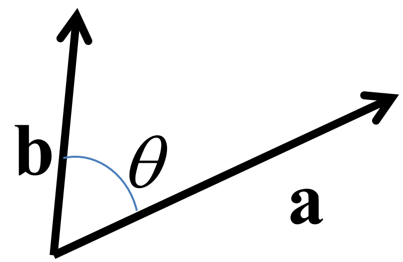
Week 2

The cross product [Textbook: 9.4]

Vector equations of lines and planes [Textbook: 9.5]

Dot product

Two vectors → Scalar



Cross product

Two vectors → Vector

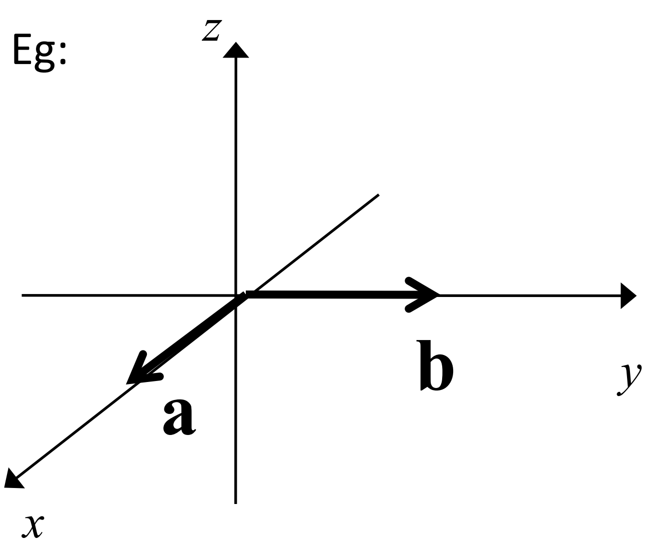
The cross product

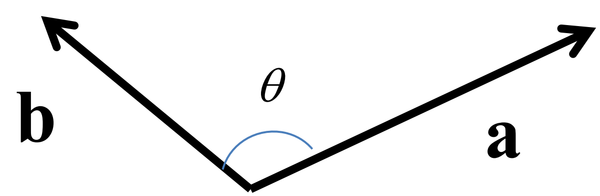
The cross product between two vectors $\mathbf{a}$ and $\mathbf{b}$ is written

$$\mathbf{a} \times \mathbf{b}$$

The cross product is a vector which *always points in direction perpendicular to both* $\mathbf{a}$ *and* $\mathbf{b}$

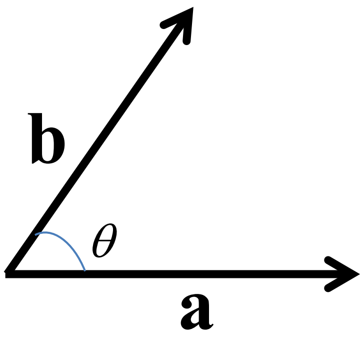
Eg:

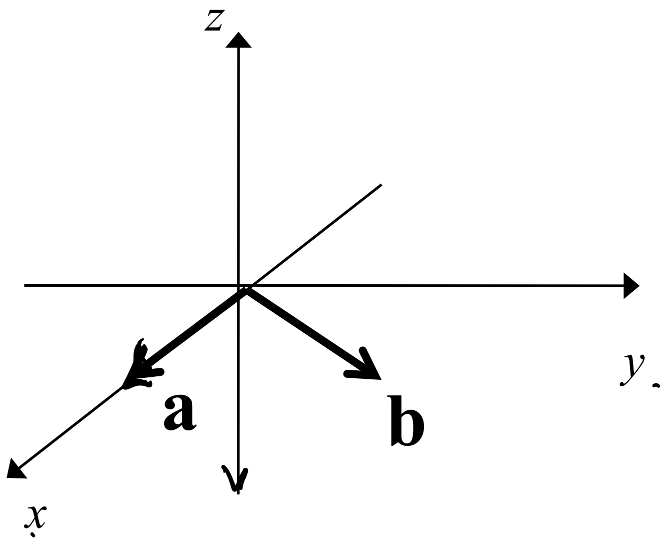


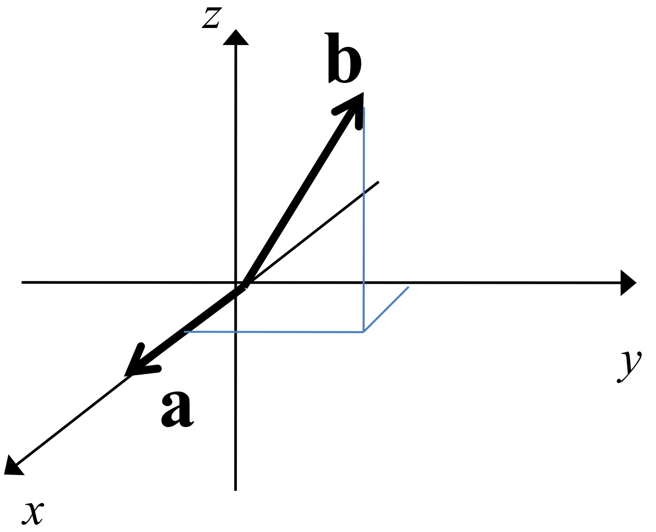


$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \, \hat{\mathbf{n}}$$

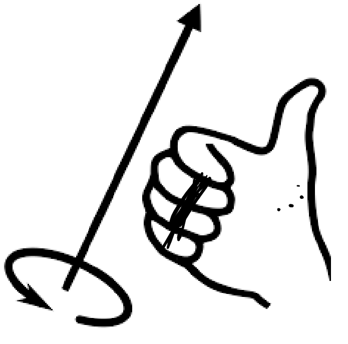
normal unit vector to both $\mathbf{a}$ and $\mathbf{b}$


b
θ
a

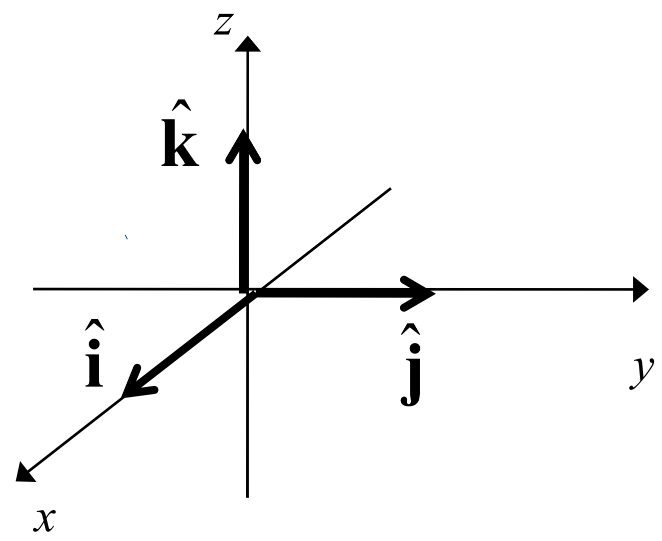




To avoid ambiguity we use the *right-hand rule:*



The unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ form a *right-handed* coordinate system:



The cross product can be calculated using the following formula

$$\mathbf{a} \times \mathbf{b} = \hat{\mathbf{i}}(a_2 b_3 - a_3 b_2) - \hat{\mathbf{j}}(a_1 b_3 - a_3 b_1) + \hat{\mathbf{k}}(a_1 b_2 - a_2 b_1)$$

Example: Calculate the cross product between $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$

$$\begin{array}{r|ccc|} & \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \mathbf{a} \longrightarrow & & & \\ \mathbf{b} \longrightarrow & & & \end{array}$$

The cross product can be calculated using the following formula

$$\mathbf{a}\times\mathbf{b} = \hat{\mathbf{i}}(a_2b_3 - a_3b_2) - \hat{\mathbf{j}}(a_1b_3 - a_3b_1) + \hat{\mathbf{k}}(a_1b_2 - a_2b_1)$$

Example: Calculate the cross product between **a** = <-1,0,1> and **b** = <1,2,2>

$$\begin{array}{c} \\ \mathbf{a} \longrightarrow \\ \mathbf{b} \longrightarrow \end{array}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ & & \\ & & \end{vmatrix}$$

Example: Calculate the cross product between **a** = <-1,3,2> and **b** = <-1,3,2>

Example: Calculate the cross product between **a** = <1,0,2> and **b** = <-1,1,1>

Example: Calculate the cross product between vectors

$$\mathbf{a} = -\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}, \qquad \mathbf{b} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$$

The cross product *does not commute*

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

The cross product is *distributive over vector addition*:

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

For any scalar k:

$$\mathbf{a} \times (k\mathbf{b}) = (k\mathbf{a}) \times \mathbf{b} = k\mathbf{a} \times \mathbf{b}$$

The cross product of a vector with itself is zero:

$$\mathbf{a} \times \mathbf{a} = \mathbf{0}$$

For any two vectors **a** and **b**:

$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$$

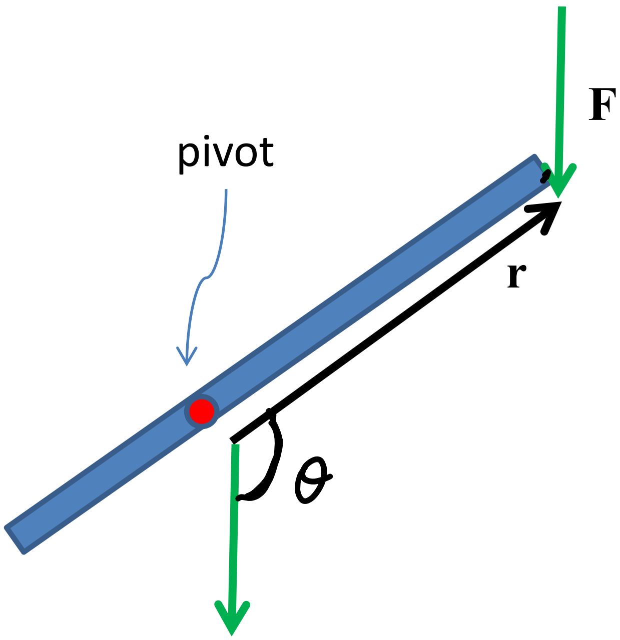
“Real-world” application: torque

Torque is defined as

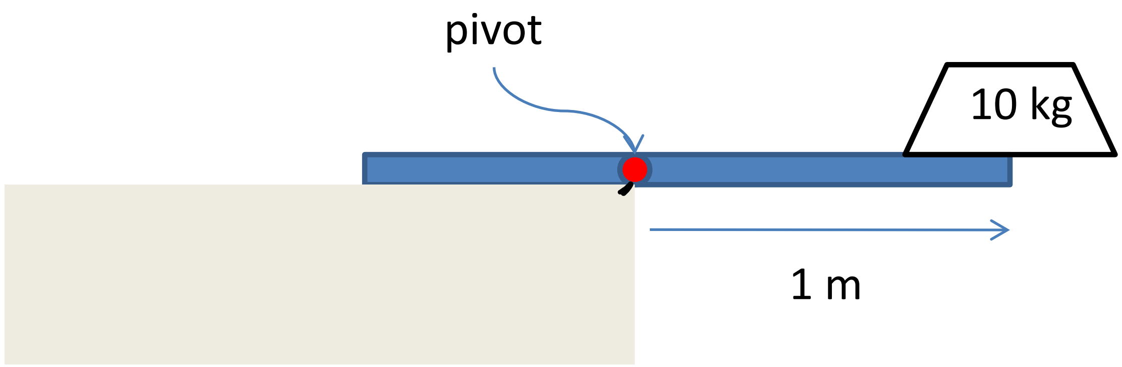
Force

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

Displacement from pivot point



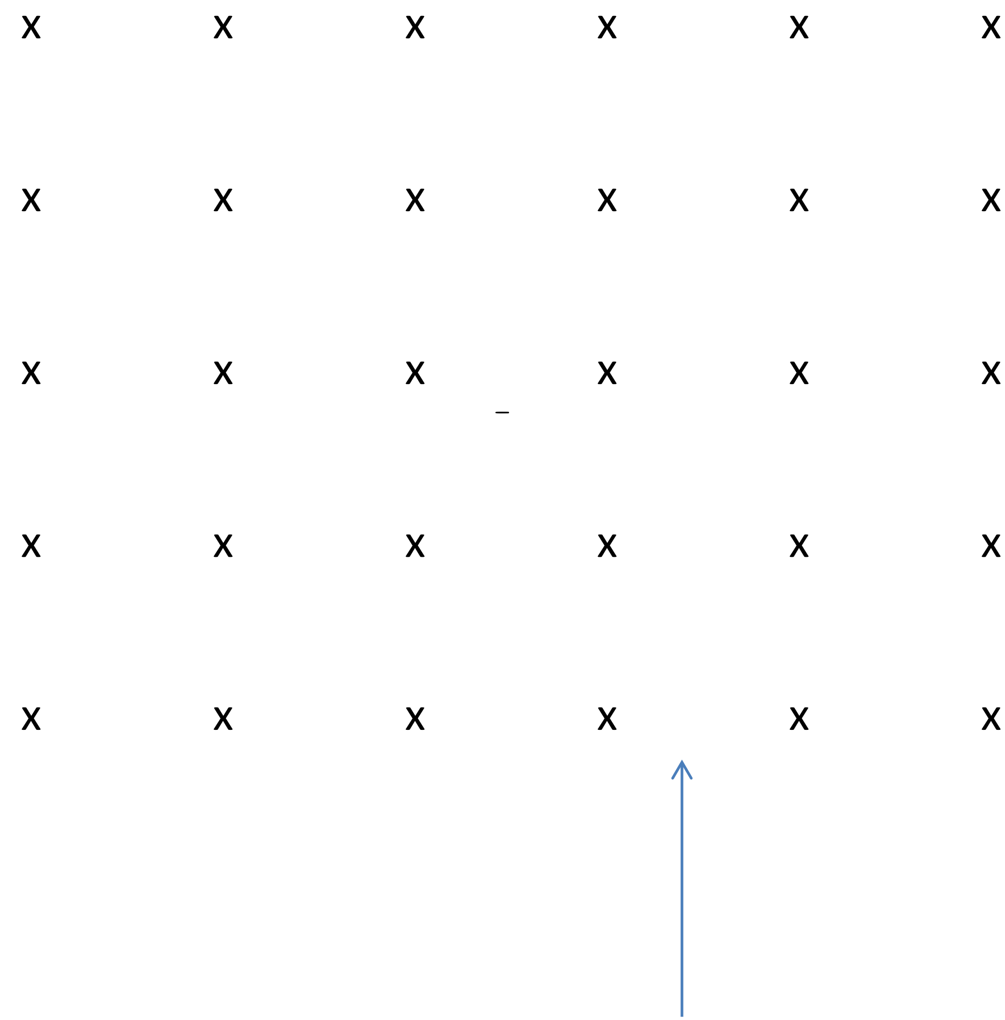
Example: Calculate the torque in the following situation:



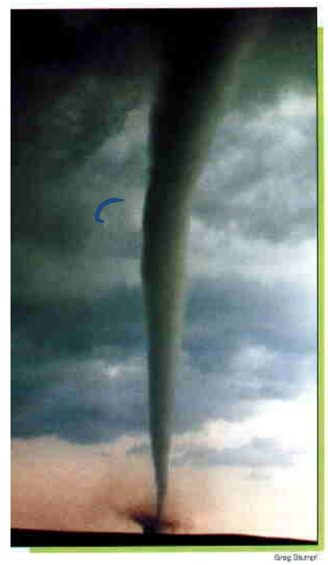
Example: charged particle in a magnetic field

The force on a particle moving with velocity **v** in a uniform magnetic field **B** is

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$



Cross products occur commonly when describing *rotations about an axis.* E.g.



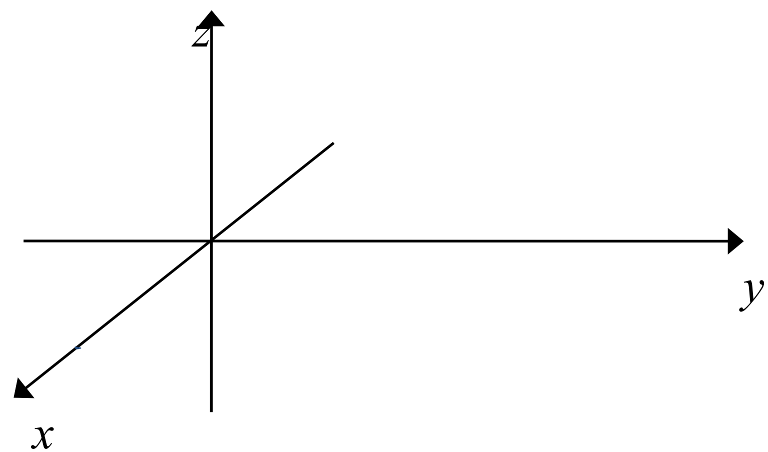
Vector representation of lines and planes

A vector representing position is usually written with the symbol **r**

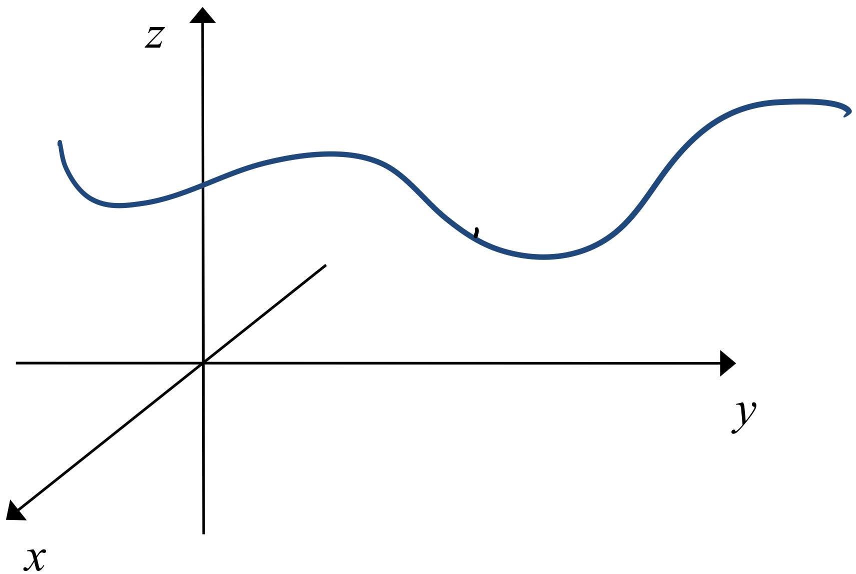
$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

The magnitude of the position vector is

|**r**| = |<x,y,z>| =



When each of the three variables x, y and z are functions of a single variable t, we obtain a *curve* in 3D space.



$$x(t) = F(t)$$
$$y(t) = G(t)$$
$$z(t) = H(t)$$

We can represent the curve as a direct relationship between x,y and z:

$$f(x) = g(y) = h(z)$$

This is called the Cartesian representation.

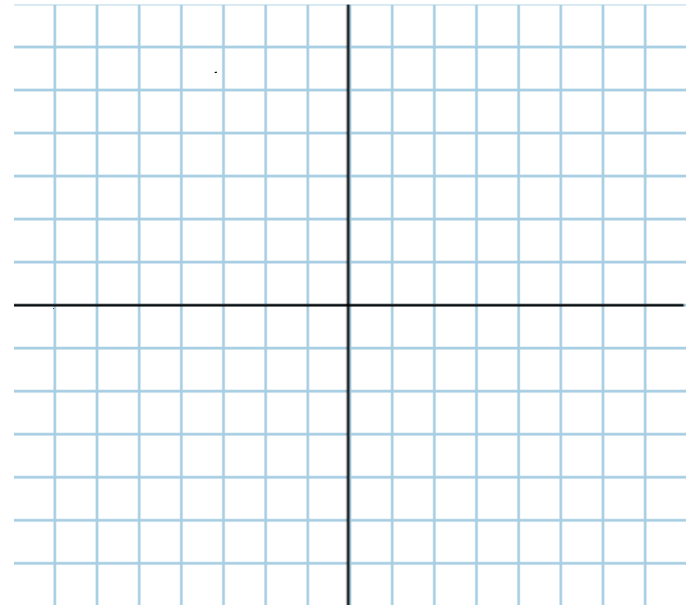
A relation involving a parameter t
is called a *parametric* representation.

Example:

For the 2D position vector $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, plot the curves

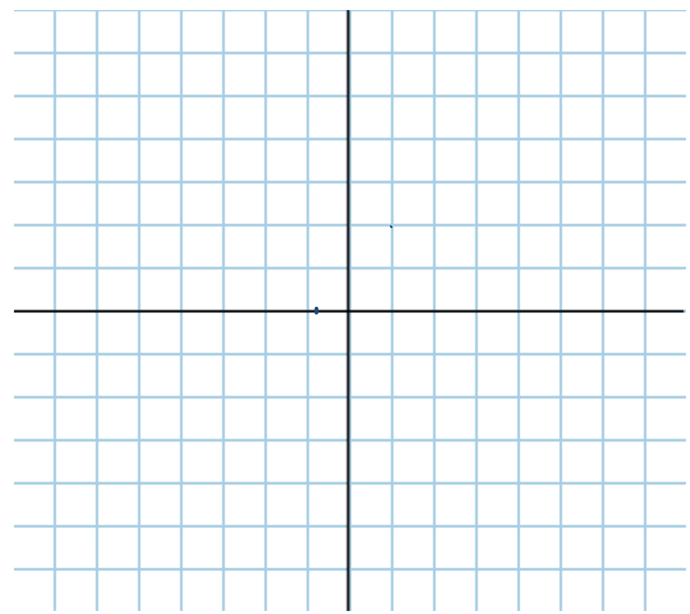
(a) $x(t) = 2\,t$

$y(t) = -t$



(b) $x(t) = \cos t$

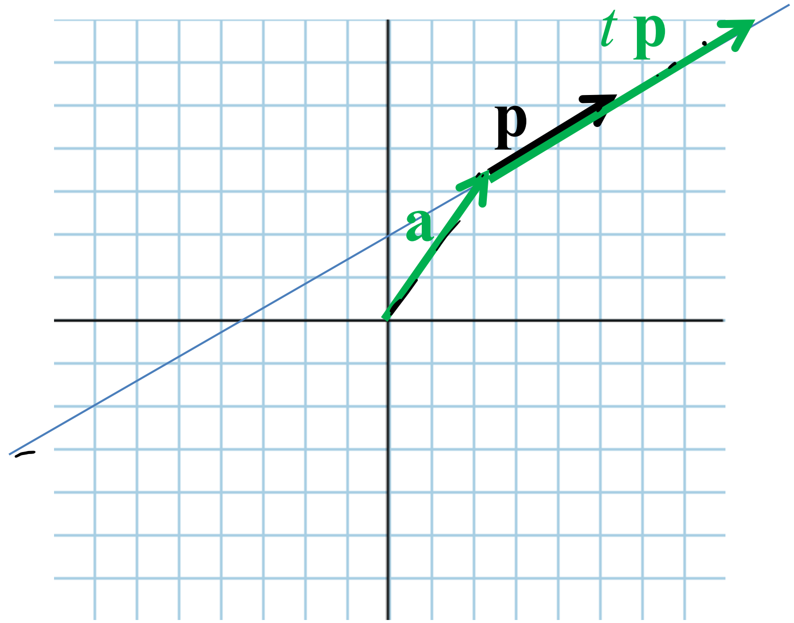
$y(t) = \sin t$



Any straight line can be written in vector form

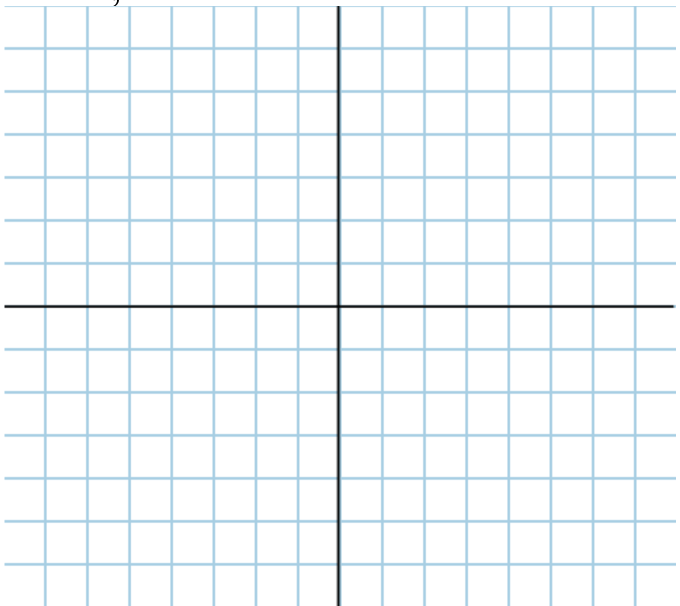
$$r(t) = a + t\,\mathbf{p}$$

where $\boldsymbol{p}$ is the direction of the line, and $\mathbf{a}$ is a point on the line



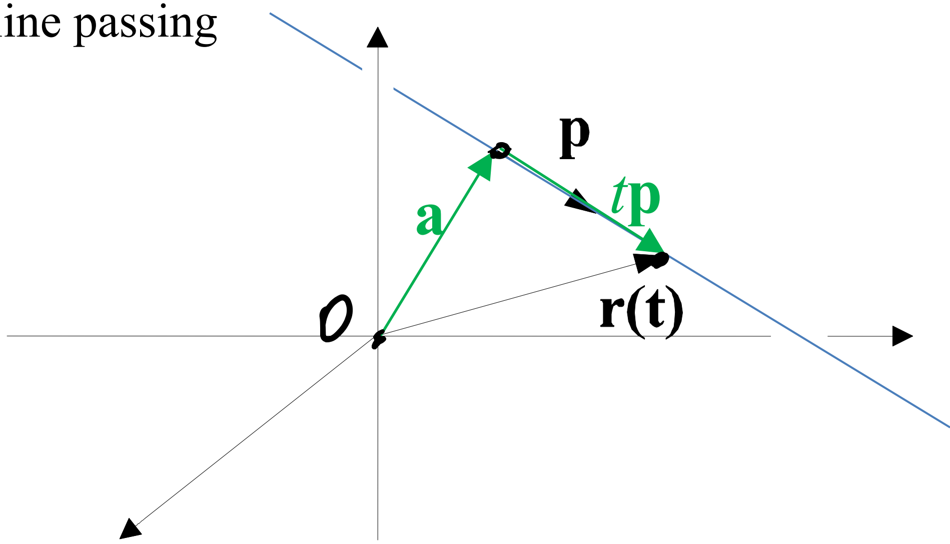
Example:

Draw the line passing through A(-1,-1 and parallel to $\mathbf{p}$ = <2,-1>



Example:

Find the vector and Cartesian equations of the line passing through the points A(1,1,3) and B(2,1,-1) .



Planes

Anything of the form

$a\,x + b\,y + c\,z = Const.$

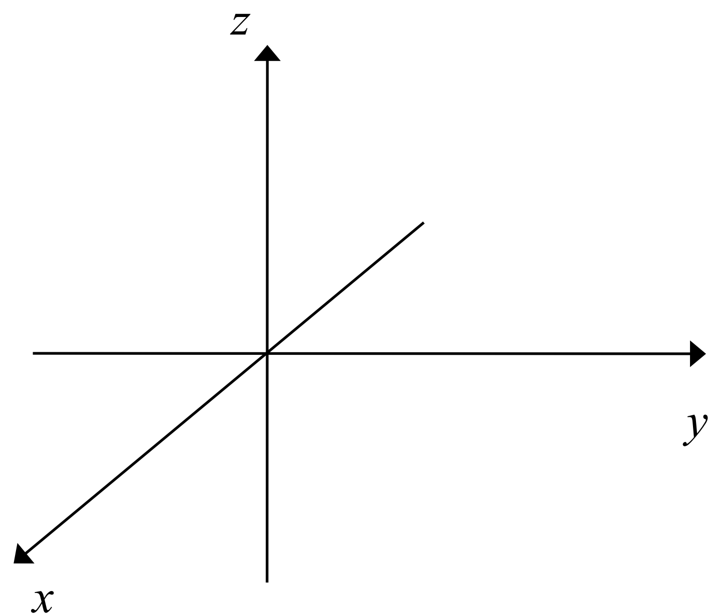
is a *plane.*

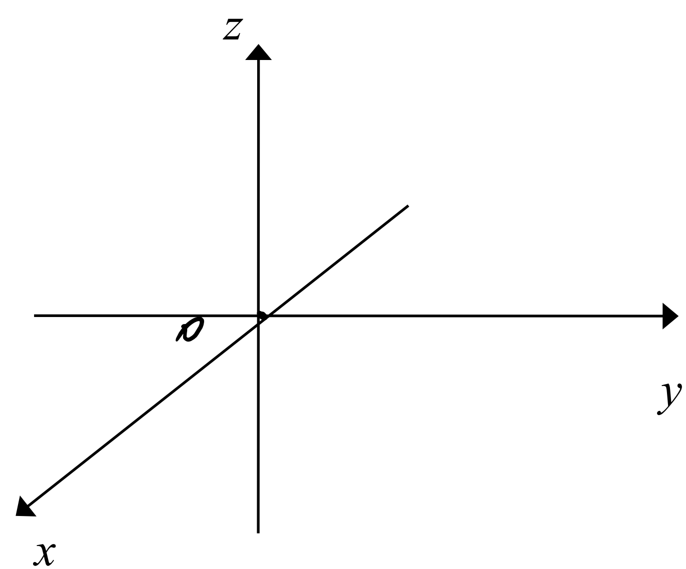
e.g:

1. $z = 1$

2. $y = 2$

3. $x + y + z = 1$





Vector representations of planes

Any two vectors $\mathbf{p}$ and $\mathbf{q}$ together define a *plane*

Any plane can be written in the form

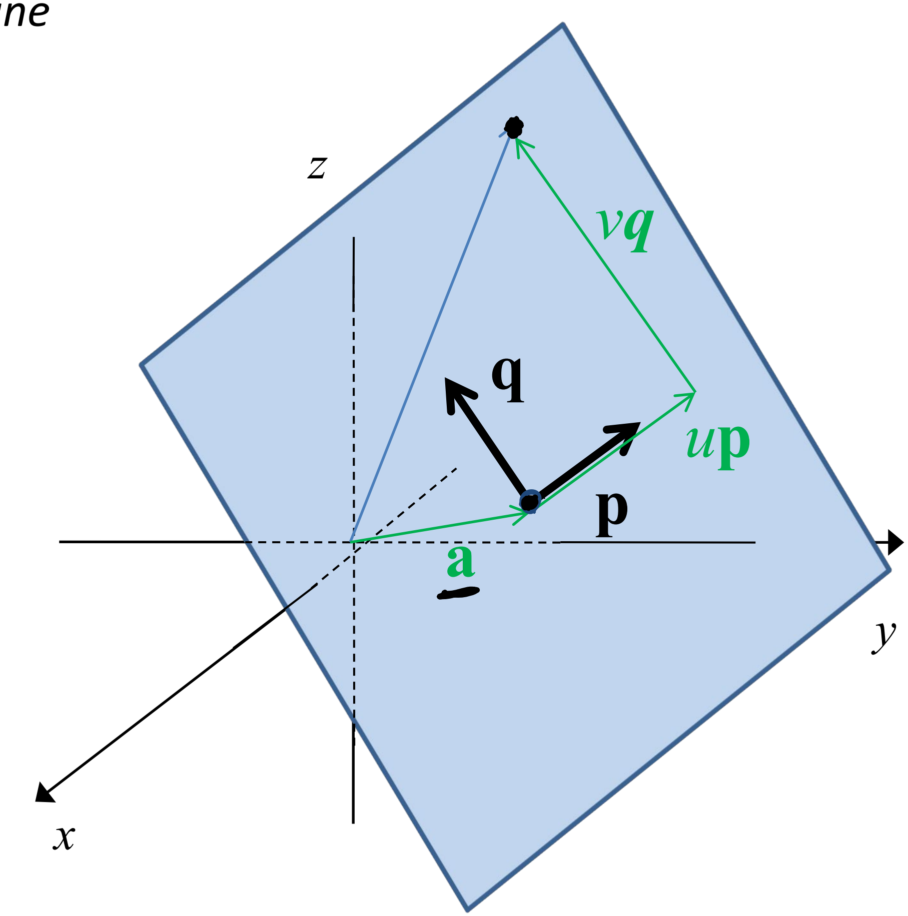
$$\boldsymbol{r(u,v) = a} + u\ \mathbf{p} + v\ \mathbf{q}$$

where:

$\mathbf{p}$, $\mathbf{q}$ are the vectors that define the plane

$\mathbf{a}$ is some point on the plane

u,v are two scalars



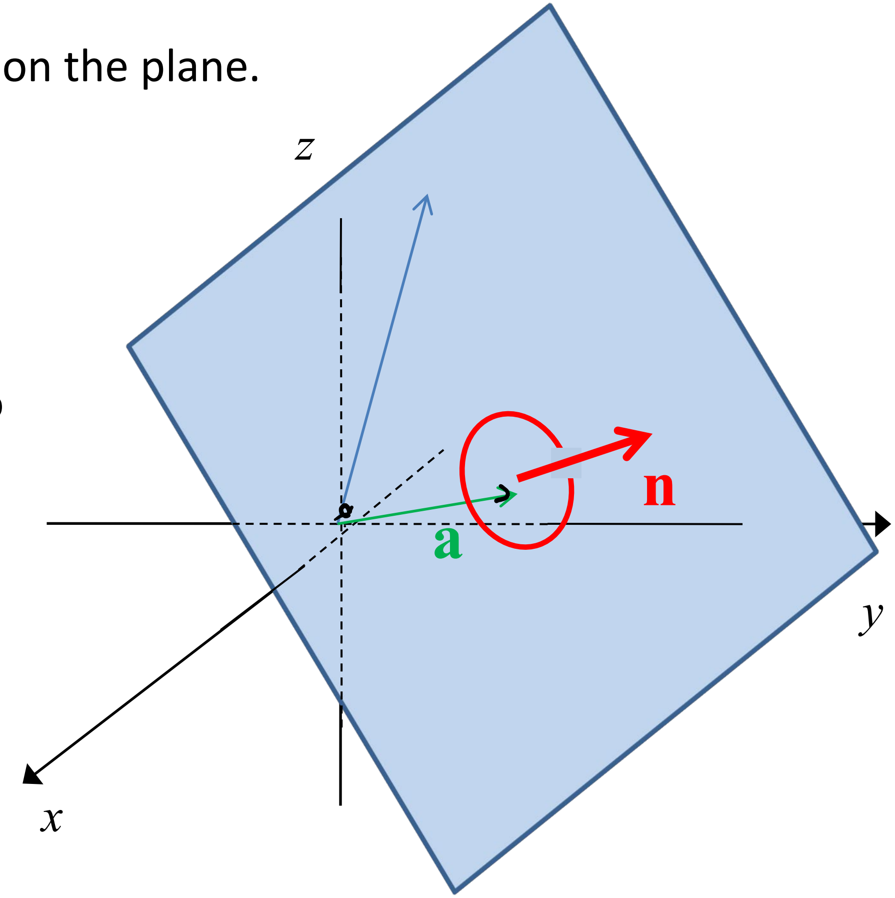
Example:

1. Find a vector equation of the plane passing through A(1,2,3) and parallel to the vectors $\mathbf{p} = \langle 0,1,-1 \rangle$ and $\mathbf{q} = \langle 1,0,-1 \rangle$

$$\boldsymbol{r}(u,v) = \boldsymbol{a} + u\,\mathbf{p} + v\,\mathbf{q}$$

Normals to the plane

A plane can also be defined by a
normal vector ***n***, together with a point ***a*** lying on the plane.



1. The vector (**r** – **a**) always lies in the plane

2. The vector (**r – a**) is always perpendicular to the normal vector

So:

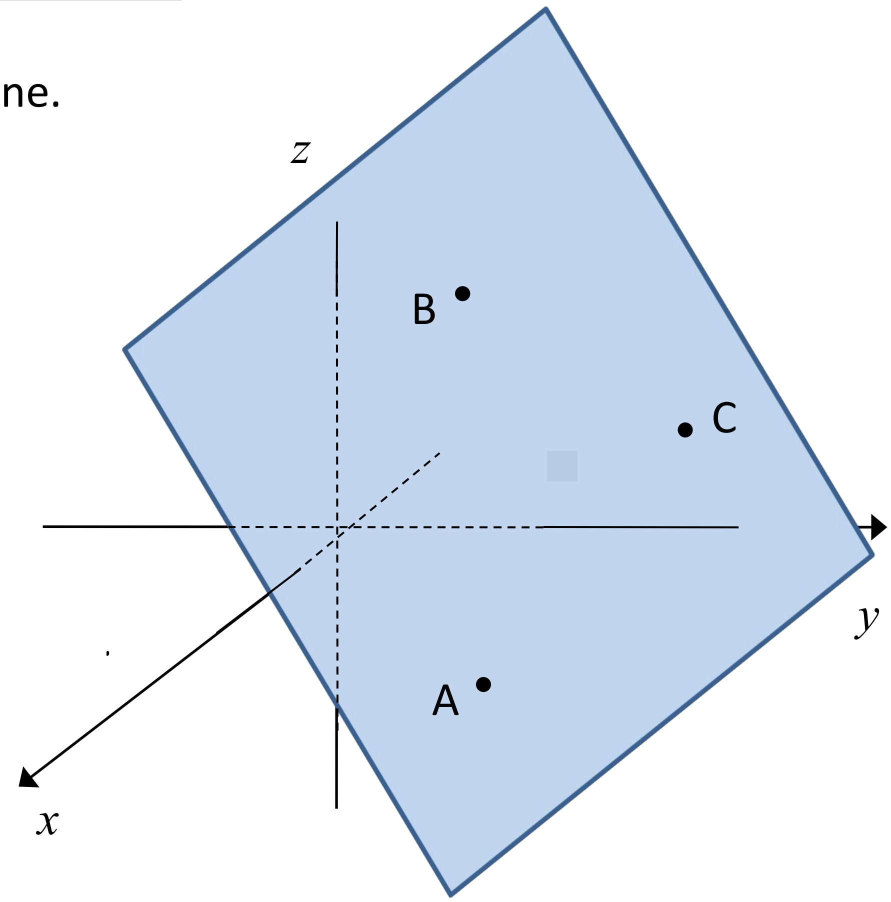
Example (using the normal to find the plane equation):

Find the equation of the plane with normal **n** = <2,3,5> and passing through the point A(8,10,1)

The normal can be found using the cross product.

Suppose points A, B and C lie on the plane.
A normal vector is given by

$$\mathbf{n} = \vec{AB} \times \vec{AC}$$



Example:

1. Find the Cartesian equation of the plane passing through the points A(3,-2,0) , B(-1,2,-1) and C(0,0,4)