Matrices

ma’tr|ix *n*. (*pl*. ~ices or ~ixes) **5.** (Math.) rectangular array of quantities in rows and columns that is treated as a single quantity.

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 0 \\ 9 & 1 & 2 \\ 1 & 3 & 4 \end{pmatrix}$$

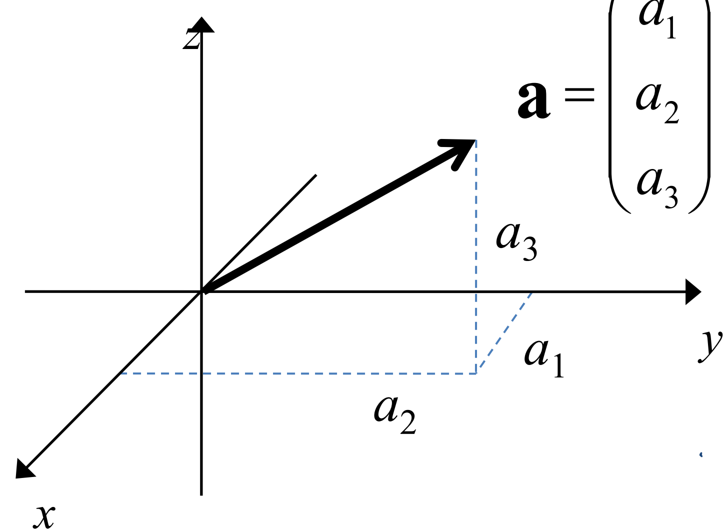
A matrix is a rectangular array of numbers.

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 0 \\ 9 & 1 & 2 \\ 1 & 3 & 4 \end{pmatrix} \qquad \mathbf{M} = \begin{pmatrix} 2 & 6 & -2 & 3 \\ -1 & 5 & 0 & 7 \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 5 \\ 2 \\ -3 \end{pmatrix}$$

A matrix has order m x n if it has m rows and n columns.

A (1 x n) matrix is known as a *row vector;*
an (m x 1) matrix is called a *column vector*.

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$



The *elements* of a matrix $\mathbf{A}$ are often written

$$a_{ij}$$

j^{th} column

i^{th} row

Eg:

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 0 \\ 9 & 1 & 2 \\ 1 & 3 & 4 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

If two matrices are *the same order*, they may be added or subtracted.

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 3 & 6 & 1 \\ -1 & 1 & 2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 3 & 6 & 1 \\ -1 & 1 & 2 & 1 \end{pmatrix} =$$

It follows that matrices are *commutative* and *associative* under addition:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$$

A zero matrix is any matrix with all elements equal to zero, and is usually written $\mathbf{0}$:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A} = \mathbf{A}$$

It is straightforward to multiply matrices by a scalar quantity *k*:

$$k\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2k & k & 0 \\ k & k & k \\ 0 & k & 2k \end{pmatrix}$$

Multiplying two matrices together is more complicated.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

<u>Rule:</u> the number of columns of the first matrix must equal the number of rows of the second.

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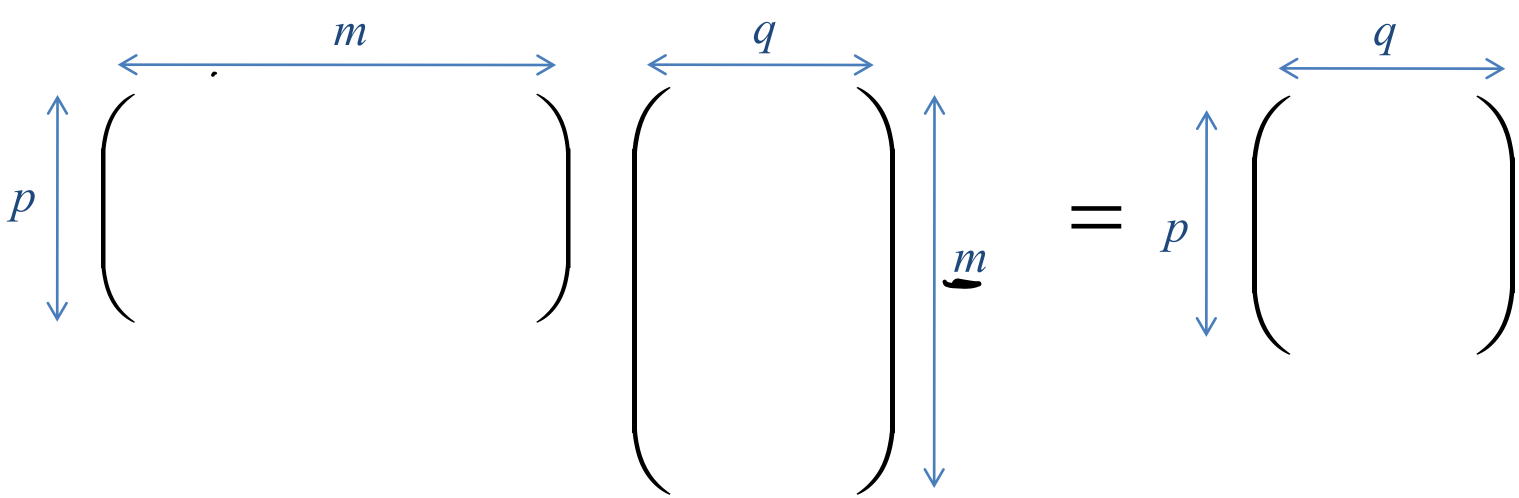

Multiplication of two matrices can be written as a sum:

$$[\mathbf{AB}]_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

We define the product of a row vector and a column vector as being the sum of the components, added together. E.g:

$$\begin{pmatrix} 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 2 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 2 & 0 & 2 \end{pmatrix} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

$$\mathbf{AB} = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 5 \\ 1 & 1 & 1 \\ 0 & 1 & 4 \end{pmatrix}$$

If the matrices have the “wrong” dimensions for multiplication, then the product *does not exist*. E.g:

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 3 & 2 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 6 & 1 \\ -1 & 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 3 \end{pmatrix}$$

Matrix multiplication is *non-commutative*, i.e. Usually,

$$\mathbf{AB} \neq \mathbf{BA}$$

$$\mathbf{AB} = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} =$$

$$\mathbf{BA} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} =$$

Matrix Algebra

Most rules for manipulating matrices are *the same* as those for regular numbers.

Equality:

If $\mathbf{A} = \mathbf{B}$ and $\mathbf{A} = \mathbf{C}$ then $\mathbf{B} = C$

Associativity under addition:

$\mathbf{A}+\mathbf{B} = \mathbf{B} + \mathbf{A}$

$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$

Matrix multiplication is *associative*

$$\mathbf{A(BC) = (AB)C}$$

but is *non-commutative*, i.e. Usually,

$$\mathbf{AB \neq BA}$$

$$\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$$

Matrix multiplication has some other “strange” properties:

1. $\mathbf{AB} = \mathbf{0}$ does not necessarily mean that $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$

$$\mathbf{AB} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ -2 & 0 \end{pmatrix} =$$

2. $\mathbf{AD} = \mathbf{AC}$ does not necessarily mean that $\mathbf{D} = \mathbf{C}$

$$\mathbf{AD} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ -3 & 0 \end{pmatrix}$$

The identity matrix $\mathbf{I}$ is the matrix with the property

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A}$$

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

The Inverse of a square matrix

The inverse of a square matrix $\mathbf{A}$ (written $\mathbf{A}^{-1}$) has the property that

$$\mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I} \qquad \text{and} \qquad \mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}$$

A matrix only has *one* inverse.

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad \det\mathbf{A} = ad-bc$$

$$\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix}$$

The *transpose* of a matrix is obtained by interchanging the rows and the columns:

$$\mathbf{A} = \begin{pmatrix} 2 & 6 & -2 & 3 \\ -1 & 5 & 0 & 7 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 2 & 1 \\ 5 & 0 \\ -1 & 3 \end{pmatrix}$$

If A and B have the same order,

$$(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$$

$$(\mathbf{A}^T)^T = \mathbf{A}$$

$$(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$$

$$\mathbf{A}^T = \begin{pmatrix} 2 & -1 \\ 6 & 5 \\ -2 & 0 \\ 3 & 7 \end{pmatrix}$$

Summary: Rules of matrix algebra

Equality:

If $\mathbf{A} = \mathbf{B}$ and $\mathbf{A} = \mathbf{C}$ then $\mathbf{B} = \mathrm{C}$

Addition:

$\mathbf{A}+\mathbf{B} = \mathbf{B} + \mathbf{A}$
$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$

Multiplication:

$(\mathbf{A} + \mathbf{B})\, \mathbf{C} = \mathbf{AC} + \mathbf{BC}$
$\mathrm{k}\, (\mathbf{A} + \mathbf{B}) = \mathrm{k}\mathbf{A} + \mathrm{k}\, \mathbf{B}$
$\mathrm{k}(\mathbf{AB}) = (\mathrm{k}\mathbf{A})\, \mathbf{B}$

Special matrices:

$\mathbf{AI} = \mathbf{IA} = \mathbf{A}$
$\mathbf{A} + \mathbf{0} = \mathbf{A}$

If A has an inverse,

$\mathbf{A}^{-1}\, \mathbf{A} = \mathbf{A}\, \mathbf{A}^{-1} = \mathbf{I}$

Transposes:

$(\mathbf{A}+\mathbf{B})^{\mathbf{T}} = \mathbf{A}^{\mathbf{T}} + \mathbf{B}^{\mathbf{T}}$
$(\mathbf{AB})^{\mathbf{T}} = \mathbf{B}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}$

Remember that in general:

$\mathbf{AB} \neq \mathbf{BA}$
$\mathbf{AB} = \mathbf{0}$ does not necessarily mean that $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$
$\mathbf{AD} = \mathbf{AC}$ does not necessarily mean that $\mathbf{D} = \mathbf{C}$

Main uses of matrices:

Numerical simulations, image processing, solving linear systems of equations …

Solving systems of Equations

Matrices are very useful for representing systems of linear equations.

E.g. Suppose we want to solve

$$\begin{array}{ccccc} 2x & + & 3y & = & 5 \\ x & - & 7y & = & 1 \end{array}$$

Overview:

To solve the system

$$\begin{array}{rcrcl} 2x_1 & + & 3x_2 & = & 5 \\ x_1 & - & 7x_2 & = & 1 \end{array}$$

Write in matrix form:

$$\begin{pmatrix} 2 & 3 \\ 1 & -7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

Multiply *on the left* by the inverse of the square matrix:

$$\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & -7 \end{pmatrix}^{-1}\begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

$$\det \mathbf{A} = ad - bc$$

$$= \frac{1}{-17}\begin{pmatrix} -7 & -3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 1 \end{pmatrix} = \frac{1}{17}\begin{pmatrix} 38 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{38}{17} \\ \frac{3}{17} \end{pmatrix}$$

Example:
Solve the linear system

$$\begin{aligned} 5x_1 &- 10x_2 &= 3 \\ 2x_1 &+ 3x_2 &= 5 \end{aligned}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 & -10 \\ 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 5 \end{pmatrix} =$$

$$\frac{1}{15-2\times(-10)} \begin{pmatrix} 3 & 10 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} =$$

$$= \frac{1}{35} \begin{pmatrix} 3\times 3+10\times 5 \\ -2\times 3+5\times 5 \end{pmatrix} = \frac{1}{35} \begin{pmatrix} 59 \\ 19 \end{pmatrix}$$

Crammer's Formula

$$\begin{array}{rcrcr} 5x_1 & - & 10x_2 & = & 3 \\ 2x_1 & + & 3x_2 & = & 5 \end{array} \Leftrightarrow \begin{pmatrix} 5 & -10 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \qquad \mathbf{Ax} = \mathbf{b}$$

$$\mathbf{A} = \begin{pmatrix} 5 & -10 \\ 2 & 3 \end{pmatrix}, \quad D = \det \mathbf{A} = 5 \times 3 - (-10) \times 2 = 35$$

D is the determinant of matrix ***A***.

$$D_1 = \det \begin{pmatrix} 3 & -10 \\ 5 & 3 \end{pmatrix} = 3 \times 3 - (-10) \times 5 = 59$$

Replace first column of matrix ***A*** with column vector ***b*** and find the determinant D_1.

$$D_2 = \det \begin{pmatrix} 5 & 3 \\ 2 & 5 \end{pmatrix} = 5 \times 5 - 3 \times 2 = 19$$

Replace the second column of matrix ***A*** with column vector $\underline{b}$ and find the determinant D_2.

$$x_1 = \frac{D_1}{D} = \frac{59}{35},$$

$$x_2 = \frac{D_2}{D} = \frac{19}{35}.$$

Determinants of 3x3 matrices

The cofactors are the signed determinants of the small matrices.

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\Delta = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\ + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Steps to compute the determinant using cofactors:

1. Remove the top row (or first column)

2. Multiply the coefficients of this row with +, -, +, -, +, - etc…

3. Multiply the coefficients of this row with the determinant of the *minor matrix* i.e. the matrix obtained by deleting the coefficient's row and column

Determinants of 3x3 matrices

The determinant of a 3 x 3 matrix is given by the formula

$$\Delta = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

and is often denoted by |square brackets| around the matrix.

Example: Calculate the determinant of $\begin{pmatrix} 1 & -2 & 2 \\ 1 & -2 & 1 \\ 3 & -2 & 1 \end{pmatrix}$

$$\det\begin{pmatrix} 1 & -2 & 2 \\ 1 & -2 & 1 \\ 3 & -2 & 1 \end{pmatrix} = 1\begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & -2 \\ 3 & -2 \end{bmatrix} =$$

$$= 2(1-3) + 2(-2+6) = 8 - 4 = 4$$

Example: Calculate the determinant of $\begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{pmatrix}$

$$\det\begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{pmatrix} = 1\begin{bmatrix} -1 & 3 \\ -1 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix} + 2\begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix} =$$

$$= (1+3) - (-2-9) + 2(-2+3) = 17$$

Tricks when dealing with determinants

If any row or column contains only zeros then the determinant is zero

e.g.

$$\begin{vmatrix} 1 & 3 & 5 \\ 1 & -2 & 3 \\ 0 & 0 & 0 \end{vmatrix}$$

If any two rows or columns are identical then the determinant is zero

e.g.

$$\begin{vmatrix} 1 & 3 & 5 \\ 1 & -2 & 3 \\ 1 & -2 & 3 \end{vmatrix}$$

To find the determinant of a matrix in *triagonal form,*
multiply down the diagonal

e.g.

$$\begin{vmatrix} 1 & 3 & 5 \\ 0 & -2 & 3 \\ 0 & 0 & 6 \end{vmatrix} \qquad \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix}$$

Use Crammer's method to find the solution of linear system

$$\begin{array}{cccccc} x_1 & -2x_2 & +2x_3 & = & 1 \\ x_1 & -2x_2 & +x_3 & = & 1 \\ 3x_1 & -2x_2 & +x_3 & = & 1 \end{array}$$

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 1 \\ 3 & -2 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{Ax} = \mathbf{b}$$

$$D = \det \mathbf{A} = \begin{vmatrix} 1 & -2 & 2 \\ 1 & -2 & 1 \\ 3 & -2 & 1 \end{vmatrix} = 4 \neq 0$$

$$D_1 = \begin{vmatrix} 1 & -2 & 2 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{vmatrix} = 0 \qquad x_1 = \frac{D_1}{D} = 0$$

$$D_2 = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{vmatrix} = -2 \qquad x_2 = \frac{D_2}{D} = -\frac{1}{2}$$

$$D_3 = \begin{vmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 3 & -2 & 1 \end{vmatrix} = 0 \qquad x_3 = \frac{D_3}{D} = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/2 \\ 0 \end{bmatrix}$$

3 x 3 systems

If we can find an inverse of a 3x3 matrix, we can solve 3x3 (or higher) systems:

$$\begin{aligned} x_1 \quad -2x_2 \quad +2x_3 &= 1 \\ x_1 \quad -2x_2 \quad +x_3 &= 1 \\ 3x_1 \quad -2x_2 \quad +x_3 &= 1 \end{aligned}$$

Put the equations in matrix form:

$$\begin{pmatrix} 1 & -2 & 2 \\ 1 & -2 & 1 \\ 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Multiply by the inverse *from the left* :

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 2 \\ 1 & -2 & 1 \\ 3 & -2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Finding the inverse of a 3 x 3 matrix

The inverse can be found by using a series of row operations.
Any system of equations is unchanged by:

- Multiplying a row by a scalar
- Adding a multiple of one row to another
- Swapping any two rows

$$\begin{array}{llll} x_1 & -2x_2 & +2x_3 & = & 1 \\ x_1 & -2x_2 & +x_3 & = & 1 \\ 3x_1 & -2x_2 & +x_3 & = & 1 \end{array}$$

Steps to find the inverse of an n x n matrix:

1. “Augment” the matrix:

$$\begin{pmatrix} 1 & -2 & 2 \\ 1 & -2 & 1 \\ 3 & -2 & 1 \end{pmatrix} \Longrightarrow \left(\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 3 & -2 & 1 & 0 & 0 & 1 \end{array}\right)$$

2. Use row operations to transform the left half into the identity. The right half will then be the inverse.

Example: find the inverse of $\begin{pmatrix} 1 & -2 & 2 \\ 1 & -2 & 1 \\ 3 & -2 & 1 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 3 & -2 & 1 & 0 & 0 & 1 \end{array}\right)$$

Alternate method for finding the inverse:

The cofactors are the signed determinants of the small matrices.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

1. Compute the matrix of cofactors $\mathbf{C}$

2. *Transpose* this to form the Adjugate matrix Adj($\mathbf{A}$)

3. The inverse is then

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \mathrm{Adj}\ \mathbf{A}$$

Example: Find the inverse of

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 3 \\ 3 & 0 & 1 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \mathrm{Adj}\ \mathbf{A}$$

$$\det \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 3 \\ 3 & 0 & 1 \end{pmatrix} = 2\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 3 & 1 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} =$$

$$= 2(1-0) - (-9) - 6 = 5$$

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 3 \\ 3 & 0 & 1 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \mathrm{Adj}\ \mathbf{A}$$

$$\mathbf{A}^c = \begin{pmatrix} \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 3 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 9 & -3 \\ -1 & -4 & 3 \\ 1 & -6 & 2 \end{pmatrix}$$

$$Adj\,\mathbf{A} = \left[\mathbf{A}^c\right]^T = \begin{pmatrix} 1 & -1 & 1 \\ 9 & -4 & -6 \\ -3 & 3 & 2 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{5} \begin{pmatrix} 1 & -1 & 1 \\ 9 & -4 & -6 \\ -3 & 3 & 2 \end{pmatrix}$$

The determinant of a 2x2 matrix is written

$$\Delta = ad - bc$$

If Δ=0, there is *no unique solution* to the equations $\mathbf{A}\,\mathbf{x} = \mathbf{b}$.

E.g. Consider the system

$$\begin{array}{ccccc} x_1 & + & x_2 & = & 5 \\ x_1 & + & x_2 & = & 1 \end{array}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

Definition:

A linear 3 by 3 system with zeros *below* the main diagonal is said to be in row-echelon form.

This can be solved using *back-substitution.*

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 11 \\ 0 & -3 & 1 & 25 \\ 0 & 0 & 2 & 70 \end{array}\right]$$

A matrix can be put into row-echelon form using *Row-operations.* This is known as Gauss elimination.

Example: For the system

$$\begin{aligned} -x_1 + x_2 + 2x_3 &= 2 \\ 3x_1 - x_2 + x_3 &= 6 \\ -x_1 + 3x_2 + 4x_3 &= 4 \end{aligned}$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 3 & -1 & 1 & 6 \\ -1 & 3 & 4 & 4 \end{array}\right]$$

Any system of equations is unchanged by:

- Multiplying a row by a scalar
- Adding a multiple of one row to another
- Swapping any two rows

Under/overspecification of a problem

Example: Does this system of equations have a (unique) solution?

$$
\begin{aligned}
2x_1 \quad &+ x_2 & &= 1 \\
x_1 \quad &+ 2x_2 &+ x_3 &= -1 \\
&+ 3x_2 &+ 2x_3 &= 1
\end{aligned}
$$

We might think so, but here the situation is more complicated than simply having three equations and three unknowns.

Under/overspecification of a problem

Example: Does this system of equations have a (unique) solution?

$$
\begin{aligned}
2x_1 \quad +x_2 \qquad\qquad &= \ 1 \\
x_1 \quad +2x_2 \quad +x_3 &= -1 \\
+3x_2 \quad +2x_3 &= -3
\end{aligned}
$$

We might think so, but here the situation is more complicated than simply having three equations and three unknowns.