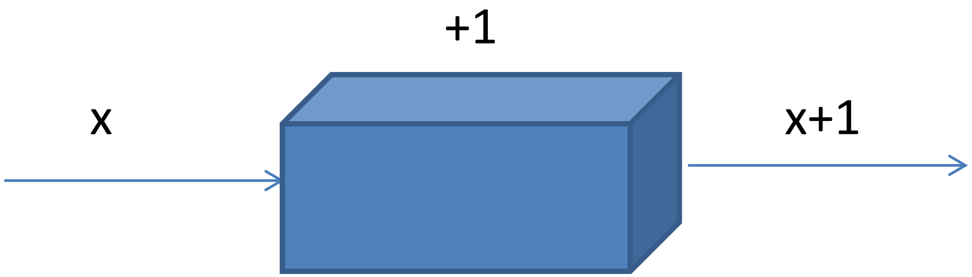
Functions and Calculus
(review)

A <u>function of one variable</u> $f(x)$ is a *rule* that transforms one number x into another number $f(x)$.

A function has an input, or *argument*, and a single output, or *value*.

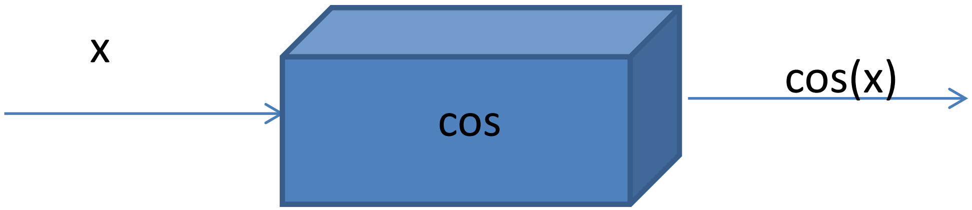


$1 * 2 * 3 \ldots * n = n!$

Product of all integers from 1 to n
Factorial function denoted as   n!
0!=1

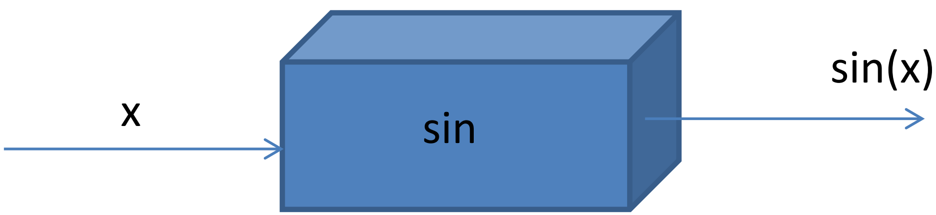
Functions tell us how *one thing depends on another*.

A <u>function of one variable</u> $f(x)$ is a *rule* that transforms one number x into another number $f(x)$.



$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$$

Taylor series



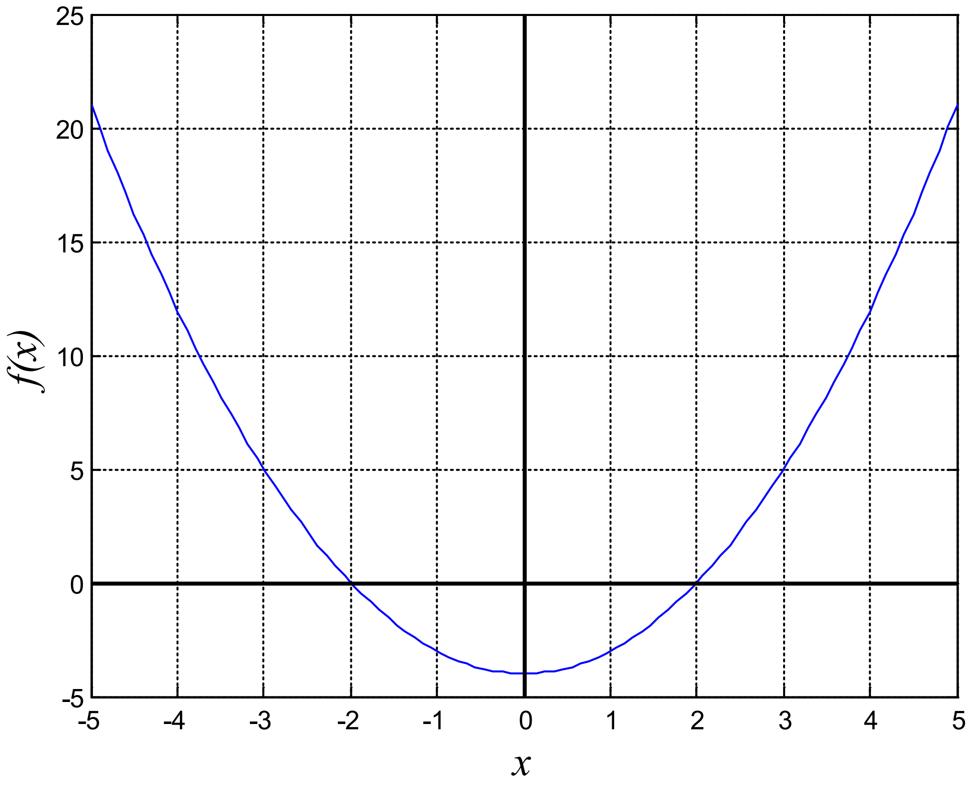
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$$

Functions can be *represented* in a number of different ways.

1. Algebraically

$$f(x) = x^2 - 3$$

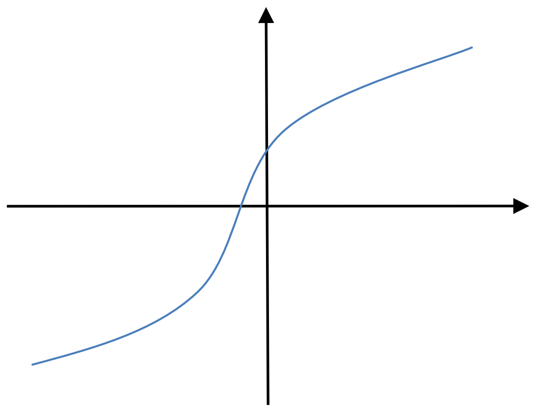
2. As a *graph*



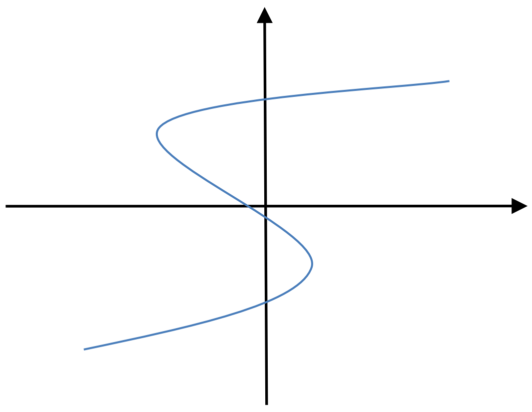
3. As a *table*

x	f(x)
-1.0	-2.0
-0.5	-2.75
0.0	-3.0
0.5	-2.75
1.0	-2.0
1.5	-0.75
2.0	1.0

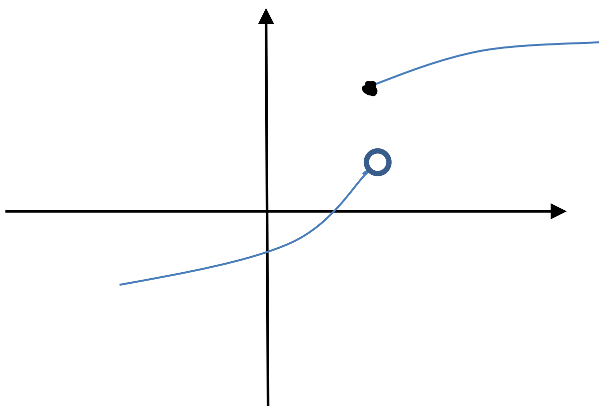
Most important property of any function: *For each x there is only one f(x)*



function



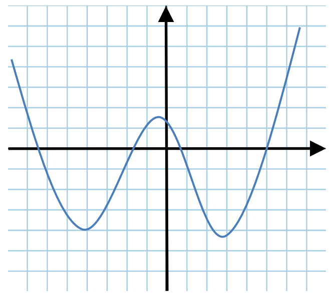
not a function



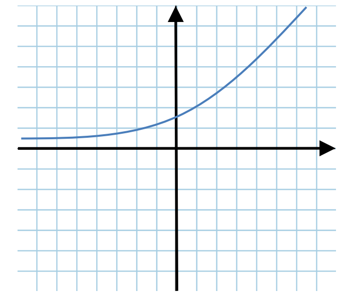
function

The basic functions used in mathematical modelling

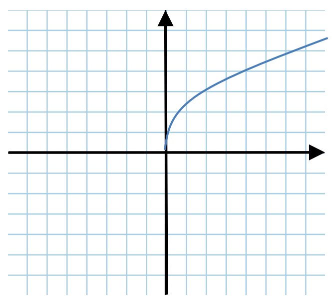
Polynomials



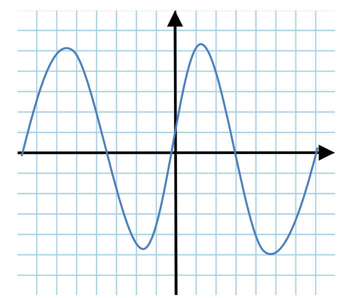
Exponential
functions



Power functions

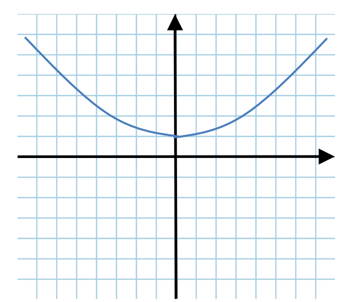


Trigonometric
functions



Rational functions-
Ratio of polynomials

Hyperbolic
functions

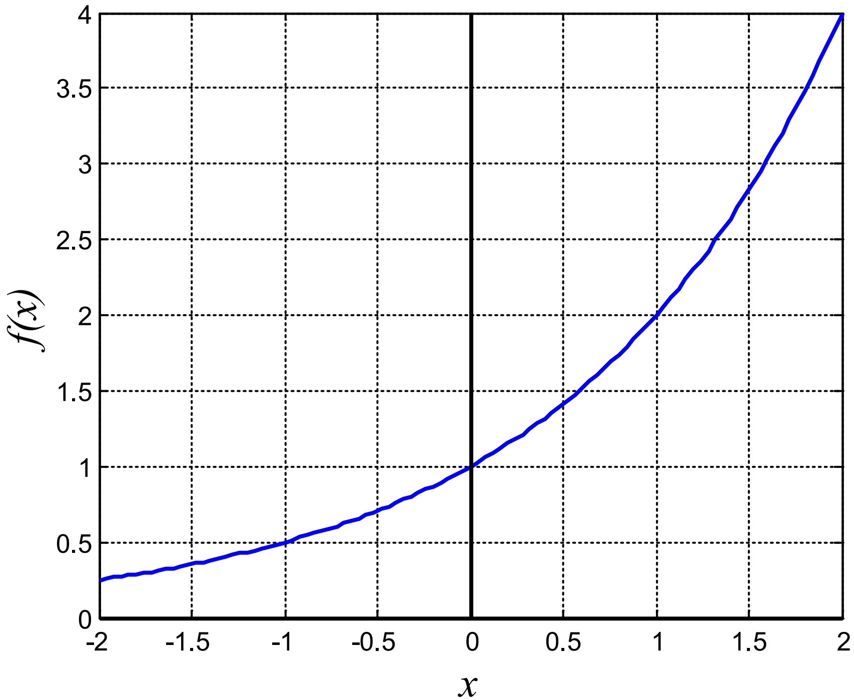


...plus inverse functions of all of these.
Anything else is called a Special Function.

Exponential functions have the general form

$$f(x) = a^x$$

where $a>0$ is a constant.



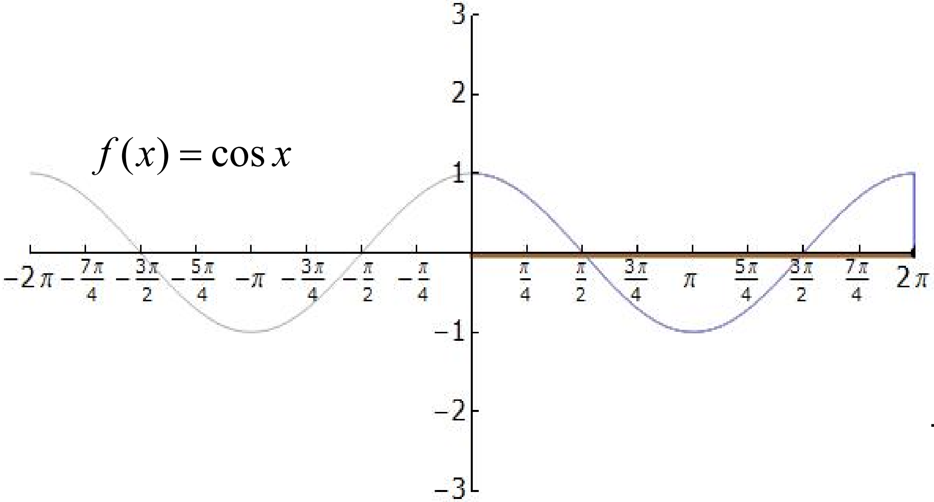
Properties:

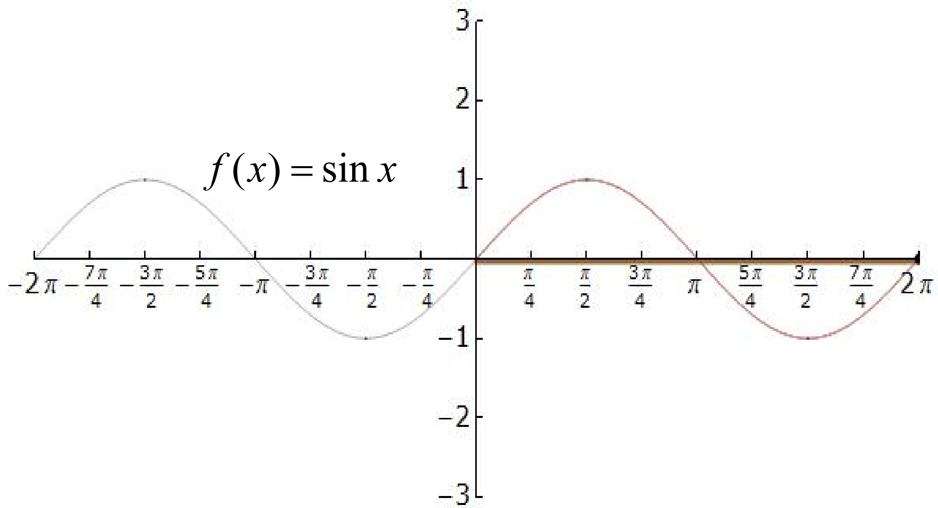
1. Always grow in one direction, decay in the other
2. Never cross the x-axis
3. Look like straight lines on a logarithmic scale

Special case: when a = 2.718281828459045. .. = e , we obtain *the* exponential function

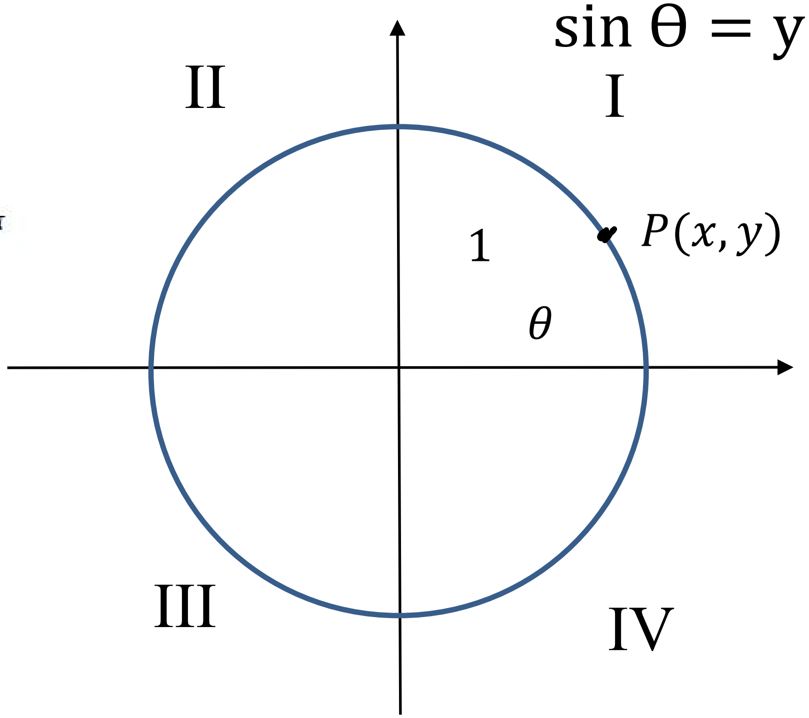
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

The trigonometric functions



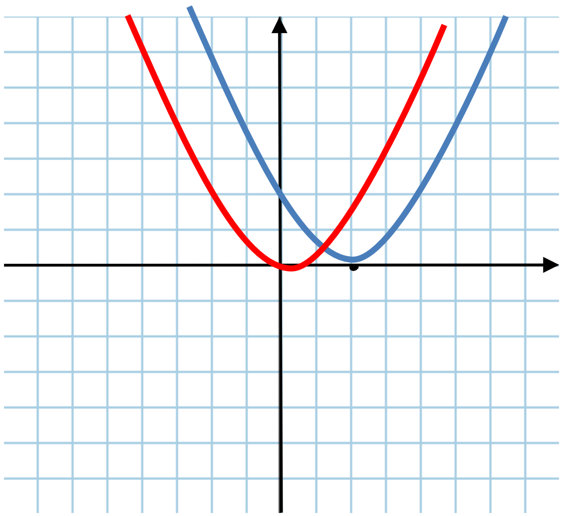


$\cos \theta = x$
$\sin \theta = y$



Building new functions:

Replacing x with $x\text{-}a$ shifts the function $a>0$ units to the right.

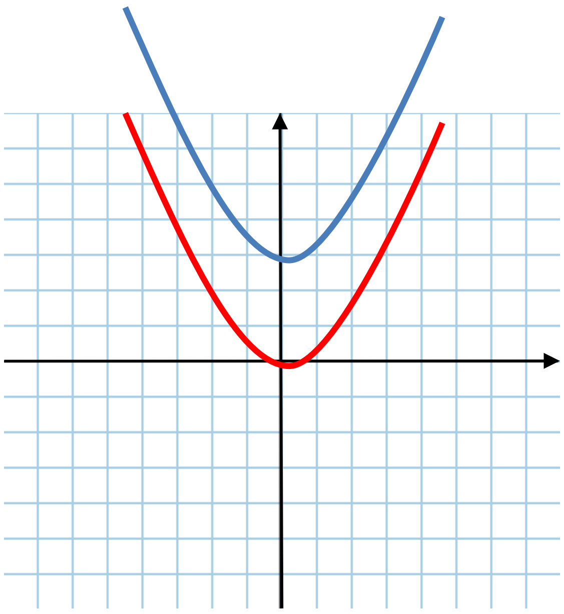


E.g.

$$f(x)=(x-2)^2$$

is the same as $f(x)=x^2$, but shifted to the right by 2.

Adding a constant a shifts the function vertically:



$$f(x)=x^2+3$$

The hyperbolic functions

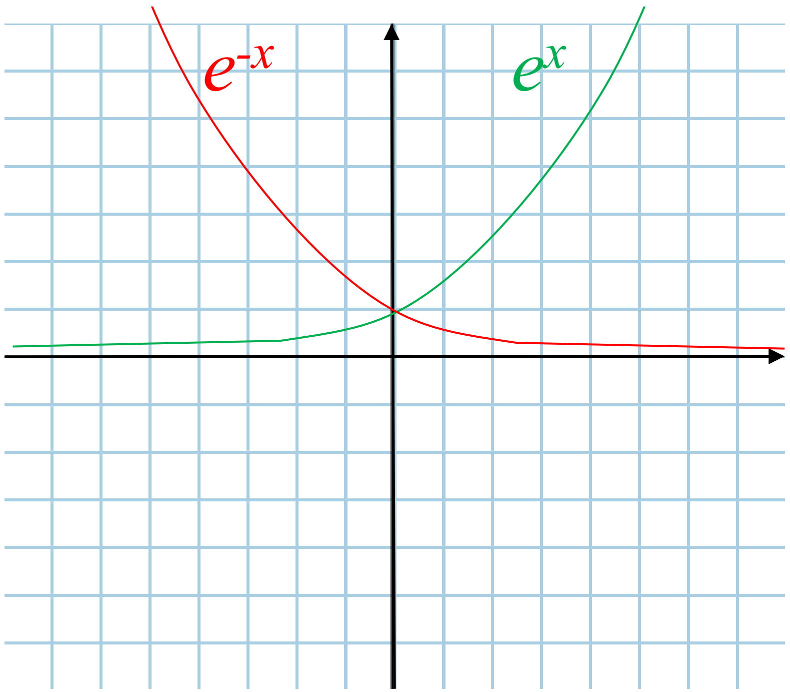
The two hyperbolic functions are defined as

$$\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

$$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

Like sine and cosine, cosh and sinh are *related:*

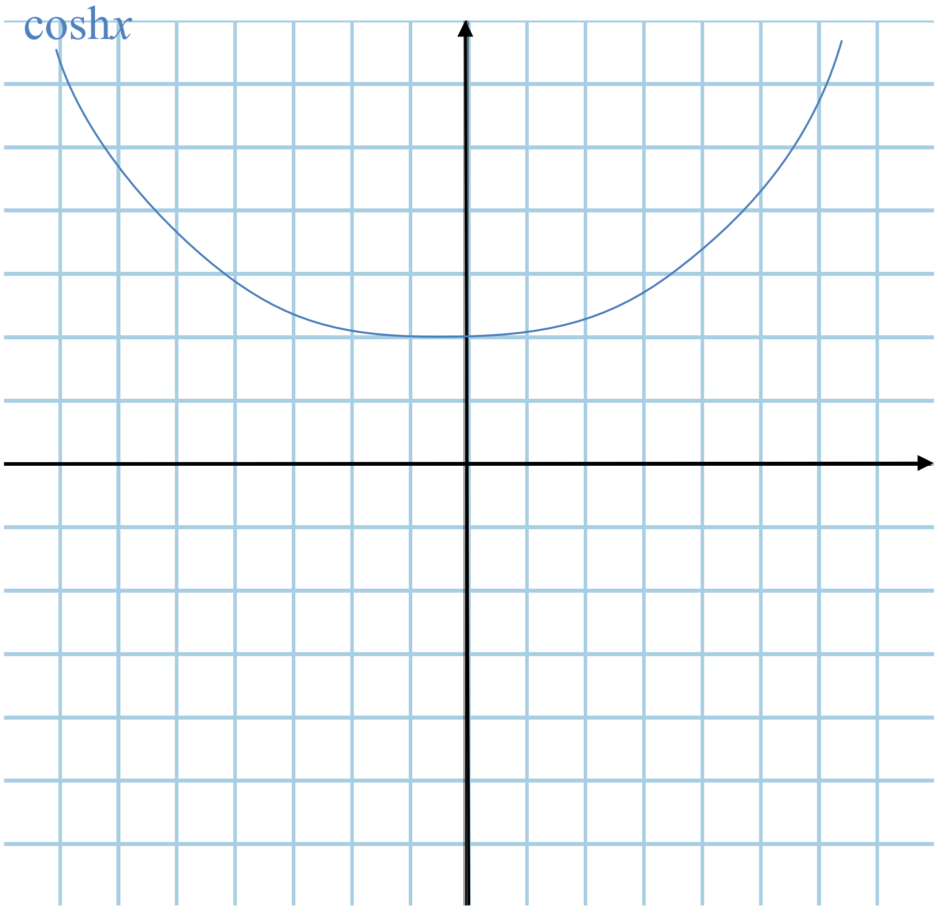
$$\cosh^2 x - \sinh^2 x = 1$$



The function $\cosh x$ is known as the *catenary (chain hanging freely from two points)* function

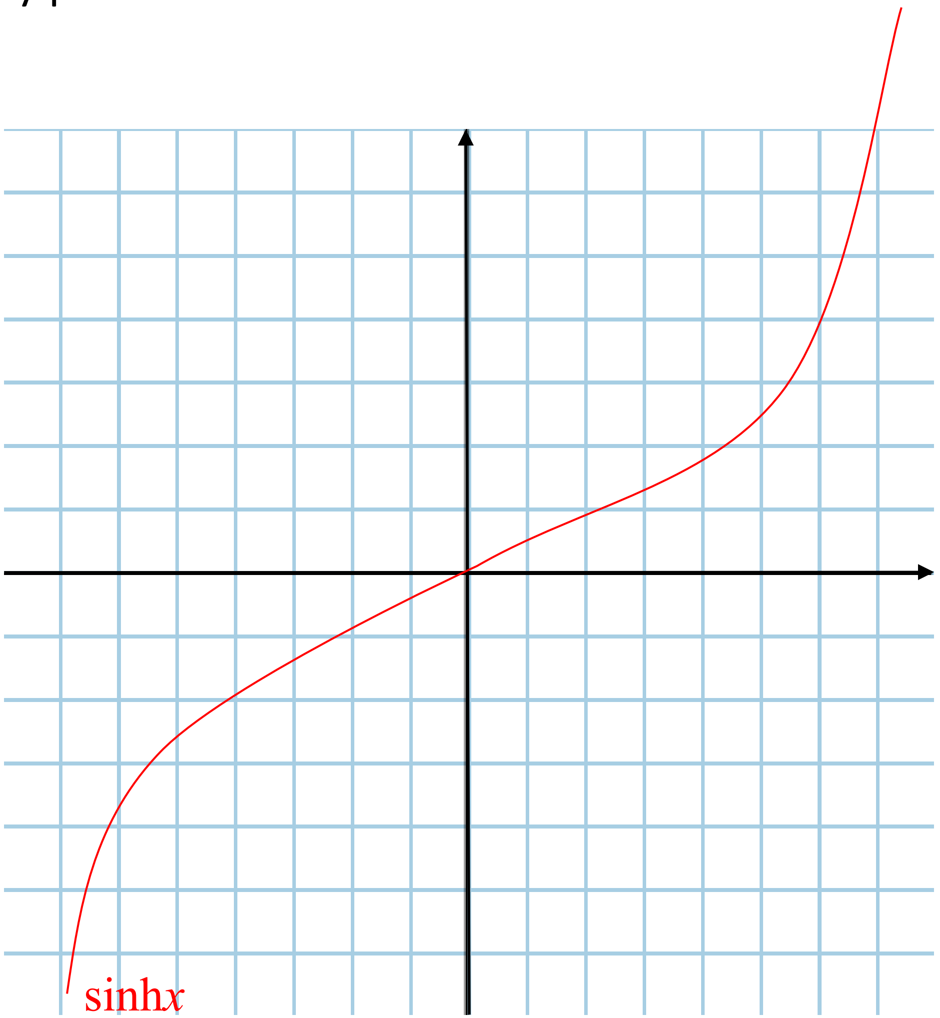
$$\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

$$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$



The “hyperbolic sine” function is usually pronounced “shine”.

Slippery dip

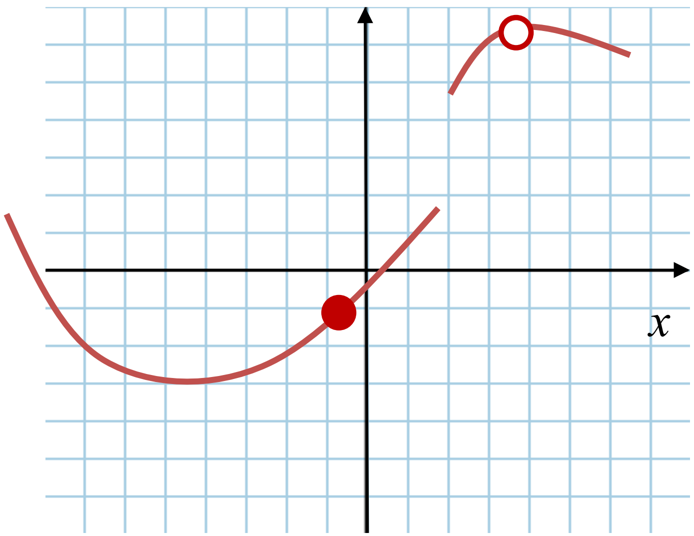


Limits

The statement

$$\lim_{x \to a} f(x) = L$$

means that *f(x) can be made arbitrarily close to the number L by taking x towards* a.



A function is <u>continuous</u> at the point $x = a$ if

$$\lim_{x \to a} f(x) = f(a)$$

We think of $\lim_{x \to a} f(x)$ as being *the number that f approaches as x gets closer and closer to* a

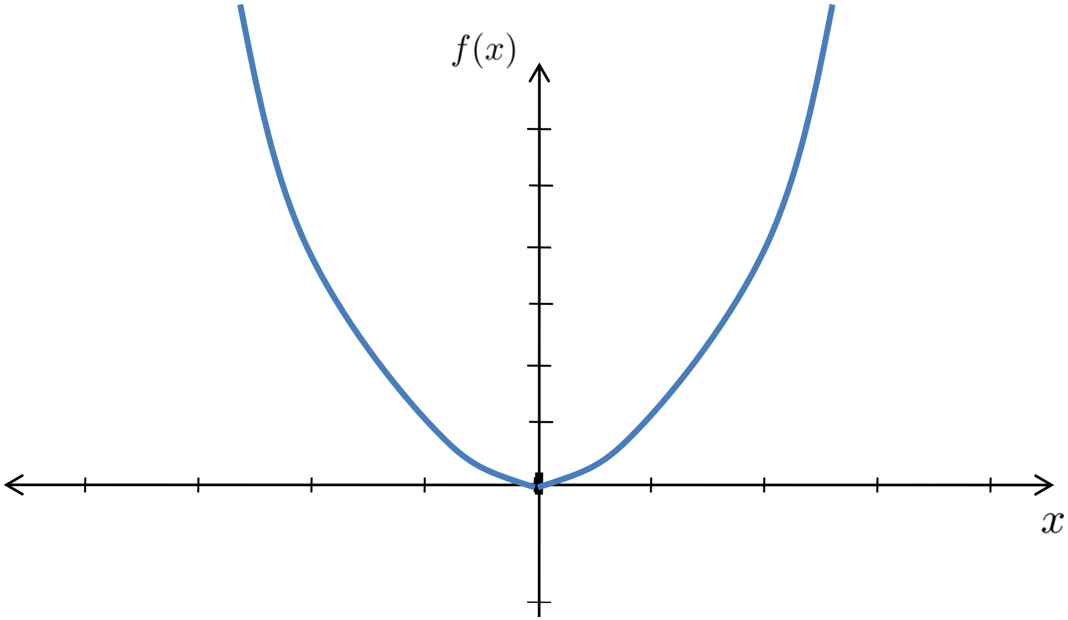
Dealing with limits

To evaluate a limit of a continuous function, you can substitute directly.
i.e. To find

$$\lim_{x \to x_0} f(x)$$

you can substitute x_0 into f(x).

Example:



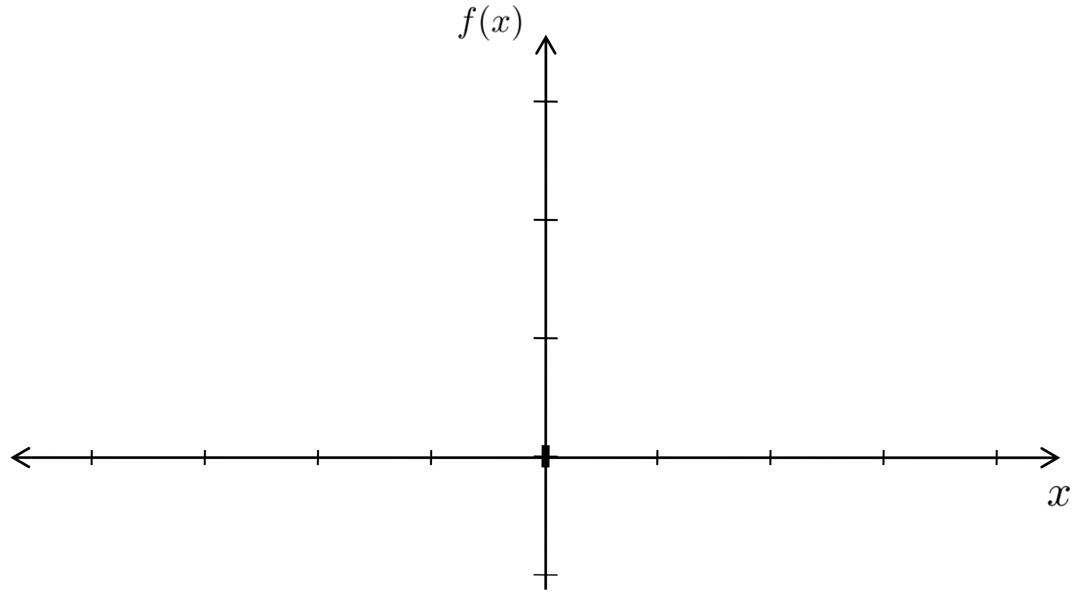
$$\lim_{x \to 2} x^2$$

$$\lim_{x \to 3} (x^3 - 1)$$

Note that the limit *does not depend on the variable x!*

Sometimes a limit will be ***different*** when approached from different sides. E.g.:

$$f(x) = \begin{cases} 2 & \text{for } x > 0 \\ -1 & \text{for } x \leq 0 \end{cases}$$



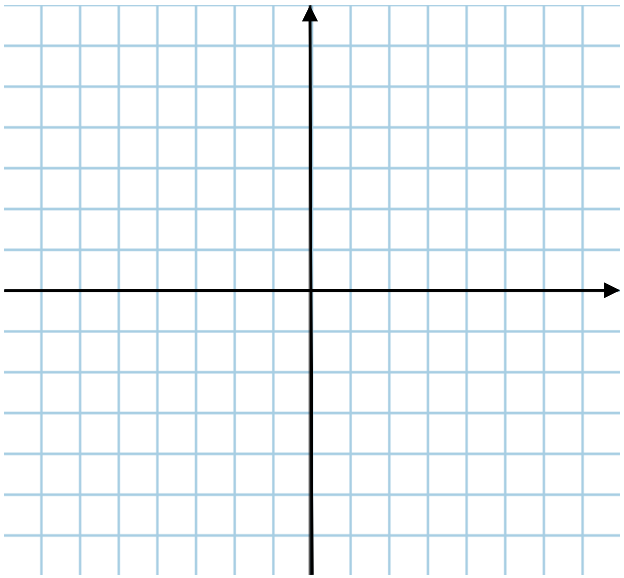
$$\lim_{x \to 0^+} f(x)$$

$$\lim_{x \to 0^-} f(x)$$

Sometimes a limit will *exist* even if the function is not continuous.

E.g.

$$f(x) = \frac{\sin x}{x}$$



$$\lim_{x \to 0} f(x) =$$

Dealing with limits: the limit laws

For any two real functions f and g such that the relevant limits exist, we can prove:

$$\lim_{x \to x_0} (f(x) + g(x)) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x)$$

$$\lim_{x \to x_0} (f(x)g(x)) = \lim_{x \to x_0} f(x) \lim_{x \to x_0} g(x)$$

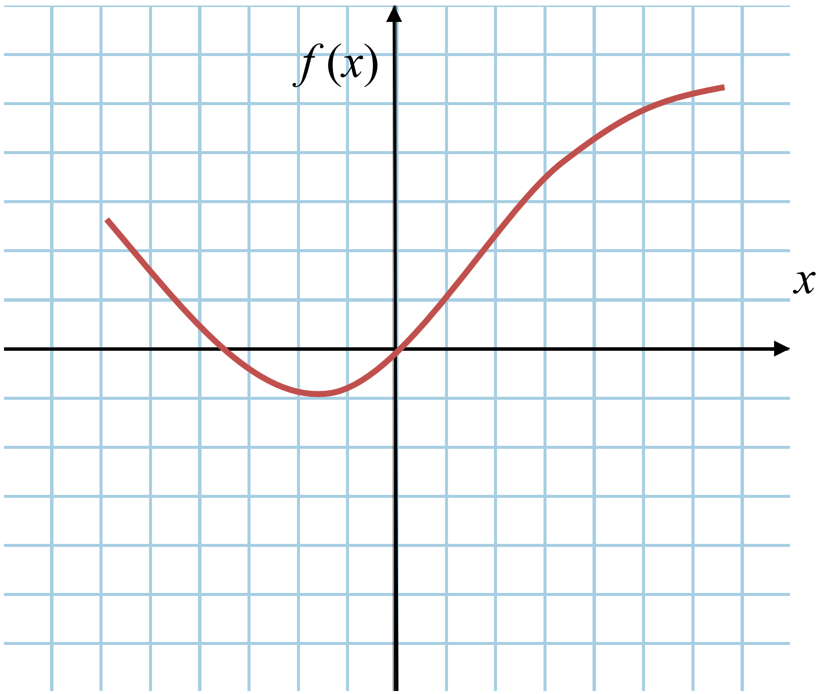
$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)}$$ (when $\lim_{x \to x_0} g(x) \neq 0$)

That is: limits obey the regular rules of algebra

$$\lim_{x \to 1} \frac{x^2 + 2}{x^3 - 3} =$$

Derivatives

The derivative of a function is the rate of change of the function at a particular point:



$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{x+h-x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \frac{df}{dx}$$

$\frac{\Delta f}{\Delta x}$ -Rate of change; Slope of the tangent

All differentiable functions are continuous, <u>but</u> not all continuous functions are differentiable.

Differentiation rules

For differentiable functions f(x) and g(x) we have the following rules:

1. Linearity

$$\frac{d}{dx}(af+bg)=a\frac{df}{dx}+b\frac{dg}{dx}$$

2. The product rule

$$\frac{d}{dx}(fg)=f\frac{dg}{dx}+g\frac{df}{dx}$$

3. The quotient rule

$$\frac{d}{dx}\left(\frac{f}{g}\right)=\frac{g\frac{df}{dx}-f\frac{dg}{dx}}{g^2}$$

4. The chain rule

$$\frac{d}{dx}f(g(x))=\frac{df}{dg}\frac{dg}{dx}$$

Using the product rule, we can show that

$$\frac{d}{dx}x^n=nx^{n-1}$$

This means that we can differentiate any polynomial function. E.g.

$$\frac{d}{dx}\left(2x^4+3x^3+1\right)=$$

The chain rule

$$\frac{d}{dx} f\left(g(x)\right) = \frac{df}{dg}\frac{dg}{dx}$$

$$f'(x) = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} = \lim_{h\to 0} \frac{f(g(x+h)) - f\big(g(x)\big)}{g(x+h) - g(x)} \frac{(g(x+h) - g(x))}{h}$$

Chain rule examples: differentiate

$$f(x) = (1-2x^2)^{10}$$

$$f(x) = \sin x^3$$

Higher order derivatives can be computed by taken the derivative twice, three times, etc. They are written

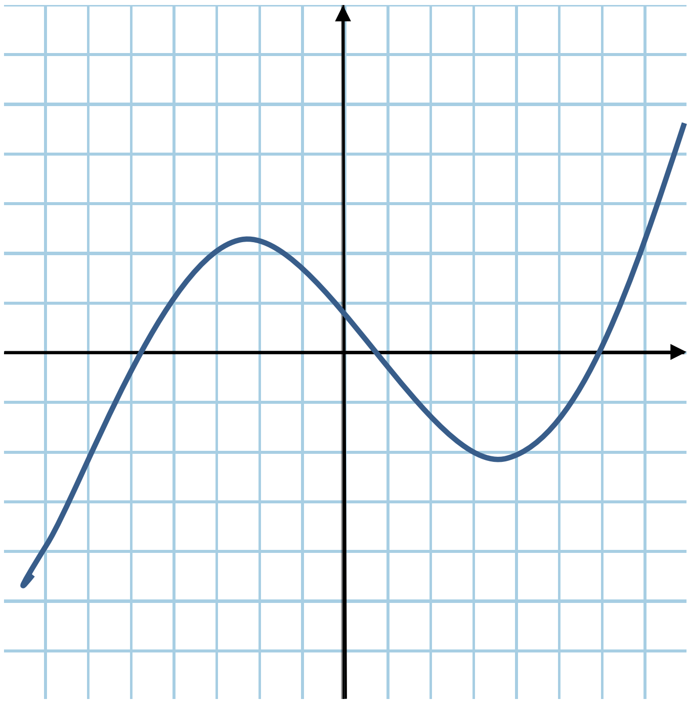
$$\frac{d^2 f}{dx^2} \quad , \quad \frac{d^3 f}{dx^3} \qquad \text{or} \qquad f''(x) \qquad f^{(3)}(x)$$

We can use derivatives to find things out about the function:

$f'(x) = 0$ ⇒ Flat (usually a maximum or minimum)

$f''(x) > 0$ ⇒ Concave up

$f''(x) < 0$ ⇒ Concave down



Derivatives of Hyperbolic functions

can be found by applying the definitions of cosh and sinh:

$$\frac{d}{dx}\cosh x =$$

$$\frac{d}{dx}\sinh x =$$

Find the first derivatives of:

a) $f(x) = x^2 \cos x$

b) $f(x) = \tan x$

c) $f(x) = \dfrac{2x}{x^2 + 3}$

d) $f(x) = e^{x^2+2x+1}$

Find the first derivatives of:

$$f(x) = \sin(\cos(x^4 + 1))$$

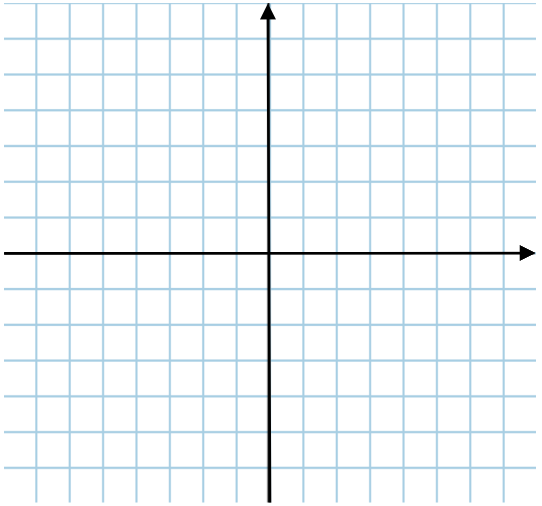
Implicit Functions

A function can be defined explicitly:

$$y = f(x) = x^2 - 2$$

or implicitly: f(x,y)=0

$$x^2 + y^2 = 4$$



The derivative of an implicit function can be found by differentiating the entire expression, then using the chain rule.

Example: Find *dy/dx* given

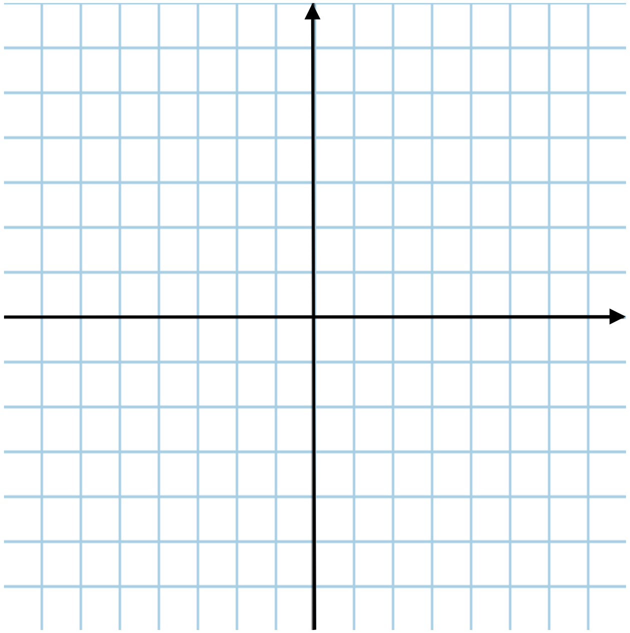
$$x + y(x)\sin x = 1$$

Applications: Using derivatives to find solutions to equations f(x)=0 (Newton's method)

$$ax^2 + bx + c = 0$$

$$x^3 + px + q = 0$$

(del Ferro, Tartaglia, Cardano, 1545)



$$\cos x = x$$

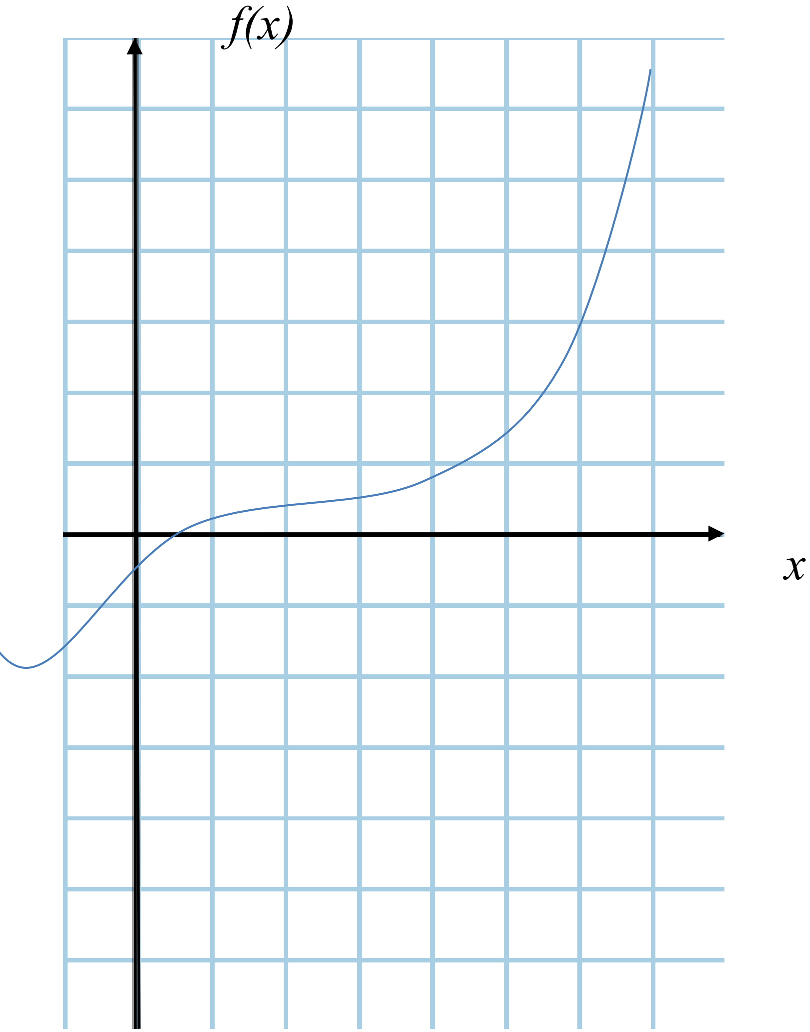
For a function f(x) we can find the solution to the equation

$$f(x) = 0$$

using *Newton's method*.

The idea is to start with an initial guess near the root, and use the derivatives to get a better guess.

Newton's Method



The algorithm:

1. Start with a point x_0

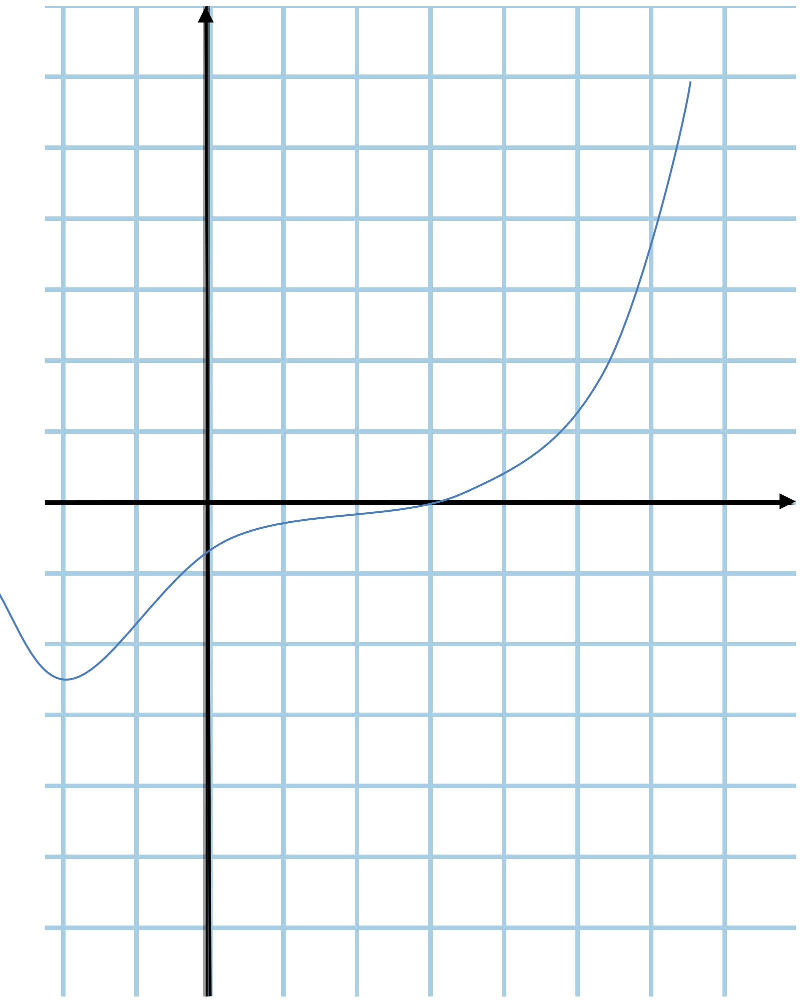
2. Draw a tangent to the curve, and find where this intersects the x axis y=0

3. This point becomes the next best guess. Repeat

$$y - f(x_0) = f'(x_0)(x - x_0)$$

$$0 - f(x_0) = f'(x_0)(x - x_0)$$

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}$$



The algorithm:

1. Start with a point x_0

2. The next point is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

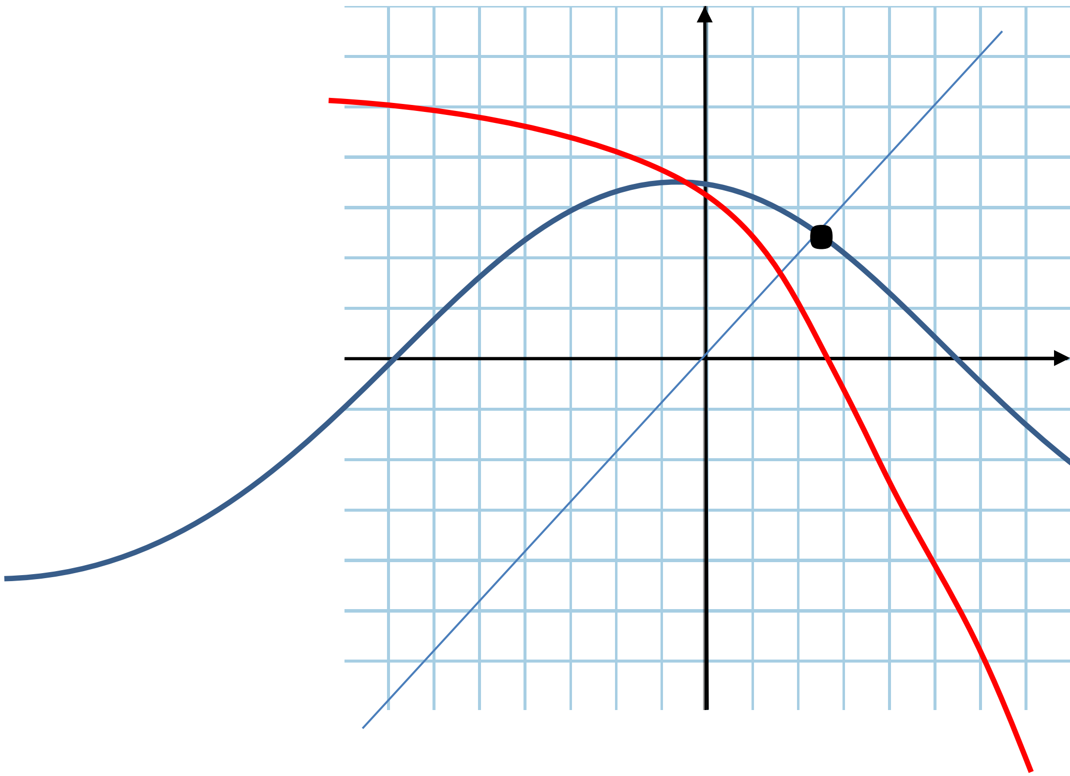
3. Repeat:

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

Example:

Find the number for which $\cos x = x$

$x_0 = \pi/4.$



$$f(x) = \cos x - x$$

$$f(x) = 0$$

$$f'(x) = -\sin x - 1$$

$$f(x) = \cos(\pi/4) - \pi/4 = \sqrt{2}/2 - \pi/4$$

$$f'(x) = -\sin(\pi/4) - 1 = -\sqrt{2}/2 - 1$$

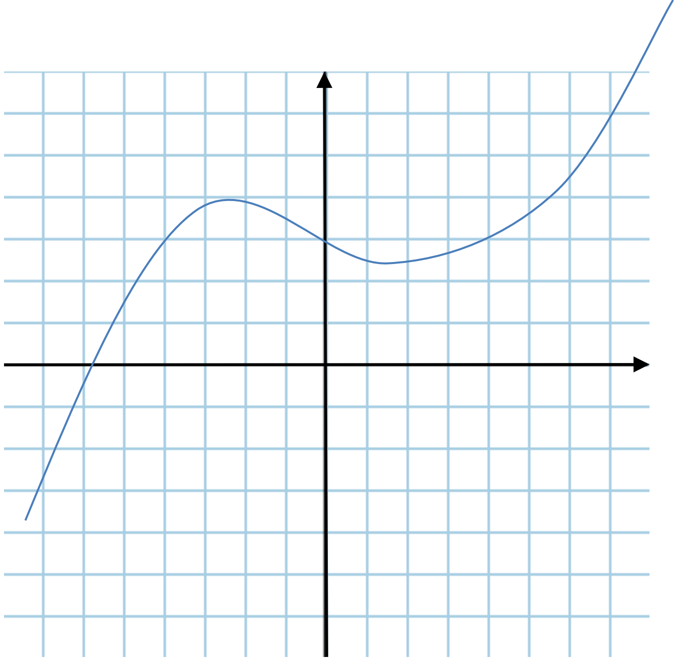
$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

$$n = 1,2,3,\ldots$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \frac{\pi}{4} - \frac{\sqrt{2}/2 - \pi/4}{-\sqrt{2}/2 - 1} = 0.74$$

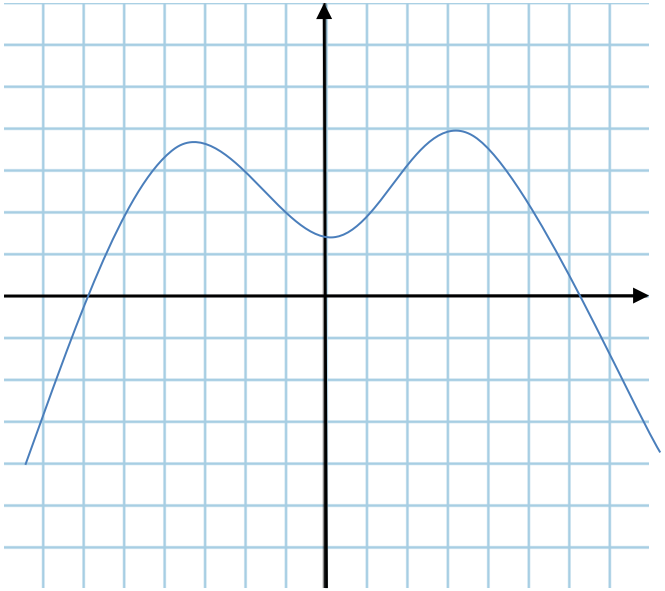
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.74 - \frac{\cos(0.74) - 0.74}{-\sin(0.74) - 1} = 0.739$$

Things that can go wrong
with Newton's method:



1. The derivative can become very small

2. You can land in a "cyclic" situation
   (see tutorial 1 in Sec. additional study
   Questions, Q.3)

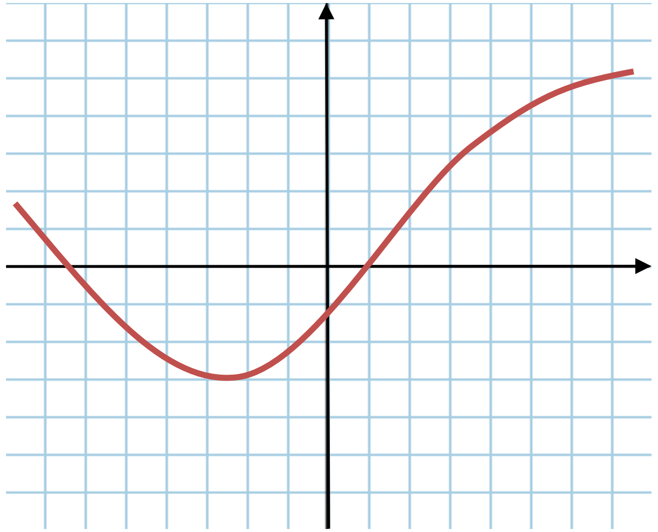


3. Move away from the solution

Applications: Derivatives and approximations

From the definition of derivatives:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$



We can then approximate

$$\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}$$

when Δx is small.

$$\Delta y \approx \frac{dy}{dx} \Delta x$$

This tells us how much y changes for a small change in x.

The main use of this is that it tells us how things *scale* for small changes.

<u>Example:</u> The radius of a sphere increases from 10.0m to 10.1m. What is approximately the increase in volume?

$$\Delta y \approx \frac{dy}{dx} \Delta x$$

1. The length of a rectangle is increasing at a rate of 3 cm/s and its width is increasing at a rate of 7 cm/s. When the length is 15 cm and the width is 7 cm, how fast is the area of the rectangle increasing?

2. Each side of a square is increasing at a rate of 2 cm/s. At what rate is the area of the square increasing when the area of the square is 49 cm^2?

3. Air is pumped into a spherical balloon at the rate of 4 cm^3/second. When the balloon's radius is 20cm, determine the rate of increase of its a) radius, and b) surface area.

4. If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 11 cm. (Give your answer correct to 4 decimal places.)