33130 Mathematics 1

Subject Coordinator:
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Course Structure:

Workshops: 2 hours per week (+pre-recorded lectures)

Tutorials: 1.5 hours per week

Skills Tests: 1 per week 30 min

Resources:

- Textbook: *Calculus: concepts and contexts,*
  James Stewart, 4^{th} Edition or later (see also subject outline)

- Slides from lectures available online.

Assumed Knowledge and skills:

1. Thorough algebra

2. Trigonometry

3. Geometry and formulas for basic shapes

4. Curve sketching, ideally at least basic calculus

Assessments

Ten weekly Class Tests worth	5% each =	50%
Final Exam:		50%

- **At least 50% must be achieved in the final exam**

If you achieved at least 50% in the final exam and your overall mark is greater than 50% then congratulations! You have passed the subject.

Course outline

Part 1: Vectors and Matrices

Part 2: Functions and Calculus

Part 3: Complex Numbers

Part 4: Differential equations

Vectors

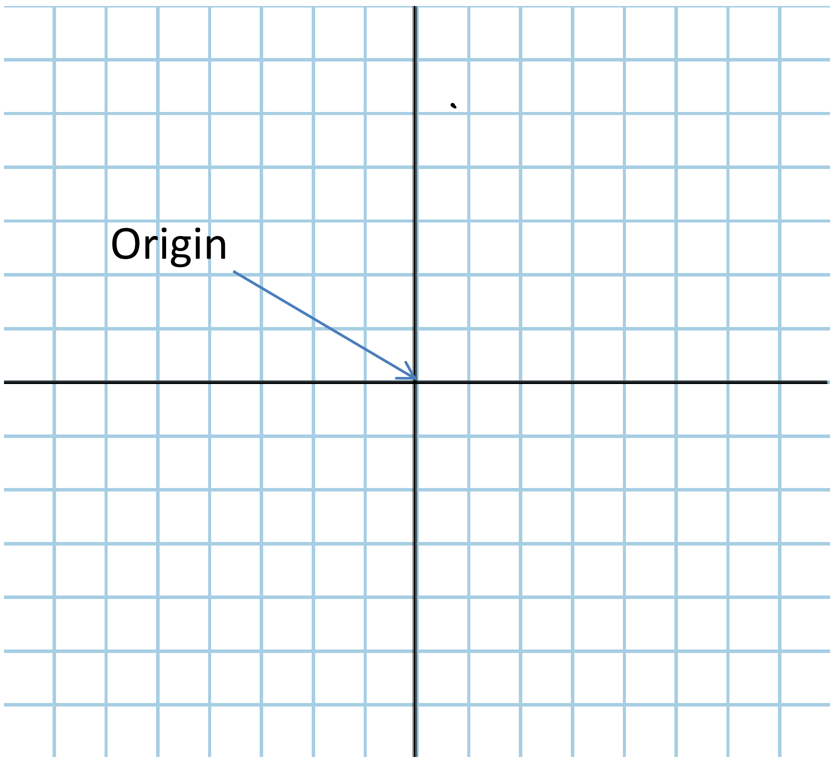
Geometry in two and three dimensions

Introduction to vectors

The dot product

Points in two dimensions

Any point in two dimensions can be represented by two numbers.
We usually refer to these numbers as the x and y coordinates of the point.



We often write points in 2D space like

$$A(2,3)$$

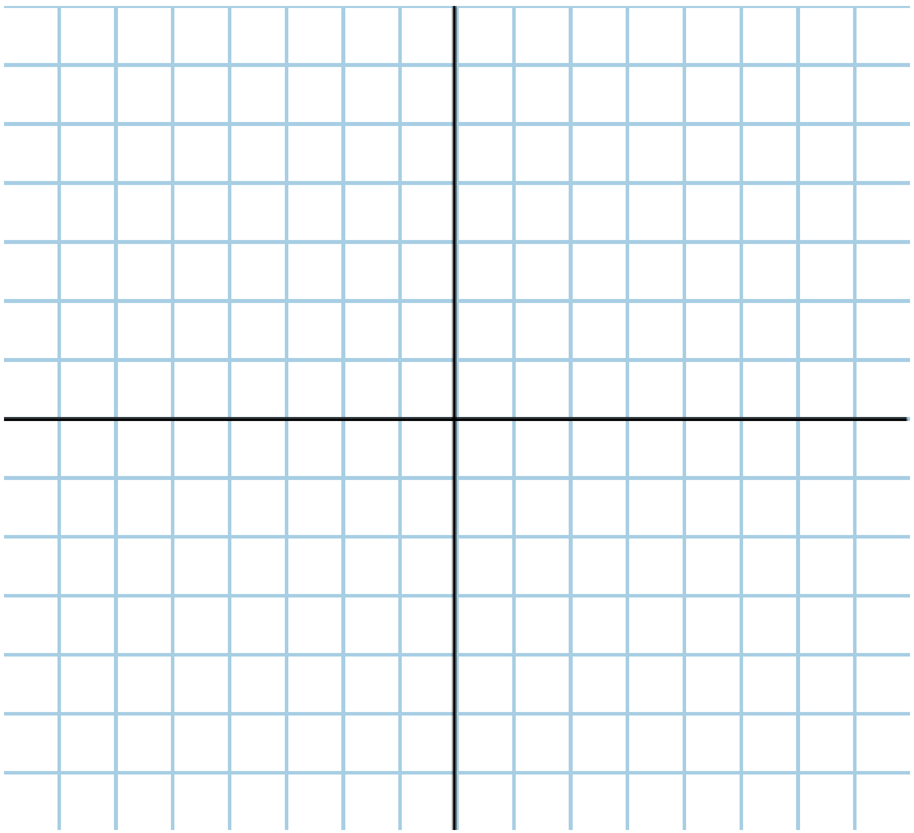
to designate a given point *A*.

Any combination (x,y) is known as an *ordered pair*. We say that $(x,y) \in \mathbb{R}^2$

$$R = \sqrt{x^2 + y^2}$$

The *distance* between two points A(x_1,y_1) and B(x_2,y_2) is

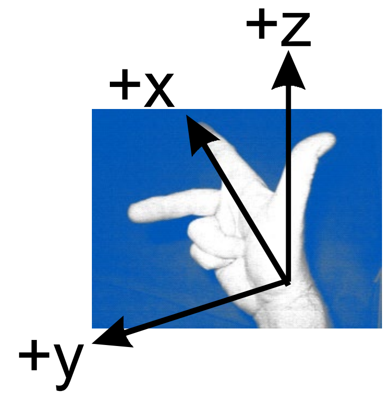
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

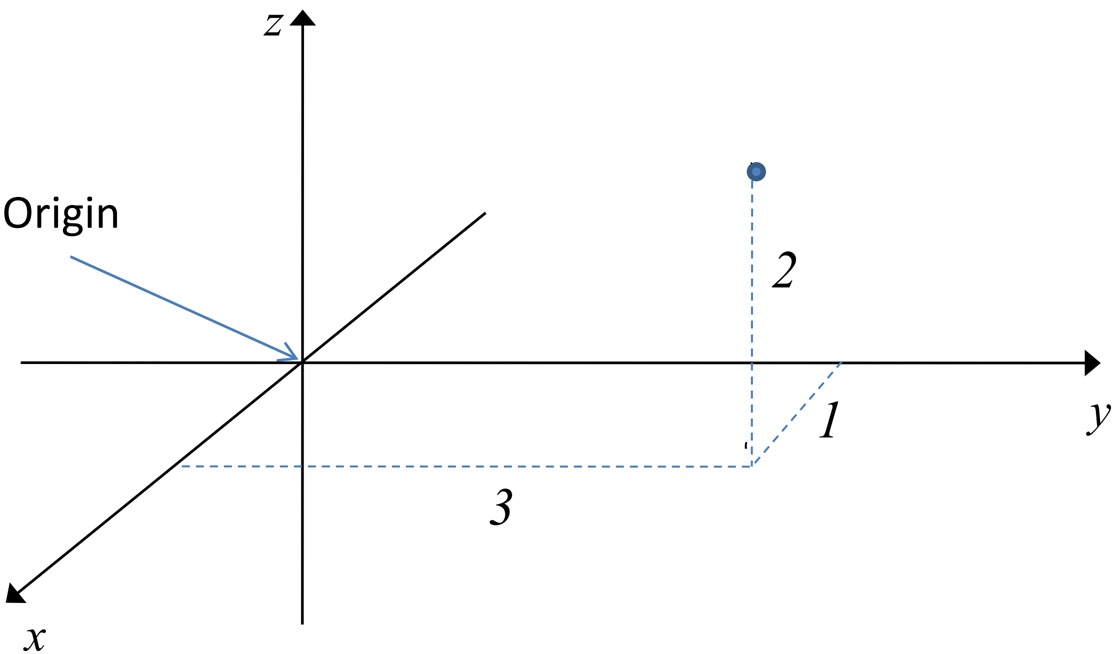


Equation of a circle centred at $A(x_1,y_1)$

Points in three dimensions

Any point in three dimensions can be represented by three numbers: the x and y and z coordinates.





We say that $(x, y, z) \in \mathbb{R}^3$

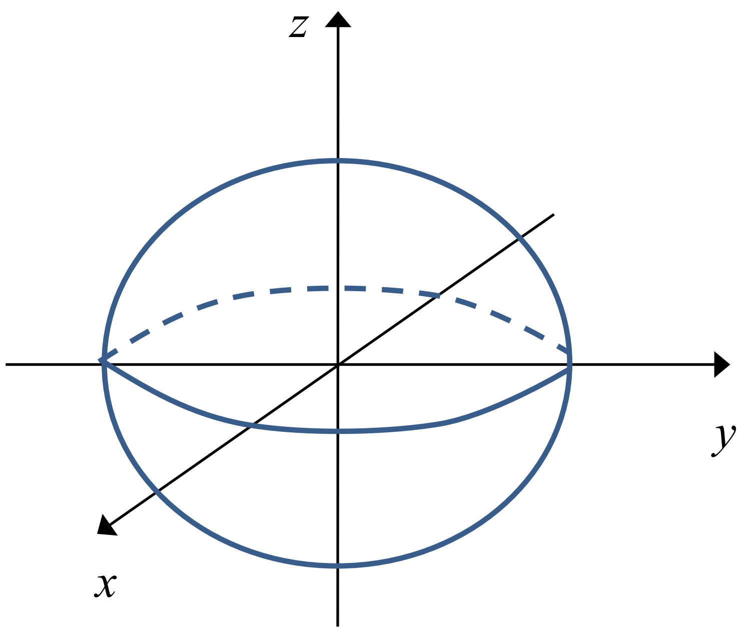
The *distance* between two points $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

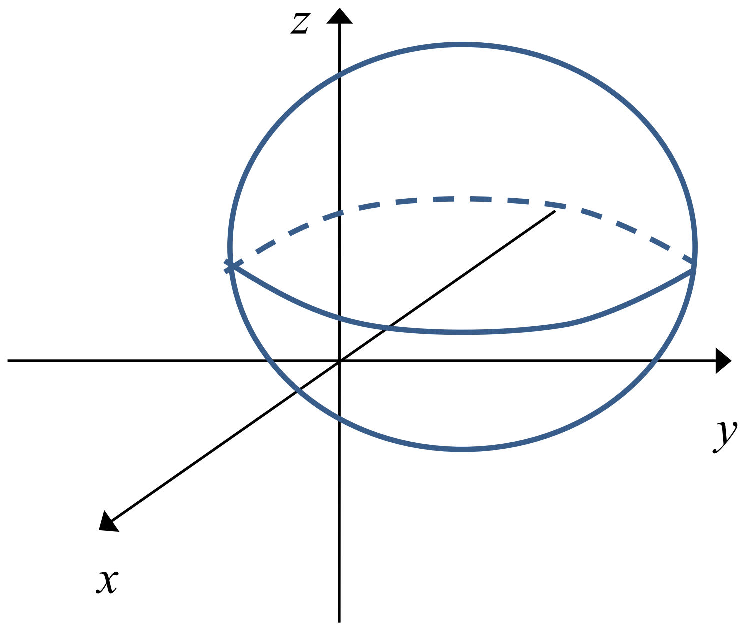
Points in three dimensions: Sphere

The *distance* between two points A(x_1,y_1,z_1) and B(x_2,y_2,z_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



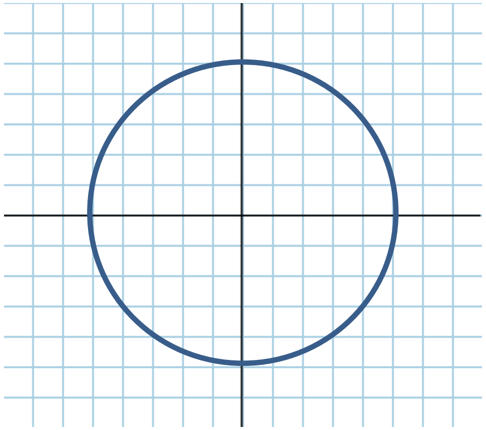
$$R = \sqrt{x^2 + y^2 + z^2}$$



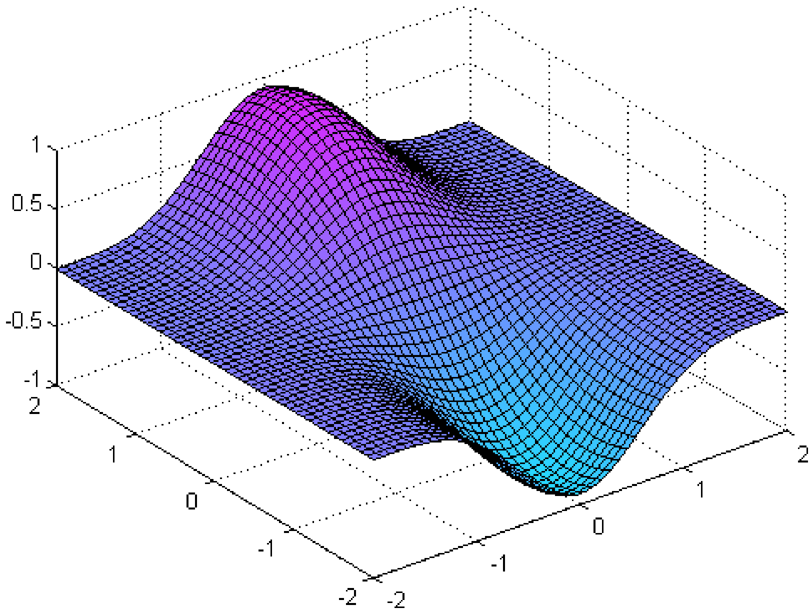
$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = R^2$$

In 2D: a general relationship between x,y describes a *curve or line:*

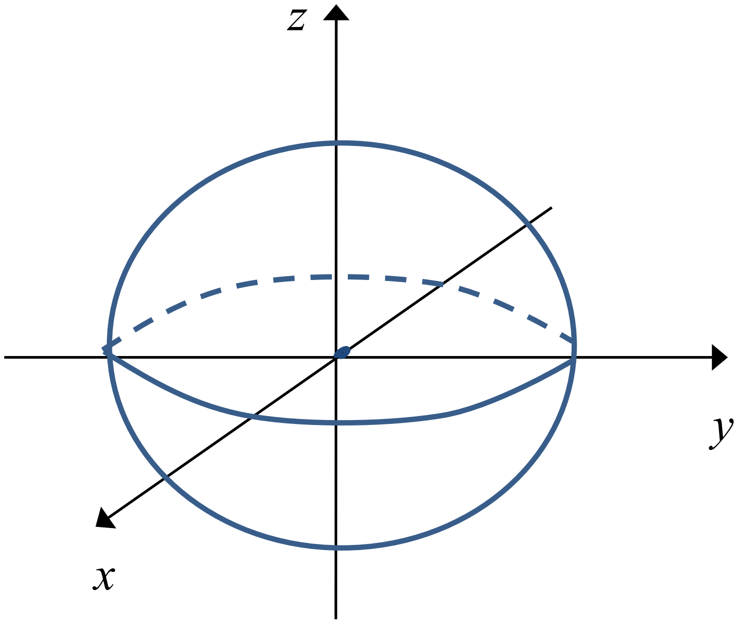
e.g.



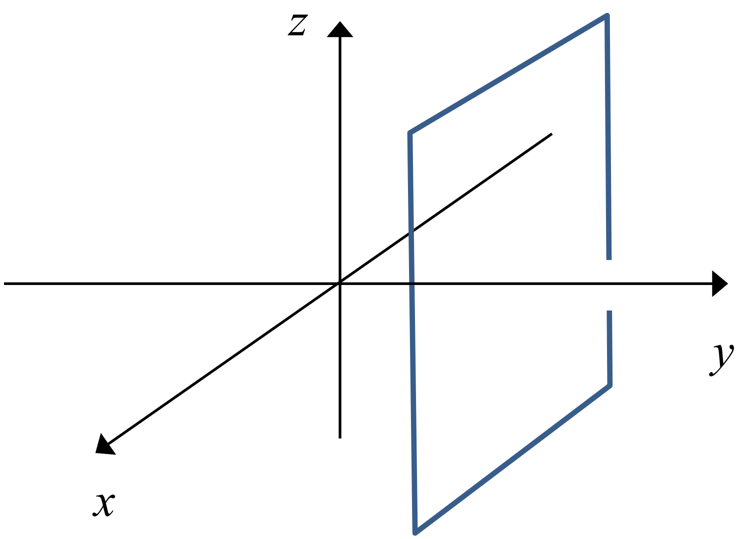
In 3D: a general relationship between x,y, and z describes a *surface*



A *sphere* is the set of points equidistant from the origin:

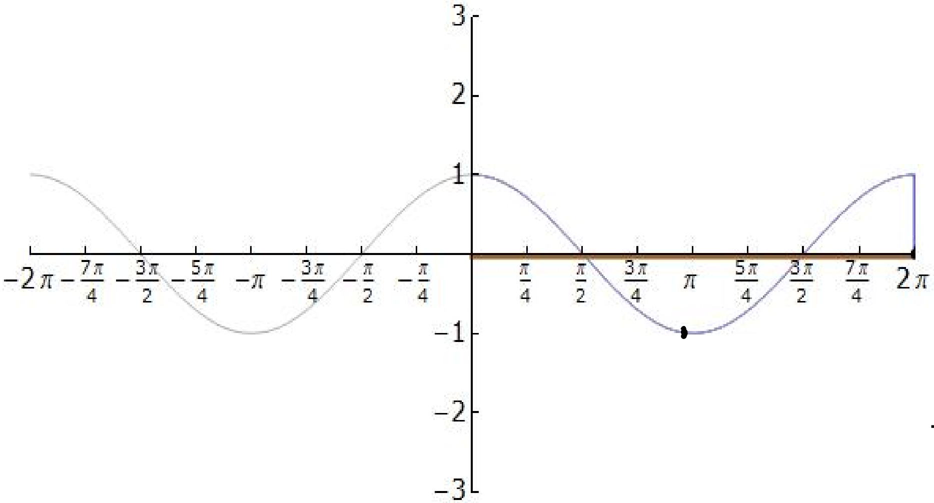


The equation y = 3 describes a *plane of constant y:*

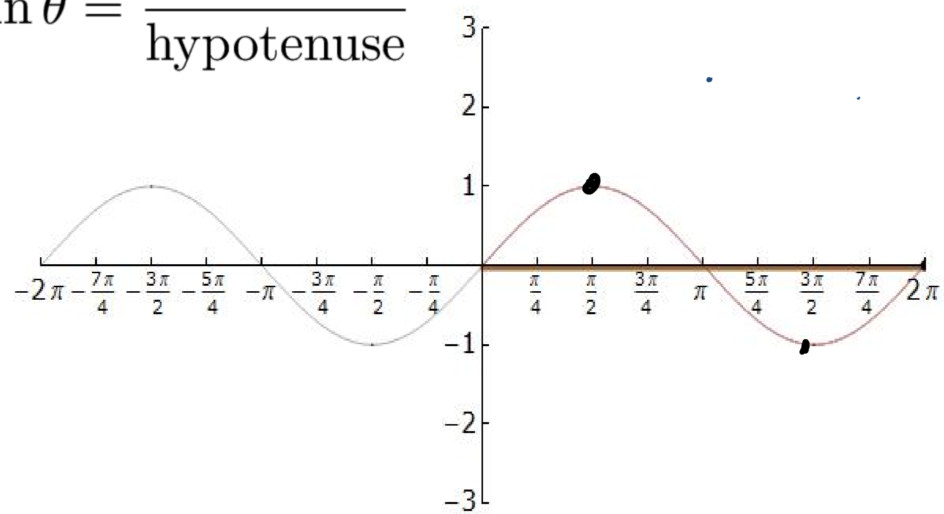


2. Properties of sine and cosine

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$



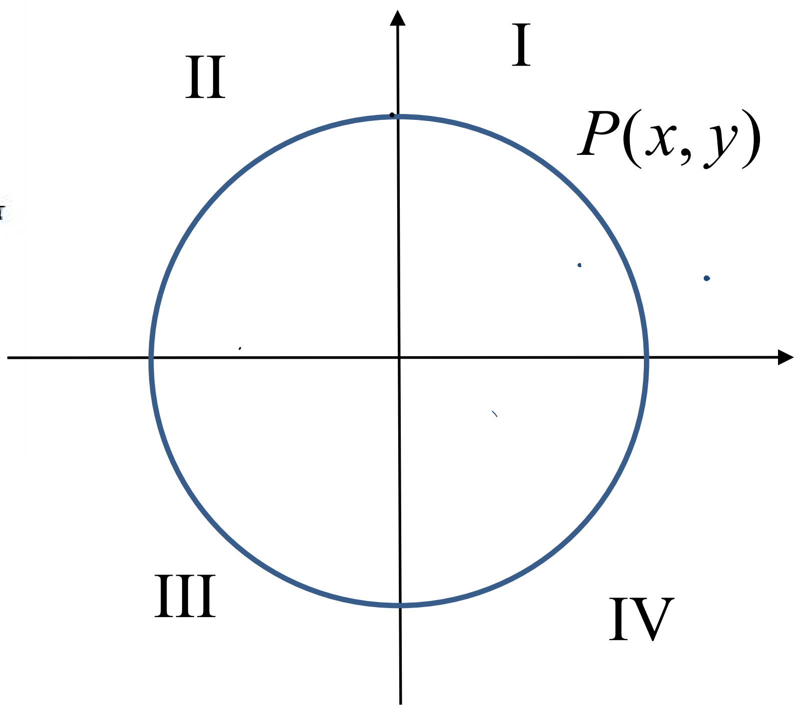
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$



$$x^2 + y^2 = 1$$

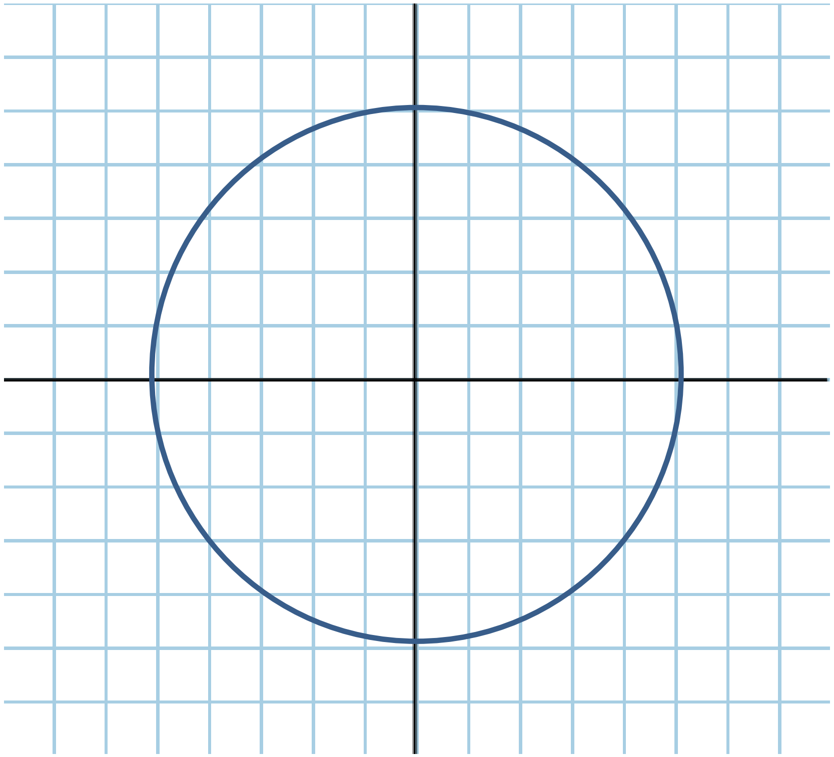
$$\cos\theta = x$$

$$\sin\theta = y$$



Recall: you really have to know:

1. The unit circle and radians



$$R = \sqrt{(x-0)^2 + (y-0)^2}$$

$$\theta = \frac{l}{R}$$

Angle for a full circle: 2¼ Radians

half a circle:

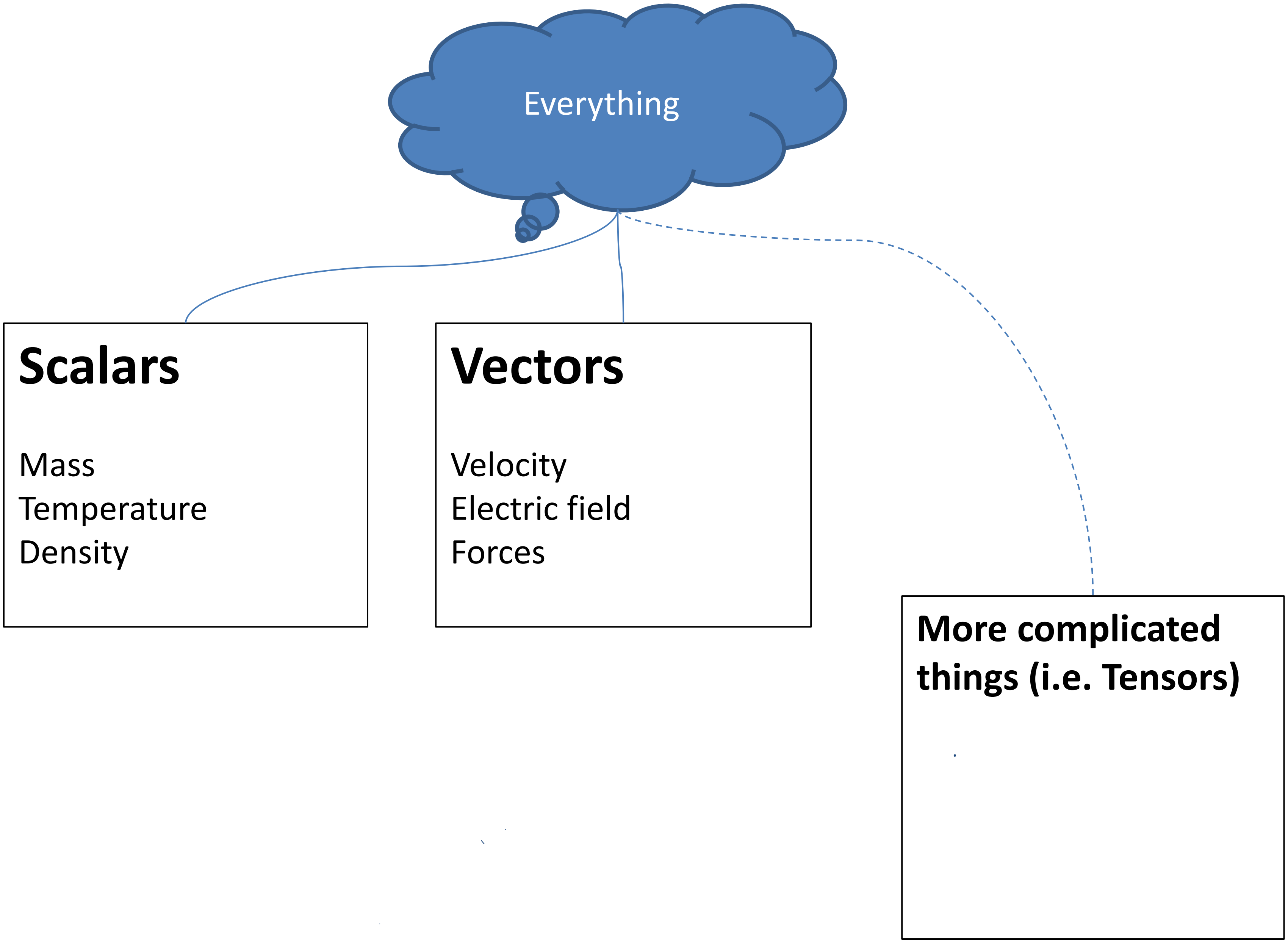
1/3 of a circle:

1/6th of a circle:

$$\theta(rad) = \frac{2\pi}{360}\theta^\circ(\deg)$$

$$\theta^\circ(\deg) = \frac{180}{\pi}\theta(rad)$$

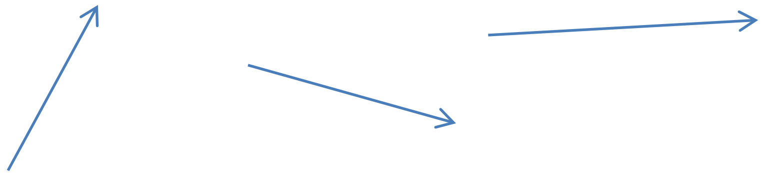
$$\theta^\circ(\deg) = \frac{180}{\pi}1(rad) \approx 57.29^o$$

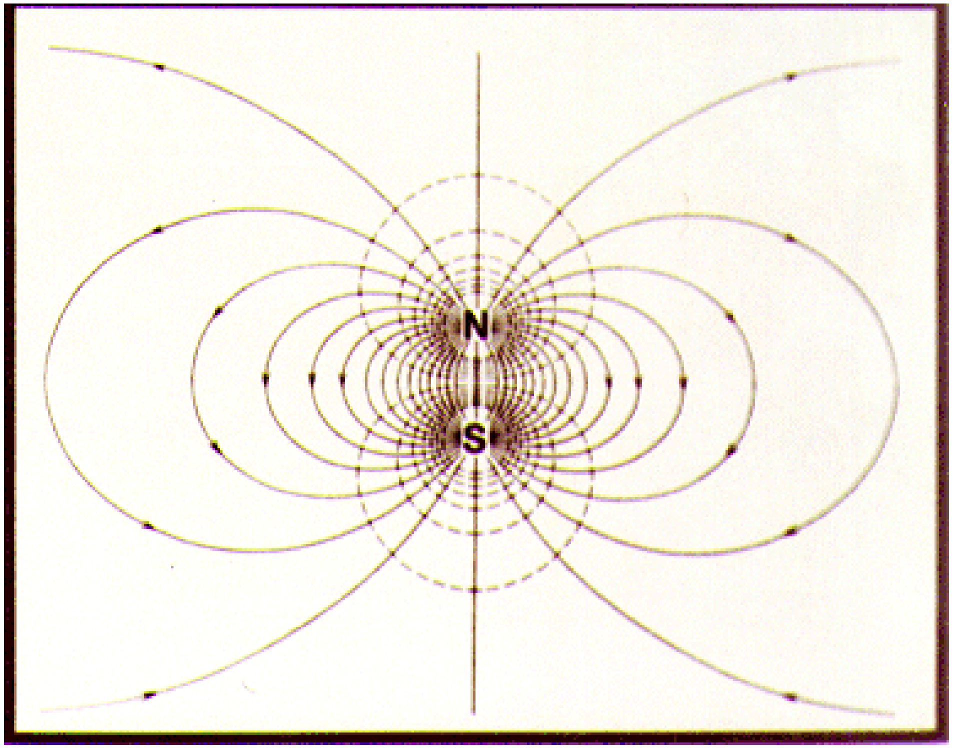

Everything
Scalars
Mass
Temperature
Density
Vectors
Velocity
Electric field
Forces
More complicated things (i.e. Tensors)

<u>Vectors</u>

A scalar quantity is completely specified by a single number

A *vector* is specified by a magnitude and a <u>direction</u>

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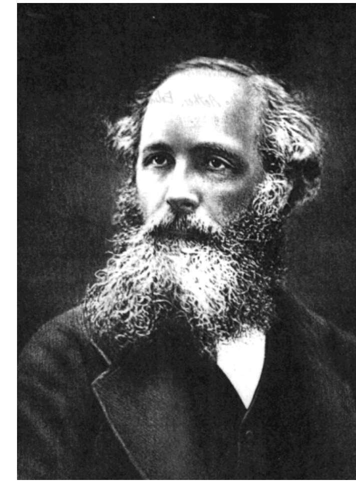
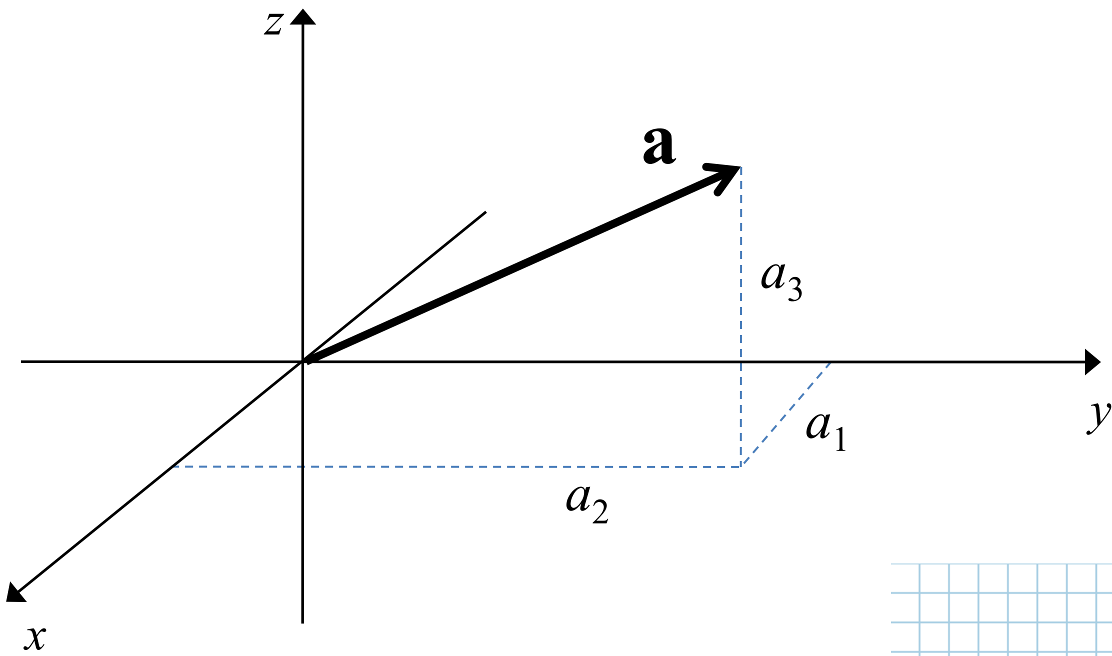
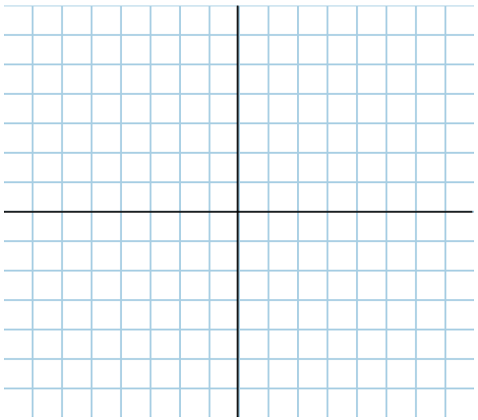


A vector can be written in terms of its *components*: $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

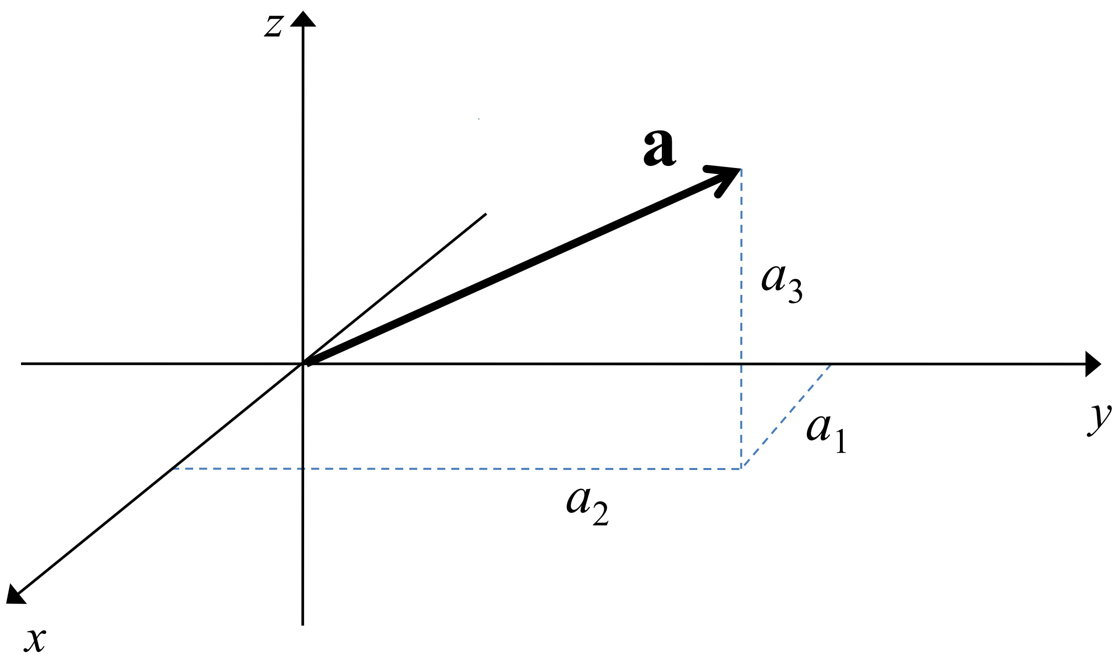




The magnitude, or length of the vector is written

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

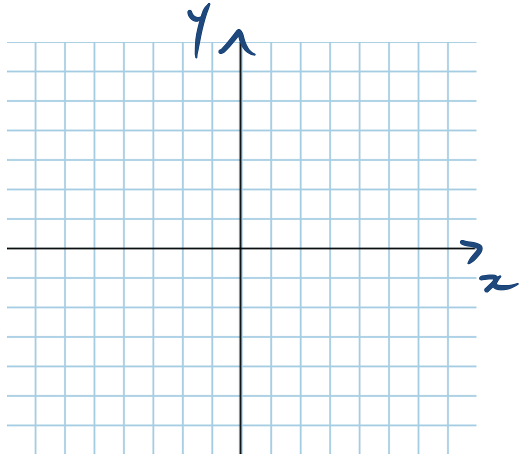
$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$



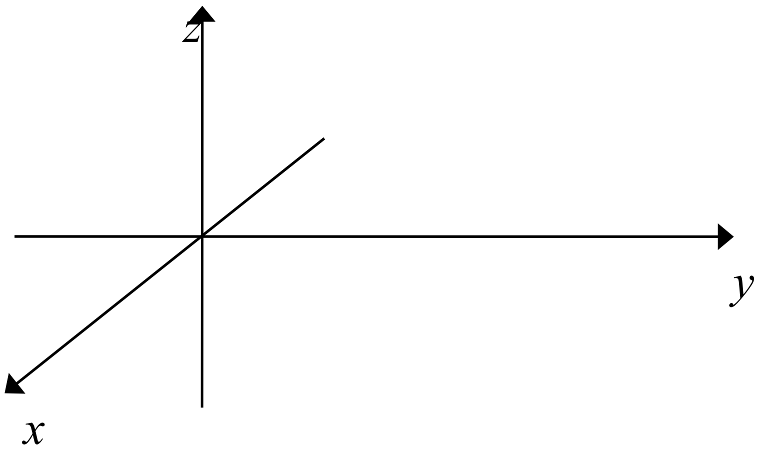
|**a**| = |< 1 , 3 , 2> |=

Example for exercise: Draw, and find the magnitude of, the following vectors:

1. **a** = < 1 , -2 >



2. **b** = < -1 , 4, 2 >



3. **c** = <1 , 2, -1>

Notation

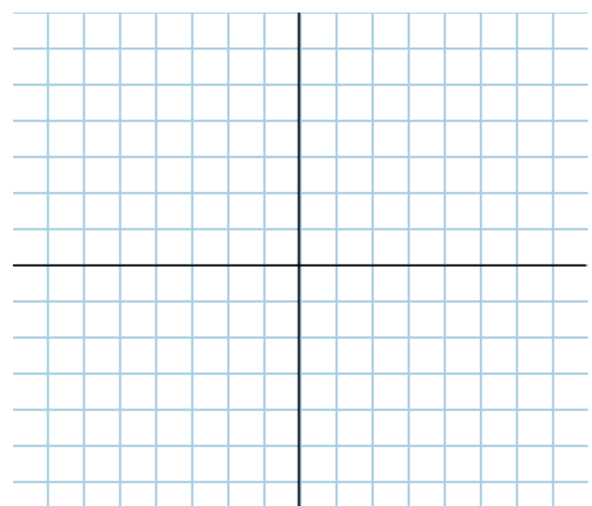
Vectors can be written in a variety of different notations:

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

$$\mathbf{a} = (a_1, a_2, a_3)$$

When a vector represents a line joining two points A and B, we write it as $\overrightarrow{AB}$.

Important: When writing a vector by hand,
we always put a "squiggle" underneath or arrow on top:



We say that two vectors are *equal* if they have the same number of components and all their components are equal. E.g.:

a = <1,3,6>

b = <1,3,6>

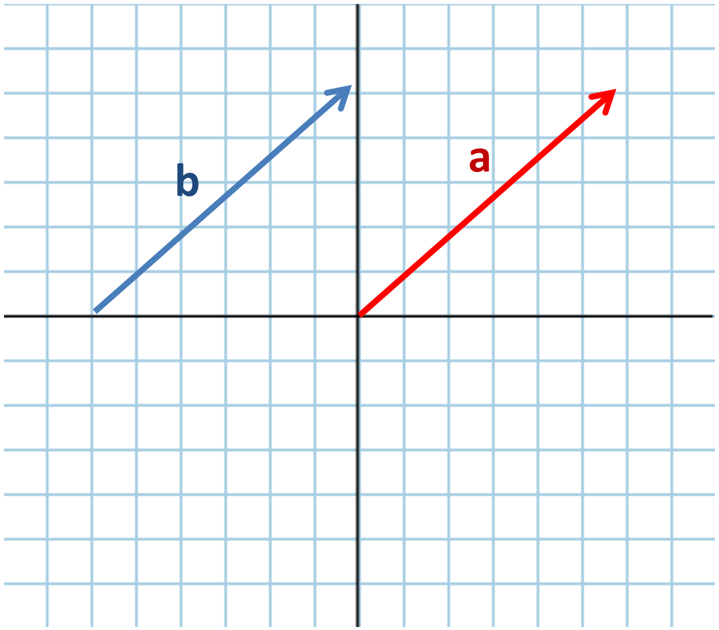
c = <2,3,6>

d = <1,3,6,0>

f = <3,6>

Example:
Find constants p and q such that **a** = <-1,0,3> and **b** = <-1,p+q,p-q> are equal.

Same vectors can have different *positions.*



Vector Algebra

Vectors can be *added* component by component:

$$\mathbf{a} = < a_1 , a_2 , a_3 >$$

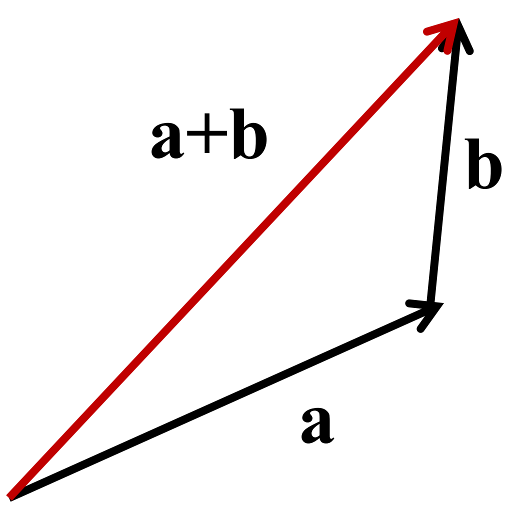
$$\mathbf{b} = < b_1 , b_2 , b_3 >$$

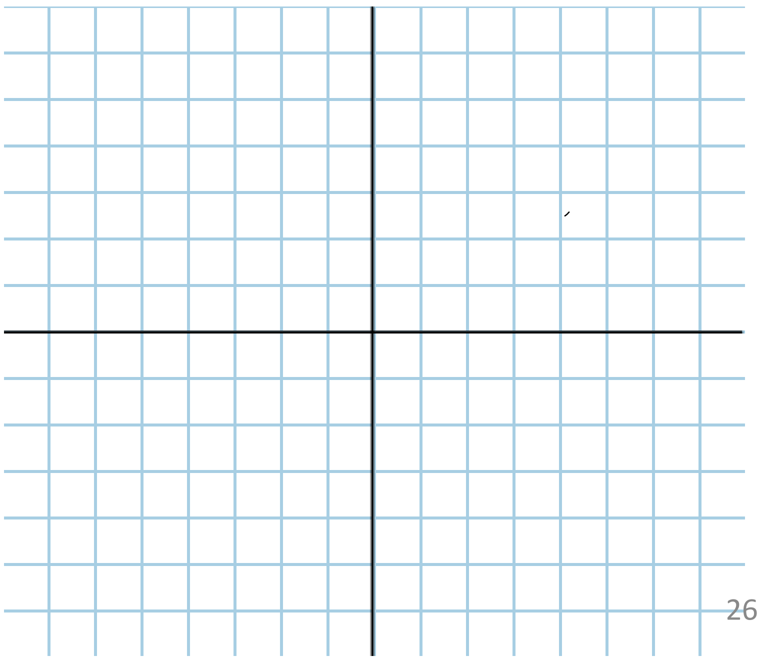
$$\mathbf{a} + \mathbf{b} = < a_1 + b_1, a_2 + b_2, a_3 + b_3 >$$

Example:

Find **c = a + b** where **a** = <2,1> and **b** =<1,2>:

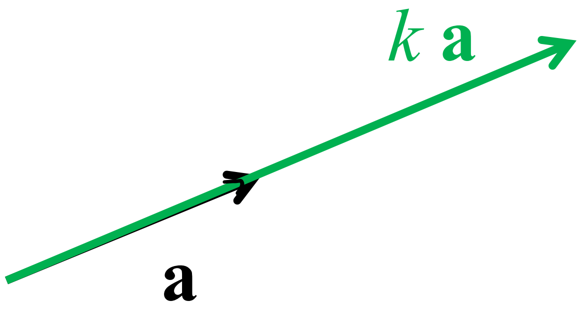
This can be visualised by putting the two vectors **a** and **b** head to tail:





We can also multiply any vector by a scalar:

$$k\mathbf{a} = k < a_1 , a_2 , a_3 >$$
$$= < k\, a_1 , k\, a_2 , k\, a_3 >$$



This has the effect of stretching the vector (if k >1) or shrinking it (if k<1).

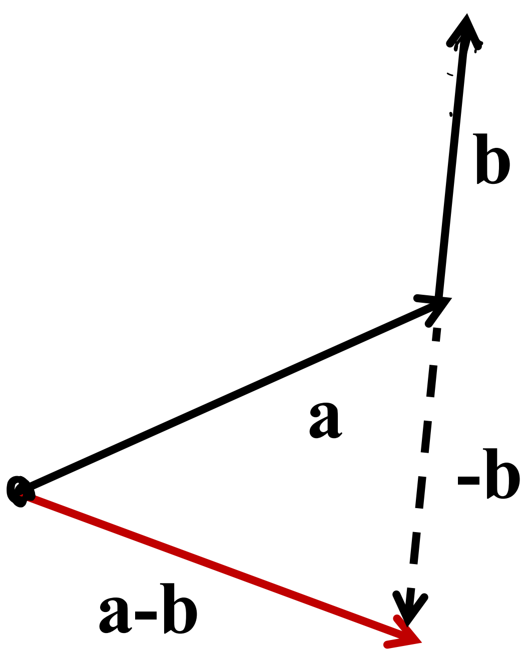
Vectors can be also *subtracted*:

$$\mathbf{a} = < a_1 , a_2 , a_3 >$$

$$\mathbf{b} = < b_1 , b_2 , b_3 >$$

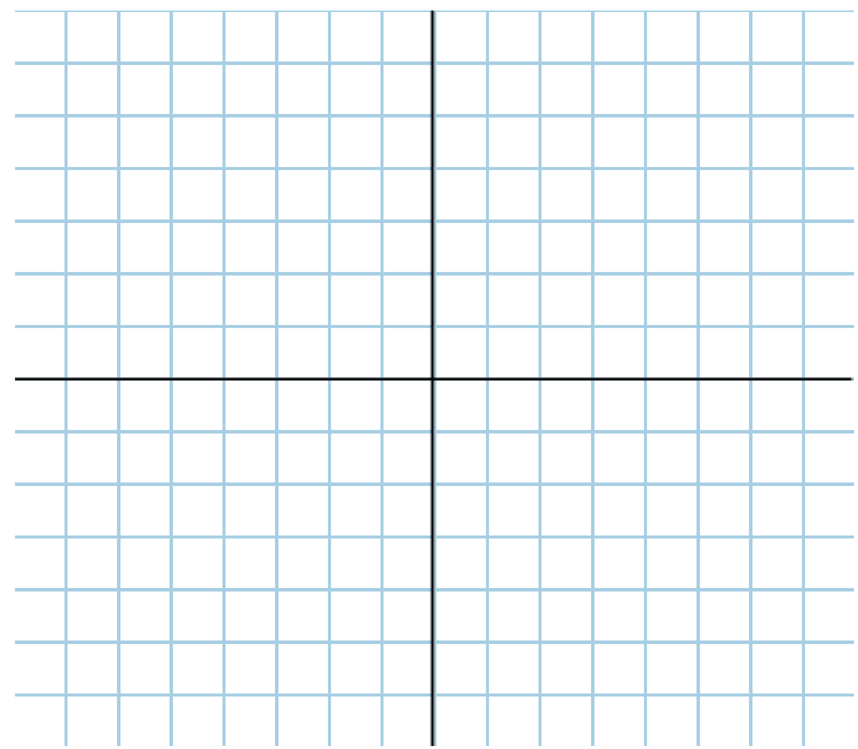
$$\mathbf{a} - \mathbf{b} = < a_1 - b_1, a_2 - b_2, a_3 - b_3 >$$

This is like adding a "negative" version of the vector:

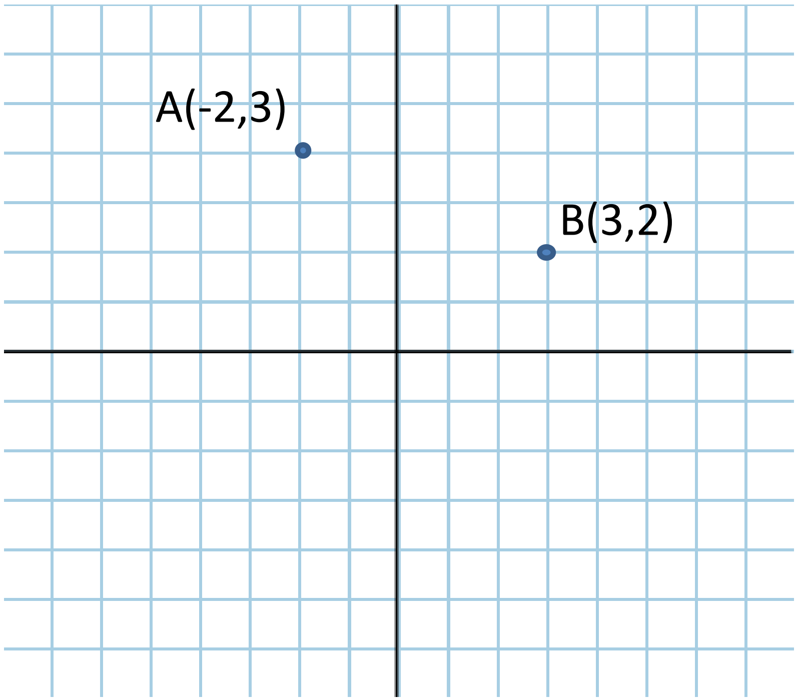


Example:

Find $\mathbf{c} = \mathbf{a} - \mathbf{b}$ where $\mathbf{a} = <2,1>$ and $\mathbf{b} = <1,2>$:



To find a vector connecting two points we can “subtract” them:

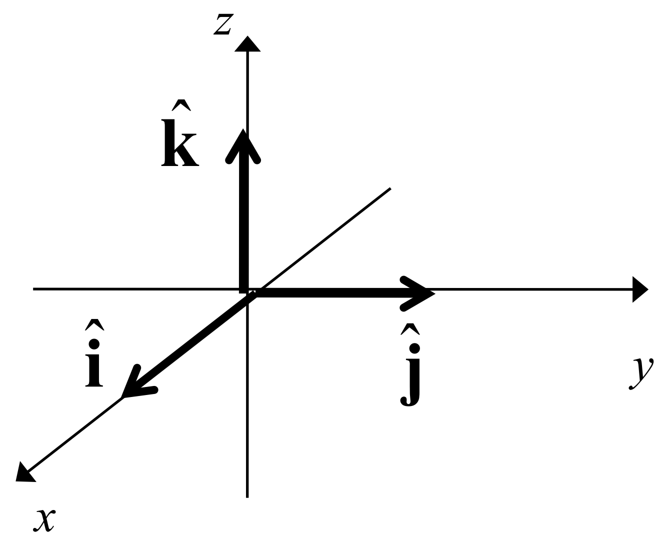


An other notation

$$\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$$

The vectors, **i**, **j** and **k** are known as the *coordinate axis vectors*:

$$\begin{aligned}\hat{\mathbf{i}} &= <1,0,0> \\ \hat{\mathbf{j}} &= <0,1,0> \\ \hat{\mathbf{k}} &= <0,0,1>\end{aligned}$$

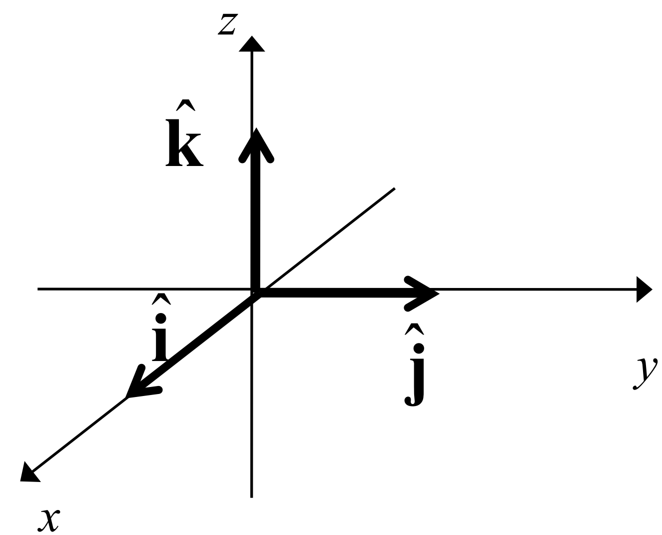


We can use these vectors as “building blocks” to write other vectors. E.g.

$\mathbf{a} = < 2,4,1> =$

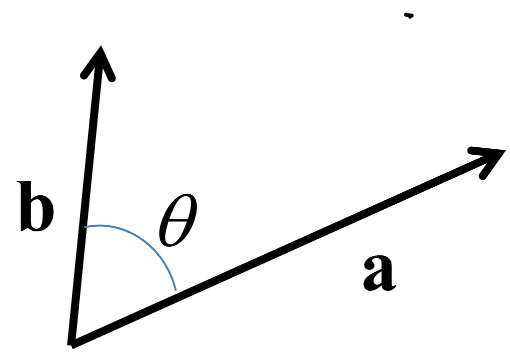
$\mathbf{a} = <-1,2,3>=$

These vectors form a *complete basis*,
i.e. *any* vector $\mathbf{a}$ can be expressed in terms of $\mathbf{i}$ $\mathbf{j}$ and $\mathbf{k}$.



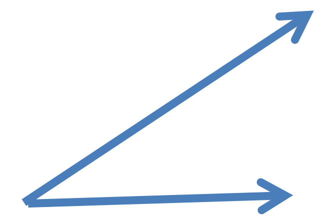
Definition of a dot product is

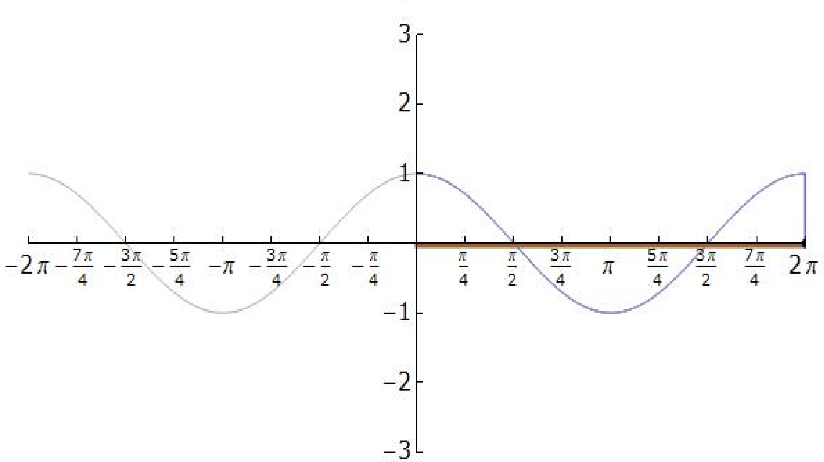
$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$



Where θ is the angle between the two vectors.

The dot product is a *scalar* quantity,
and is sometimes called the *scalar product* or *inner product*.





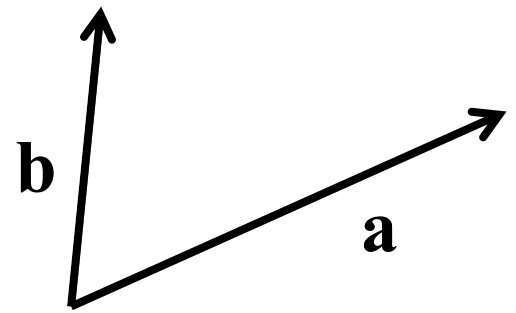
$$\mathbf{i} \cdot \mathbf{j} = |\mathbf{i}||\mathbf{j}| \cos\frac{\pi}{2} = 0, \qquad \mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$$

$$\mathbf{i} \cdot \mathbf{k} = |\mathbf{i}||\mathbf{k}| \cos\frac{\pi}{2} = 0$$

$$\mathbf{j} \cdot \mathbf{k} = |\mathbf{j}||\mathbf{k}| \cos\frac{\pi}{2} = 0$$

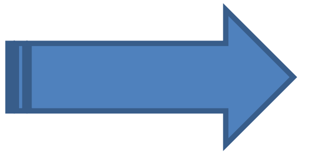
The dot product

The dot product between two vectors **a** and **b** can be written as



$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

The dot product is a *scalar* quantity,
and is sometimes called the *scalar product* or *inner product*.

Two vectors  Scalar

Example:

a = < 1 , 0 , 3>
b = < 2 , 1 , -4>

The dot product of a vector with itself is

$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}||\mathbf{a}|\cos 0 = |\mathbf{a}|^2$$

The dot product is *commutative*:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

The dot product is *distributive over vector addition*:

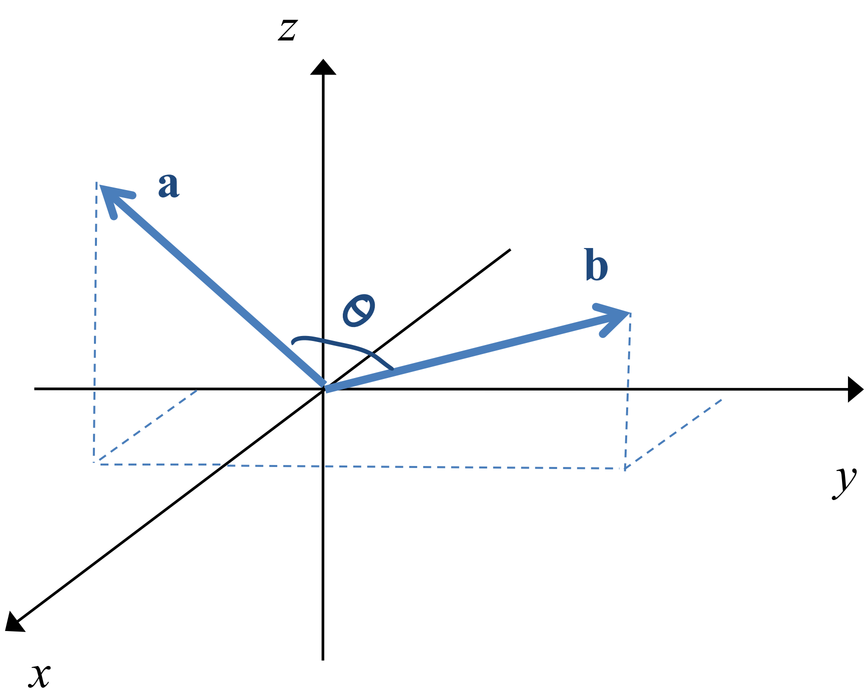
$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

We can also show that for any scalar k:

$$\mathbf{a} \cdot (k\mathbf{b}) = (k\mathbf{a}) \cdot \mathbf{b} = k\mathbf{a} \cdot \mathbf{b}$$

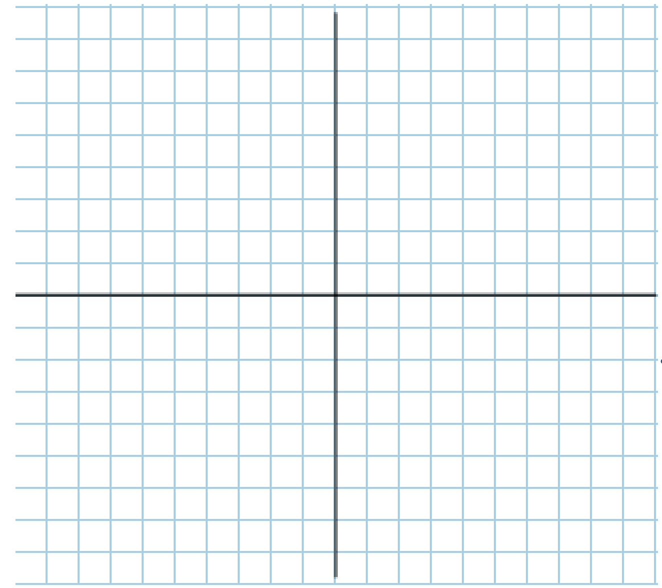
$$\mathbf{a} \cdot \mathbf{b} = 0 \Rightarrow \mathbf{a} \perp \mathbf{b}$$

Example: Find the angle between the vectors **a** = <1,-1,2> and **b** = <1,2,1>



Example 1:

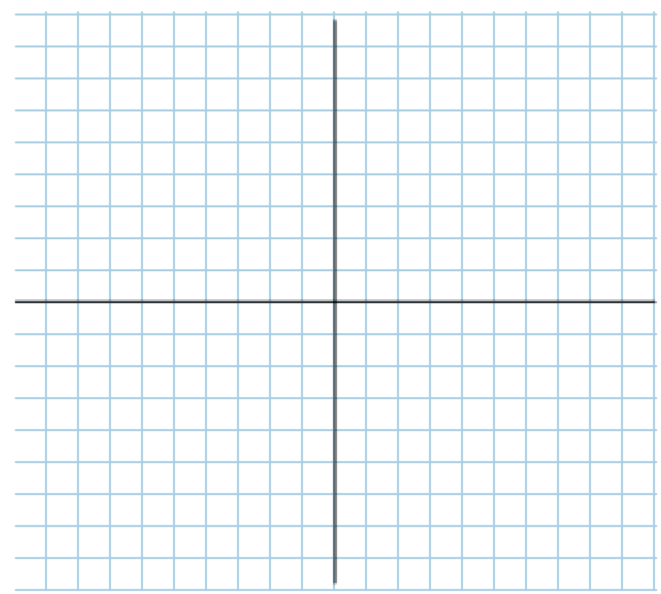
Find the dot product of **a** = <3,1> and **b** = <5,0>



[Ans: 15]

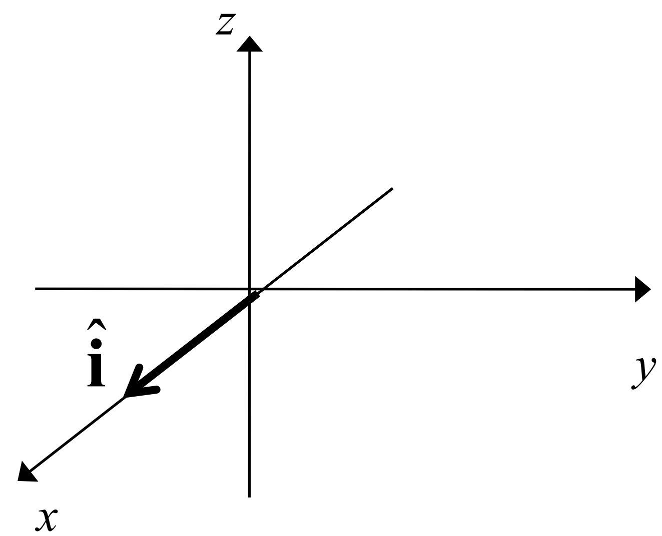
Example 2:

Find the dot product of **a** = <2,2> and **b** = <-1,1>



[Ans: 0]

A *unit vector* is a vector of length 1. They are usually written with a “hat” above the vector symbol.



eg:

$$\hat{\mathbf{i}} = <1, 0, 0>$$

is the unit vector in the x direction.

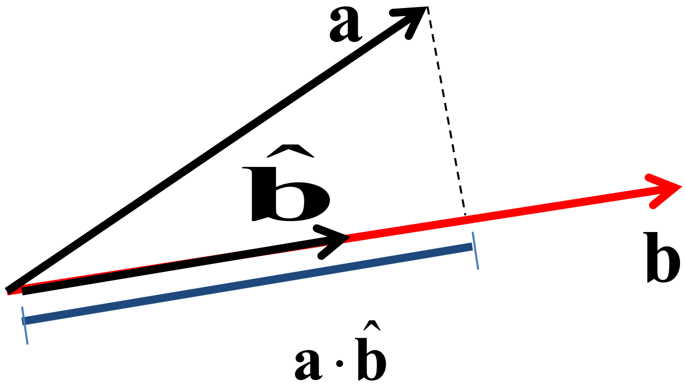
Any vector can be made into a unit vector by dividing it by its own length:

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

Example: Find the unit vector pointing in the same direction as **a** = <-1,2,3>

The *scalar projection* of a vector $\mathbf{a}$ onto a vector $\mathbf{b}$ is defined as

$$\mathbf{a} \cdot \hat{\mathbf{b}}$$



$$comp_{\hat{\mathbf{b}}}\mathbf{a} = \mathbf{a} \cdot \hat{\mathbf{b}} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}$$

$$\hat{\mathbf{b}} = \frac{\mathbf{b}}{|\mathbf{b}|}$$

The *scalar projection* of a vector $\mathbf{b}$ onto a vector $\mathbf{a}$ is

$$comp_{\hat{\mathbf{a}}}\mathbf{b} = \mathbf{b} \cdot \hat{\mathbf{a}} = \frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|}$$

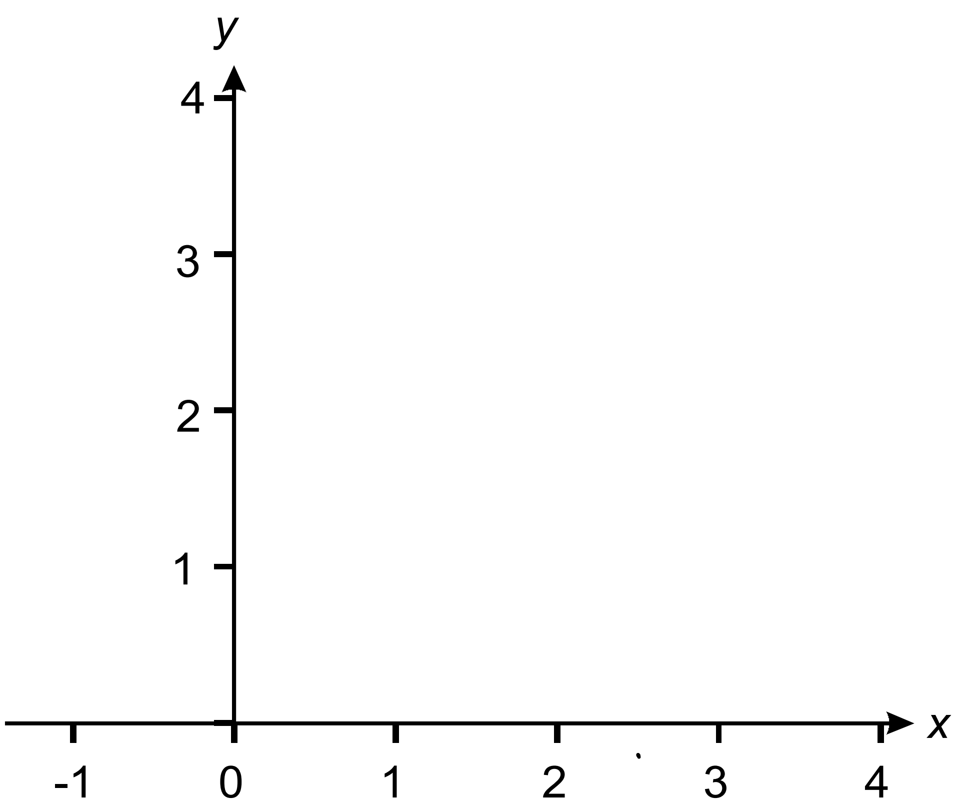
$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

Example

Draw the 2D vectors $\boldsymbol{a} = <1, 3>$ and $\boldsymbol{b} = <4, 3>$.

Find the scalar projection of **a** onto ***b*** and illustrate what you have found

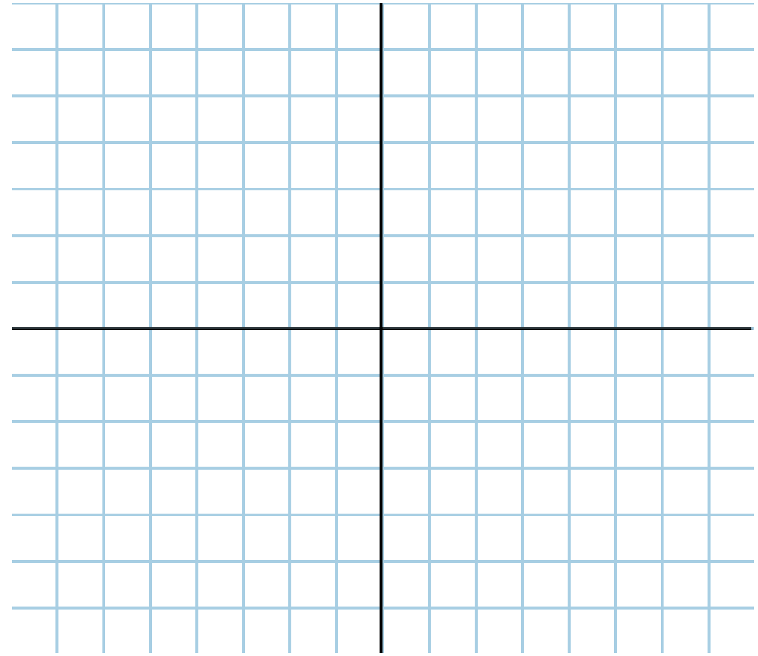
$$comp_{\hat{\mathbf{b}}}\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}$$



We can write any vector in two ways:

$$\mathbf{a} = < a_1, a_2 > = a_1\mathbf{i} + a_2\mathbf{j}$$

$\mathbf{a} \cdot \hat{\mathbf{i}}$ is the *projection* of $\mathbf{a}$ onto the x axis.



$$\mathbf{a} = < a_1, a_2, a_3 > = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$$

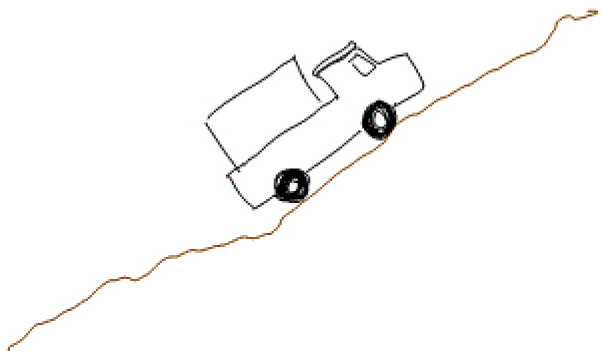
The quantity

$$comp_{\mathbf{i}}\mathbf{a} = \mathbf{a} \cdot \mathbf{i}$$

is the *projection* of $\mathbf{a}$ onto the x axis.

Example:

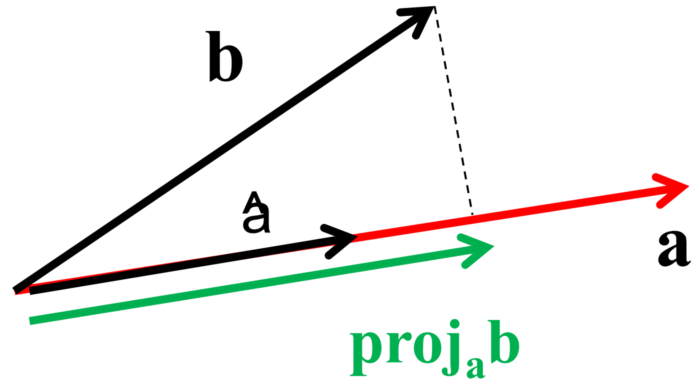
A 10,000 kg truck is on a 30 degree incline. Compute the *magnitude* of the force due to gravity pulling the truck downhill.



The scalar projection gives the *magnitude* of the force.   What if we want the force itself, which is a vector.

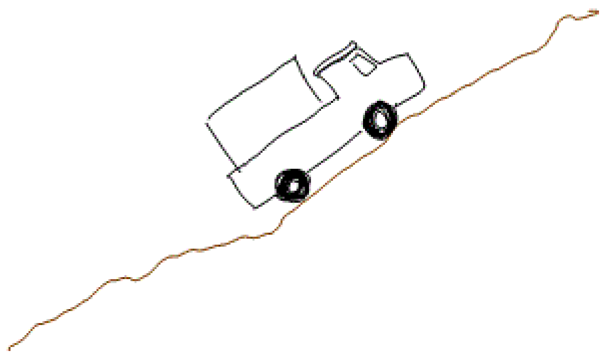
The *vector projection* of $\mathbf{b}$ onto $\mathbf{a}$ is

$$proj_{\widehat{\mathbf{a}}}\mathbf{b} = (comp_{\widehat{\mathbf{a}}}\mathbf{b})\,\widehat{\mathbf{a}} = (\mathbf{b}\cdot\widehat{\mathbf{a}})\widehat{\mathbf{a}}$$



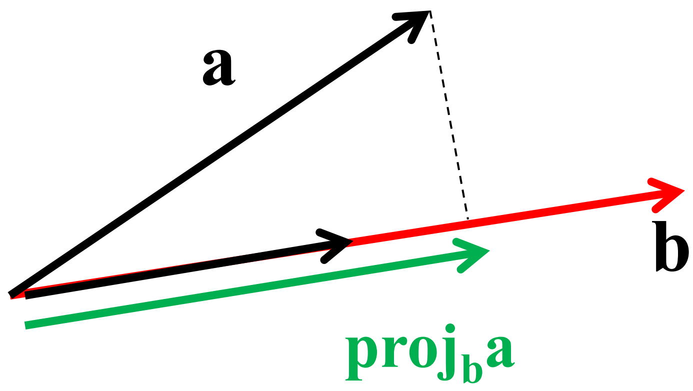
Example:
In the previous example, we had $\hat{\mathbf{a}} = < \frac{\sqrt{3}}{2}, \frac{1}{2} >$ and found

$$\mathbf{F}\cdot\hat{\mathbf{a}} = -\frac{mg}{2}$$



What is the vector force acting down the slope?

The *vector projection* of $\mathbf{a}$ onto $\mathbf{b}$ is



$$proj_{\hat{\mathbf{b}}}\mathbf{a} = (comp_{\hat{\mathbf{b}}}\mathbf{a})\,\hat{\mathbf{b}} = (\mathbf{a}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}} = \frac{(\mathbf{a}\cdot\mathbf{b})\mathbf{b}}{|\mathbf{b}|^2}$$

$$\hat{\mathbf{b}} = \frac{\mathbf{b}}{|\mathbf{b}|}$$

For vectors ***a*** = (2,1), **b** = (1,3) **c**=(-1,4) calculate:

$$comp_{\hat{\mathbf{a}}}\mathbf{b}$$

$$comp_{\mathbf{c}}\mathbf{a}$$

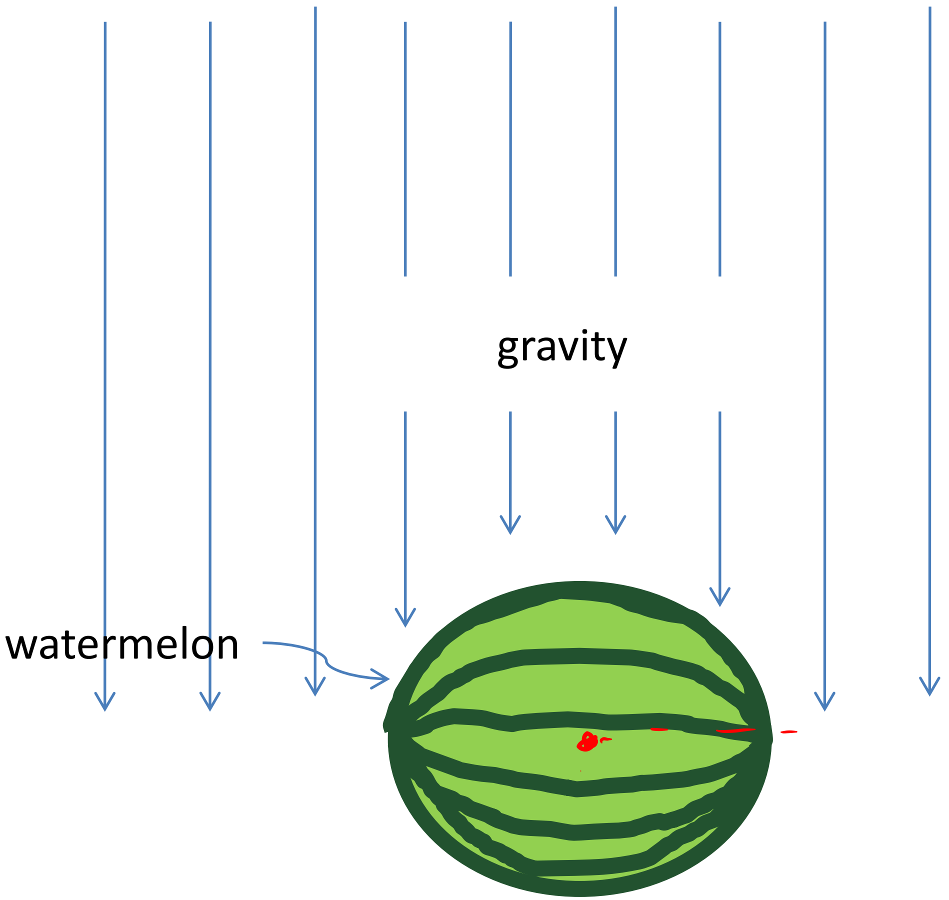
$$proj_{\widehat{\mathbf{a}}}\mathbf{c} = (comp_{\widehat{\mathbf{a}}}\mathbf{c})\,\widehat{\mathbf{a}} = (\mathbf{c}\cdot\widehat{\mathbf{a}})\widehat{\mathbf{a}}$$

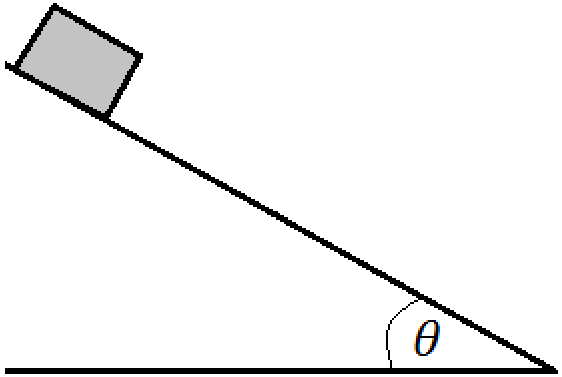
For vectors ***a*** = **i** + 3 **j** - **k** and **b** = 2**i** + **j** - **k** find |**a-b**|, -**a**+2**b**

If $\mathbf{r} = < x, y >$ and $\mathbf{r}_0 = < 2, -2 >$, then find the Cartesian equation for $|\mathbf{r}-\mathbf{r_0}|=2$

The *Work done* on an object is defined by

$$W = \mathbf{F} \cdot \mathbf{d}$$





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