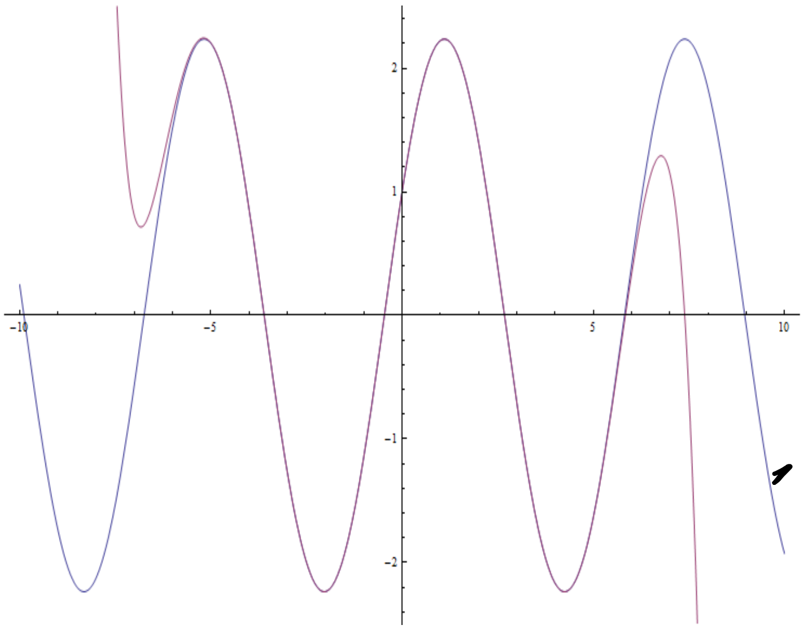
Another way of *representing functions* is as an infinite sum of *powers of x.*

E.g. instead of writing

$$f(x) = \cos(x) + 2\sin(x)$$

We can write

$$f(x) = 1 + 2x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{24}x^4 + \cdots$$



We can write any smooth function this way. Such a representation is called a *power series.*

All we need to find are the *coefficients* in the series.

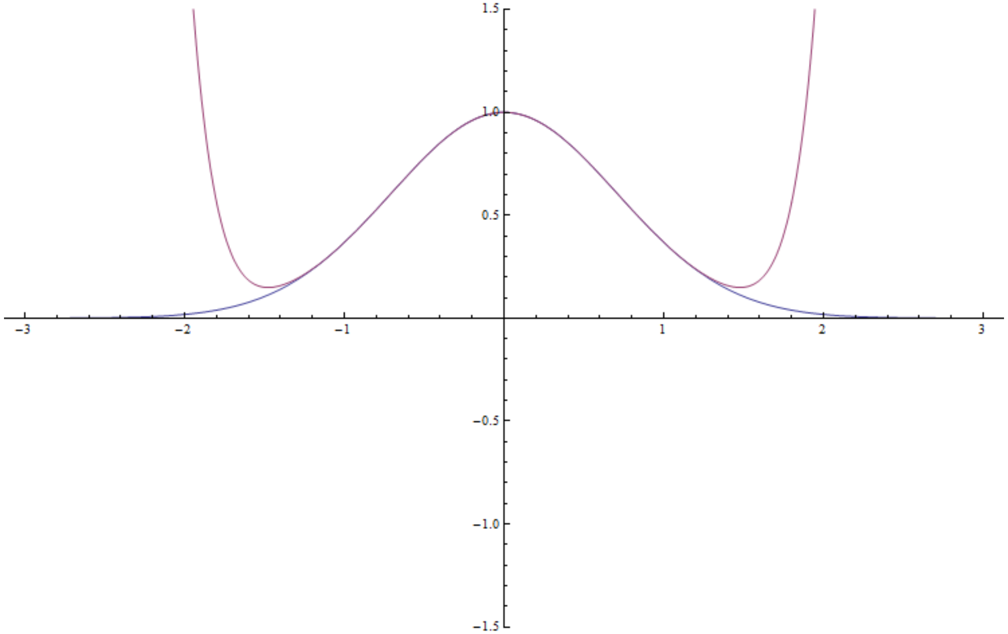
Another example:

The function

$$f(x)=e^{-x^2}$$

Can be represented by the series

$$f(x) \approx 1-x^2+\frac{x^4}{2}$$



Why would we do this?

1. Because the series representation is *often much simpler to deal with*.

Example:
If we know the coefficients of the series, then we can differentiate/integrate very easily.

f(x)=e^{-x^2}

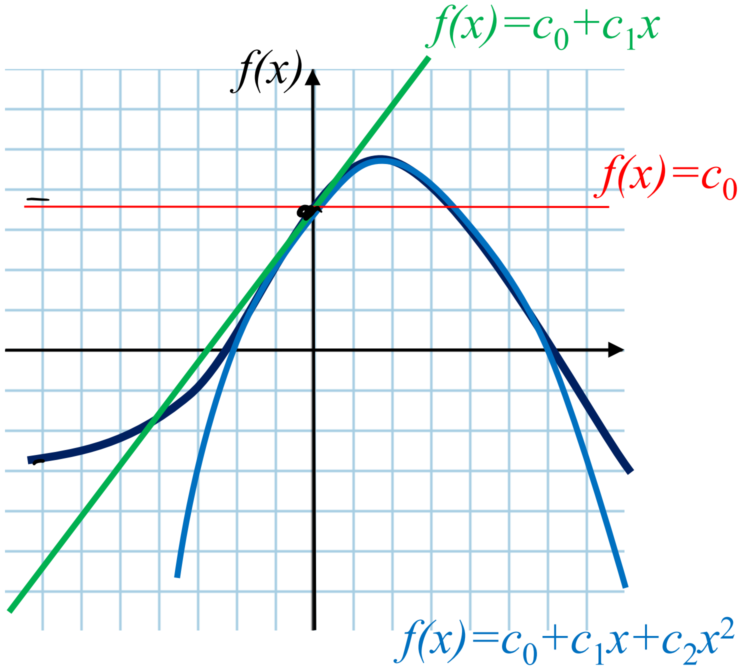
f(x) $\approx 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6}$

2. It gives us a powerful way to evaluate functions. (This is in fact how most functions are evaluated)

E.g. f(0.1) $\approx 1 - 0.1^2 + \frac{0.1^4}{4} - 0.1^6/6$

We can think of a series representation as a sum of polynomials.

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots$$



The 1st term specifies the *value* of f at x = 0.

The 2nd term specifies the *first derivative* of f at x = 0.

The 3rd term specifies the *second derivative* of f at x = 0.

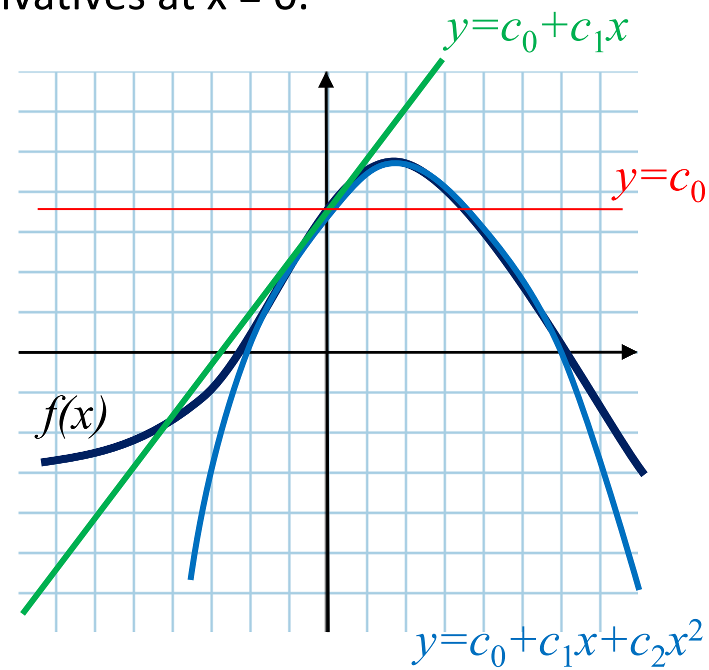
As we add more terms, the series converges to the "real" function.
A truncated series is *most accurate near the point x = 0.*

Finding the coefficients

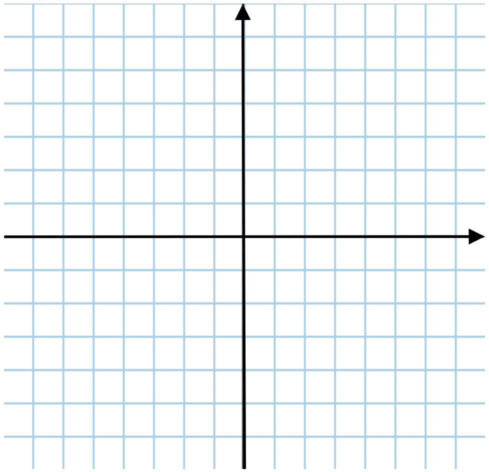
Suppose that we can evaluate the function and all its derivatives at x = 0.
Then we can find the coefficients as follows:

Let

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots$$



Example: Find the complete Taylor series expansion of $f(x) = e^x$ about the point $x = 0$.

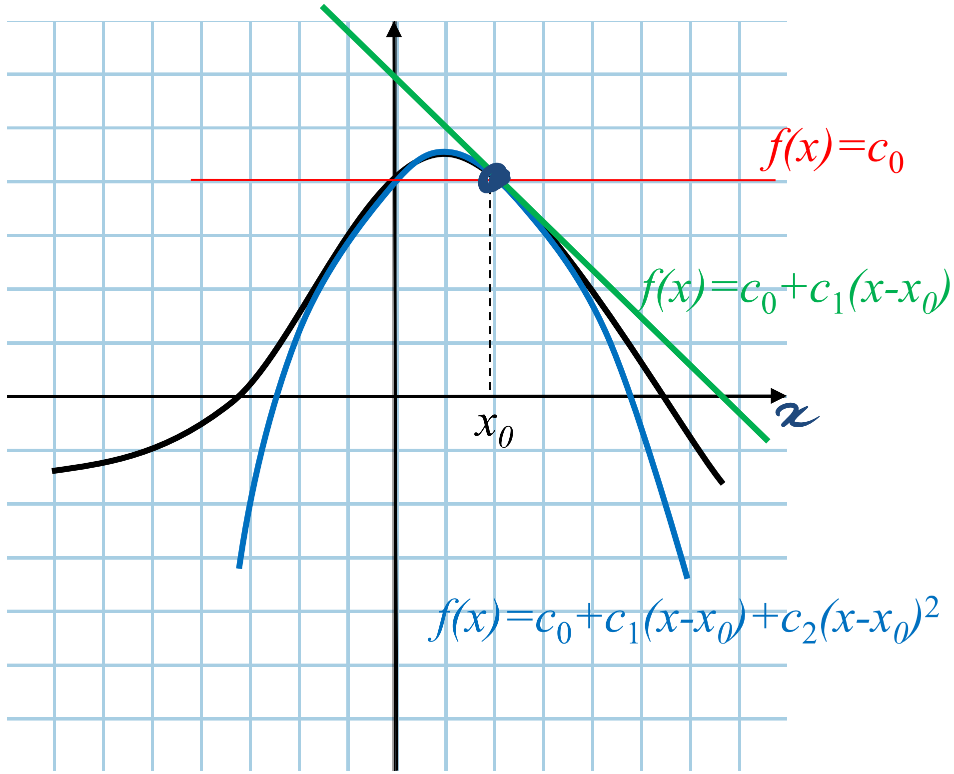


Taylor Series

The power series expansion of a function $f(x)$ about a point $x=x_0$ is called the *Taylor series* of $f(x)$ at x_0.

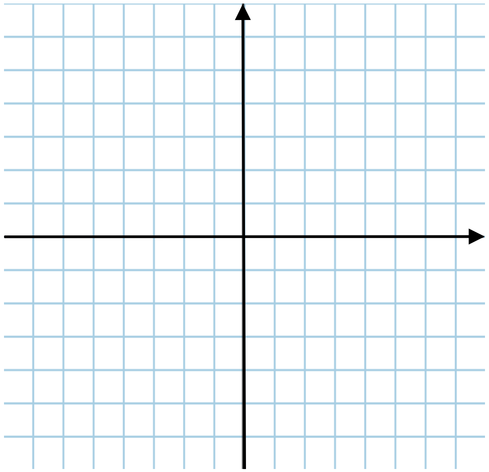
$$f(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \cdots$$

$$f(x) = c_0$$
$$+ c_1(x - x_0)$$
$$+ c_2(x - x_0)^2$$
$$+ \cdots$$

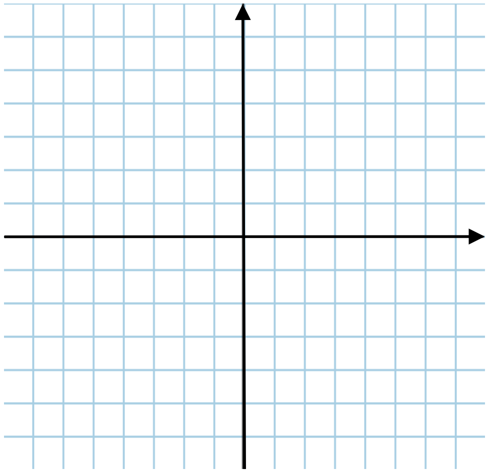


The Taylor series expansion of a function f(x) about x_0 is:

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k$$

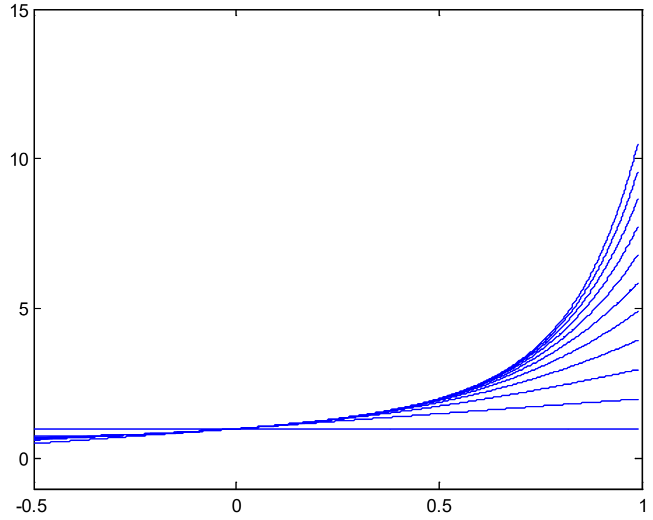


Exercise: Expand $f(x) = \sin(x)$
near x=π/2:



Examples:

$$f(x) = 1 + x + x^2 + \ldots = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$



$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = \cos x$$