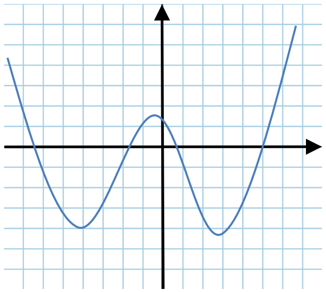
Inverse functions

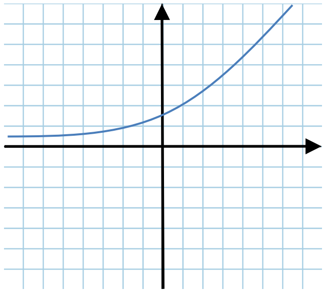
[Textbook: 1.6]

The basic functions that are used in mathematical modelling are:

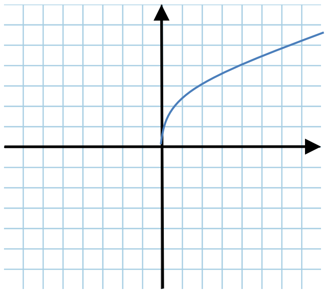
Polynomials



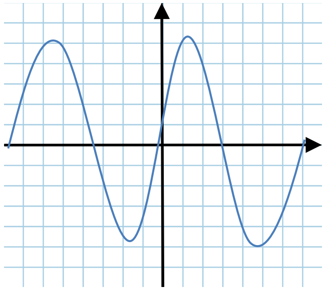
Exponential functions



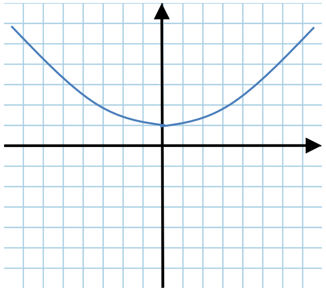
Power functions



Trigonometric functions



Hyperbolic functions

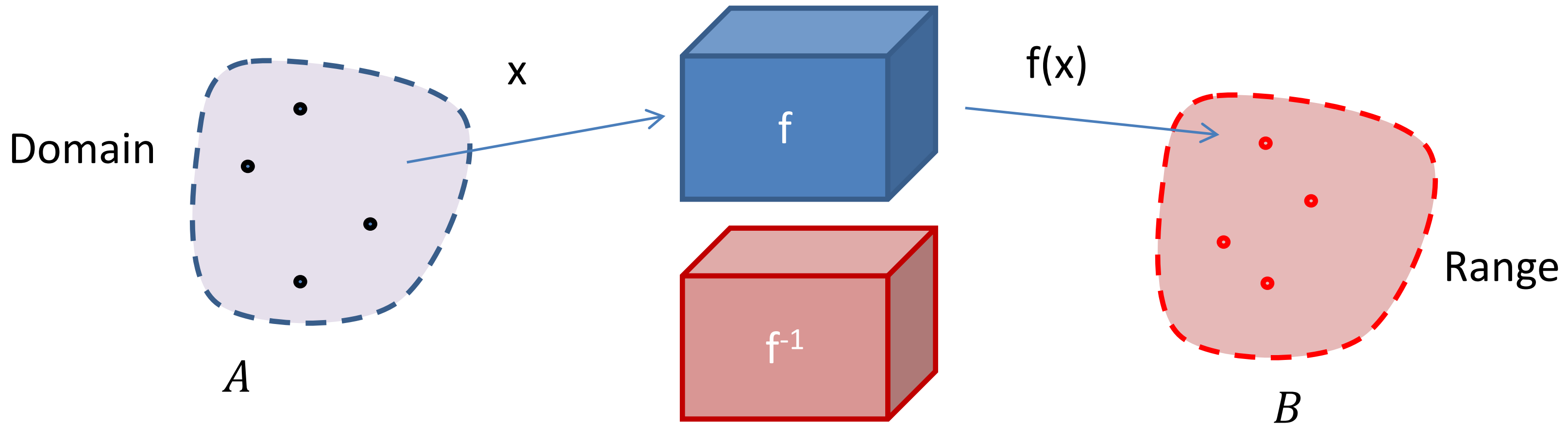


...plus inverse functions of all of these.
Anything else is called a Special Function.

Inverse functions

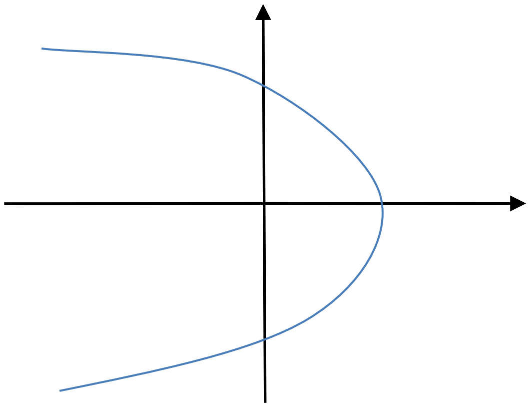
$$f^{-1}(f(x)) = x$$

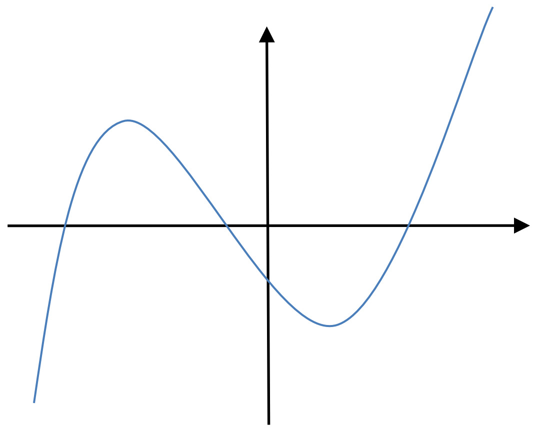
The inverse function "undoes" the action of the function.
It has domain equal to the range of f.

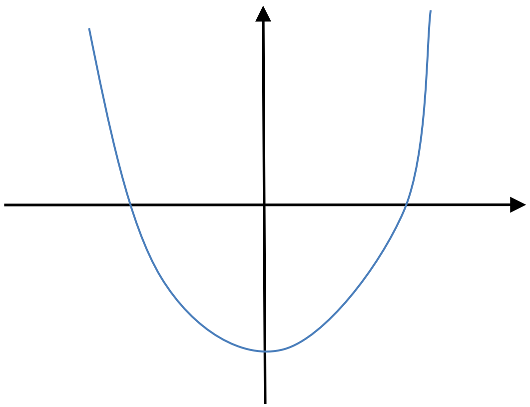


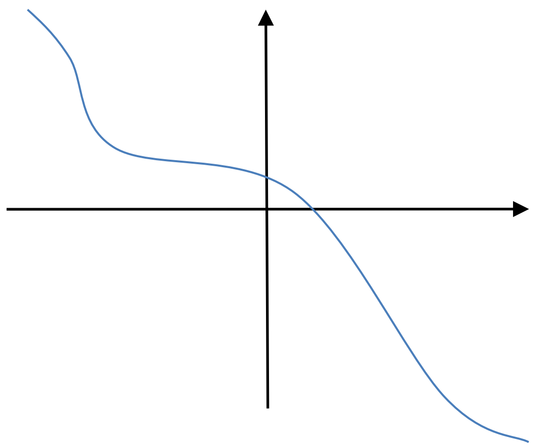
If f is not one-to-one then we cannot find the inverse for the whole range – we may have to leave out some points.

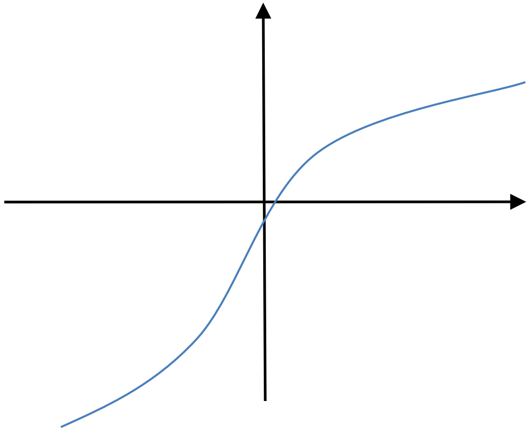
If for each y there is *only one x* then the function is called an **injective,** or **one-to-one** function.

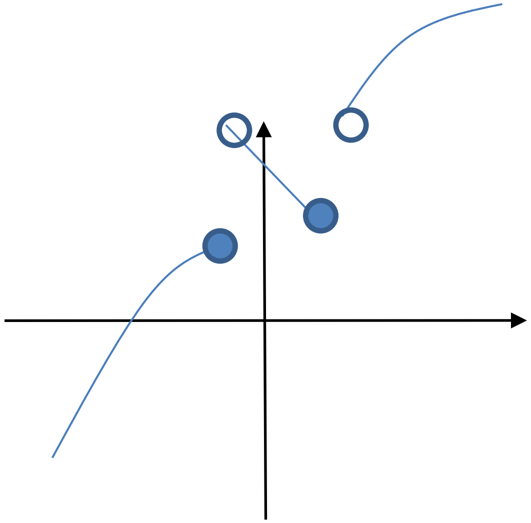












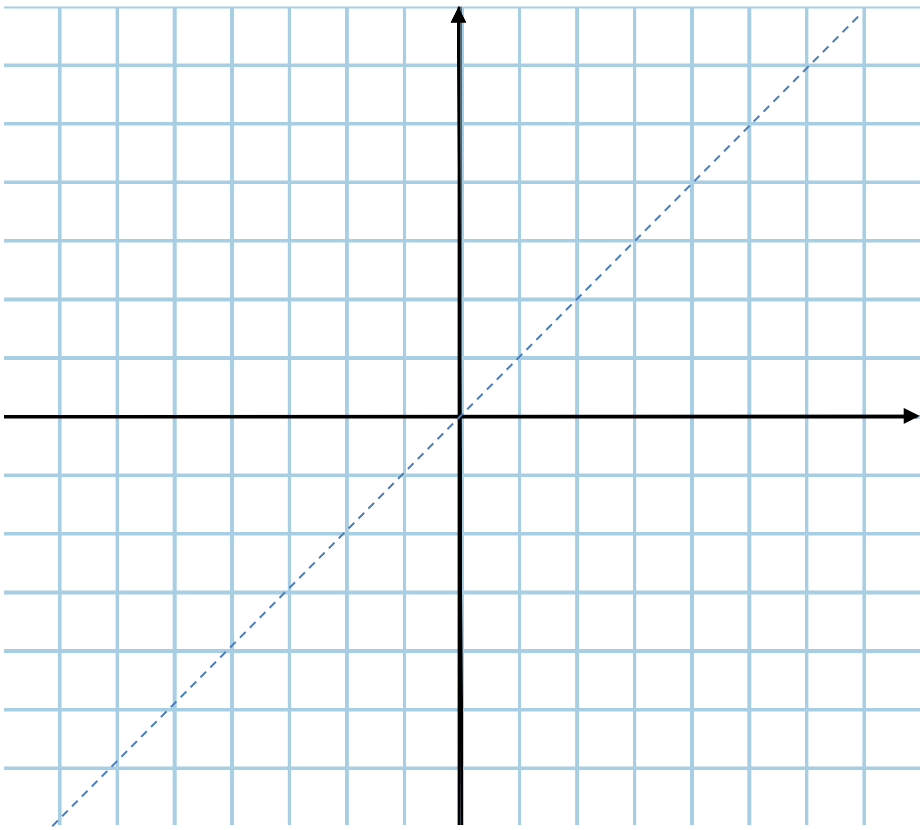
If a function's derivative does not change sign (monotonic function) then it is one-to-one.

However a function can be one-to-one even if this condition is not satisfied.

Finding an inverse function

When a function y = f(x) is represented algebraically, the inverse function is found by *making x the subject of the equation, then swapping x and y.*

Example: Find the inverse of $f(x) = x^3 + 1$

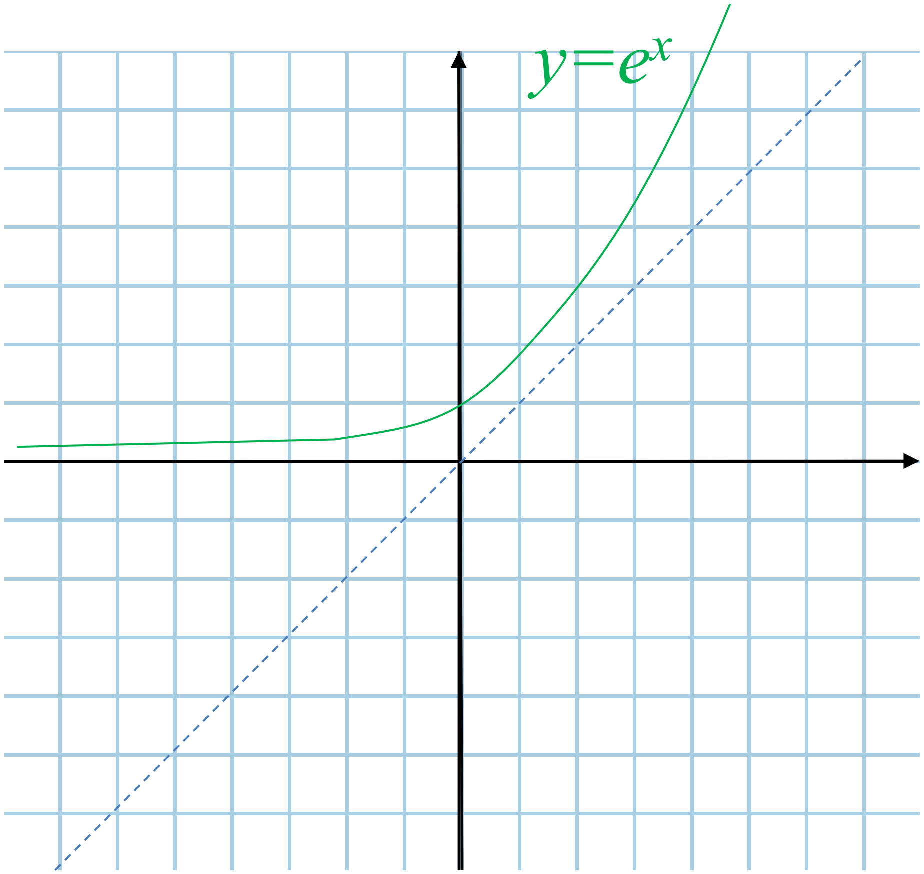


Example: Inverse of exponential function

$$f(x) = e^x$$

Make x the subject:

$$y = e^x$$



Finding an inverse function

When the function is represented as a table: $f(x) = 2^x$

$f(x)$

x	-3	-2	-1	0	1	2	3	4	5
y	1/8	1/4	1/2	1	2	4	8	16	32

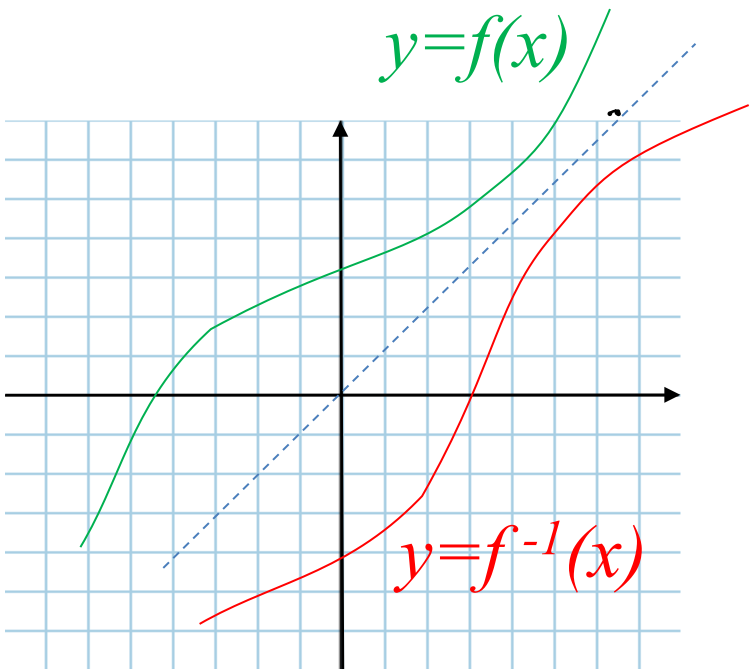
$x \leftrightarrow y$

$f^{-1}(x)$

x	1/8	1/4	1/2	1	2	4	8	16	32
y	-3	-2	-1	0	1	2	3	4	5

Finding an inverse function geometrically

When the function is represented as a graph, we *reflect the curve about the line* *y = x*



The logarithmic function

The logarithm to the base a is

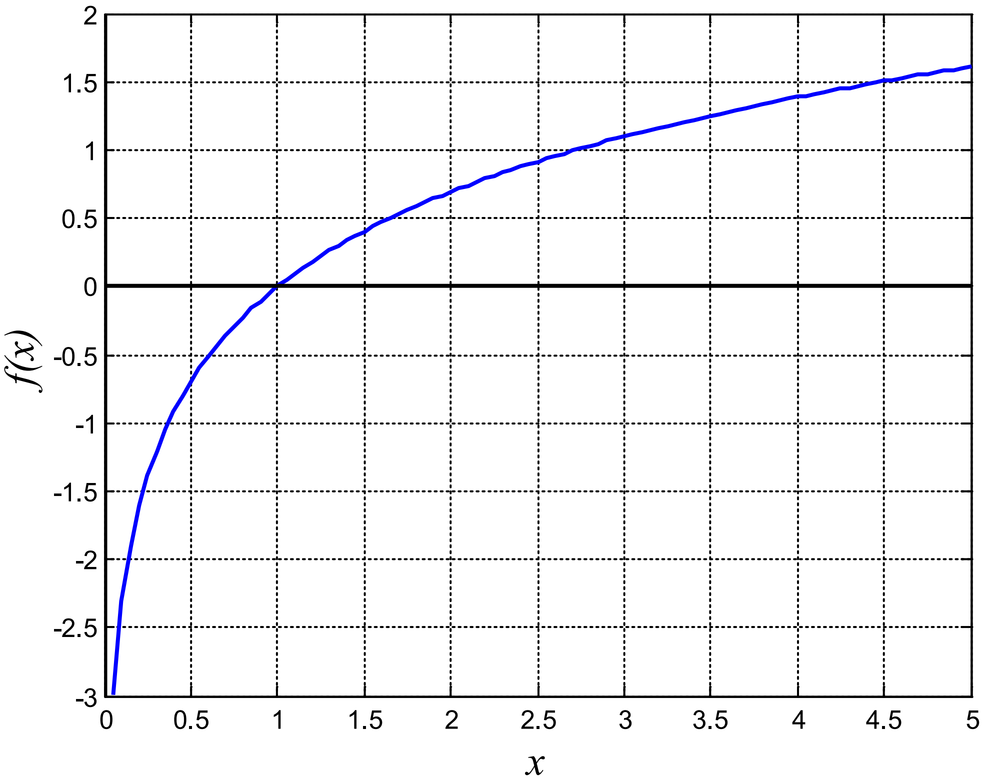
$$f(x) = \log_a x$$

This is the inverse function of

$$g(x) = a^x$$

When a = 2.7182818... = e, we write

$$f(x) = \ln x$$



Inverse trigonometric functions

Cosine is not a one-to-one function.

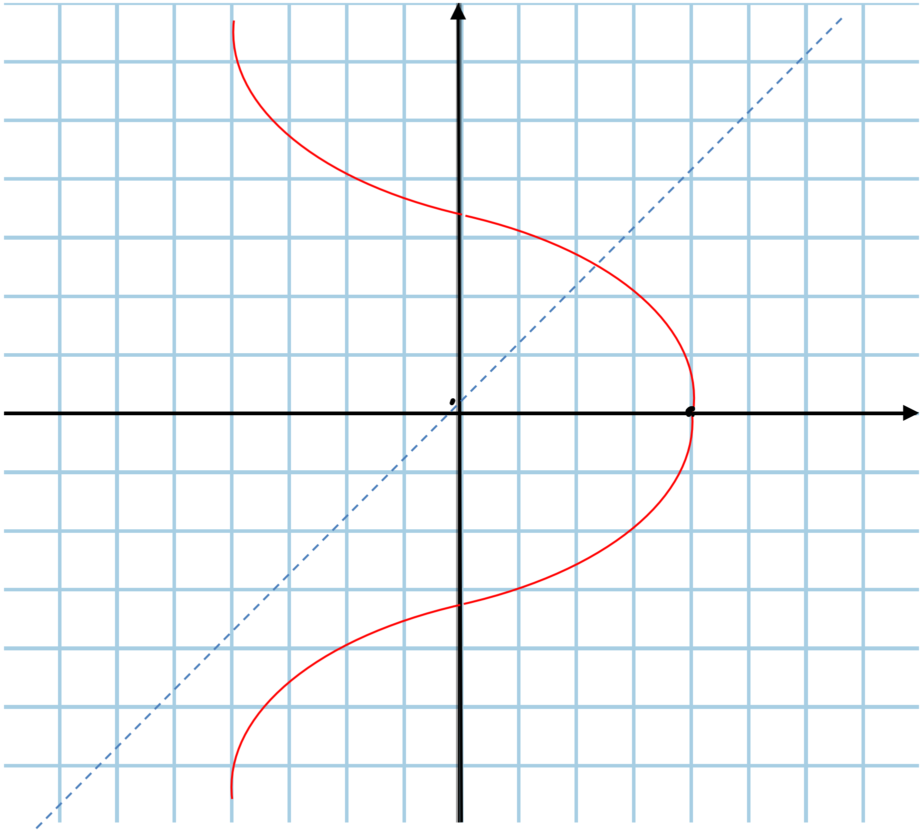
We can define an inverse by restricting the domain of cosine to $0 \le x \le \pi$

The inverse function of this restricted cosine is written

$$f^{-1}(x) = \cos^{-1} x = \arccos x$$

The *domain* of $\cos^{-1} x$ is $-1 \le x \le 1$

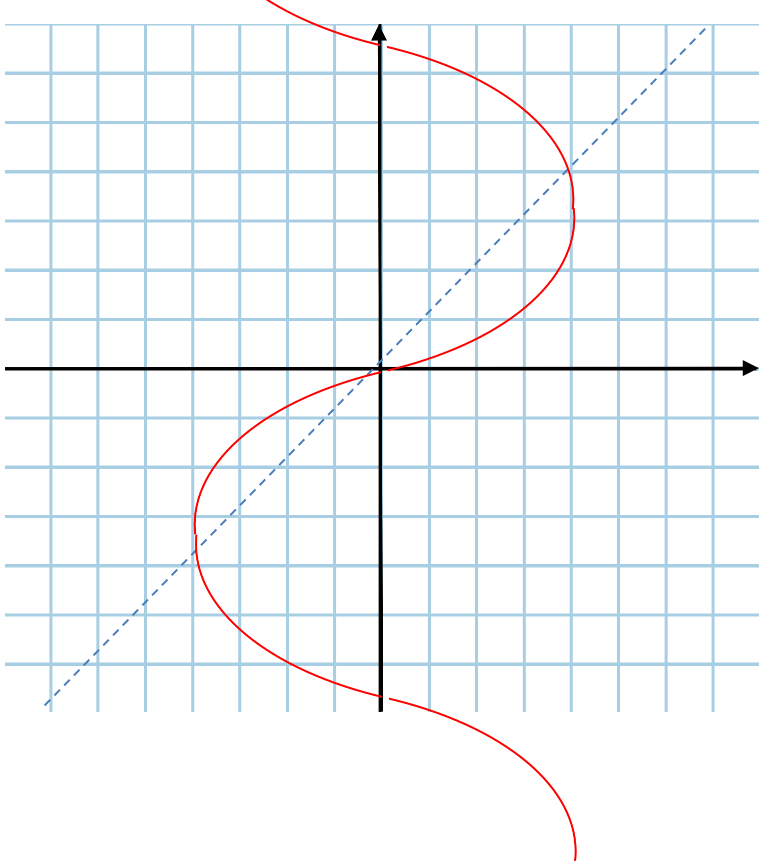
The *range* of $\cos^{-1} x$ is $0 \le \cos^{-1} x \le \pi$



$$\cos x = a$$

To define an inverse for $f(x)=\sin x$,
We restrict the domain to

$$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$$



$$f^{-1}(x) = \sin^{-1} x = \arcsin x$$

The *domain* of $\sin^{-1}$ x is $-1 \le x \le 1$

The *range* of $\sin^{-1}$ x is $-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$

$$\sin x = a$$

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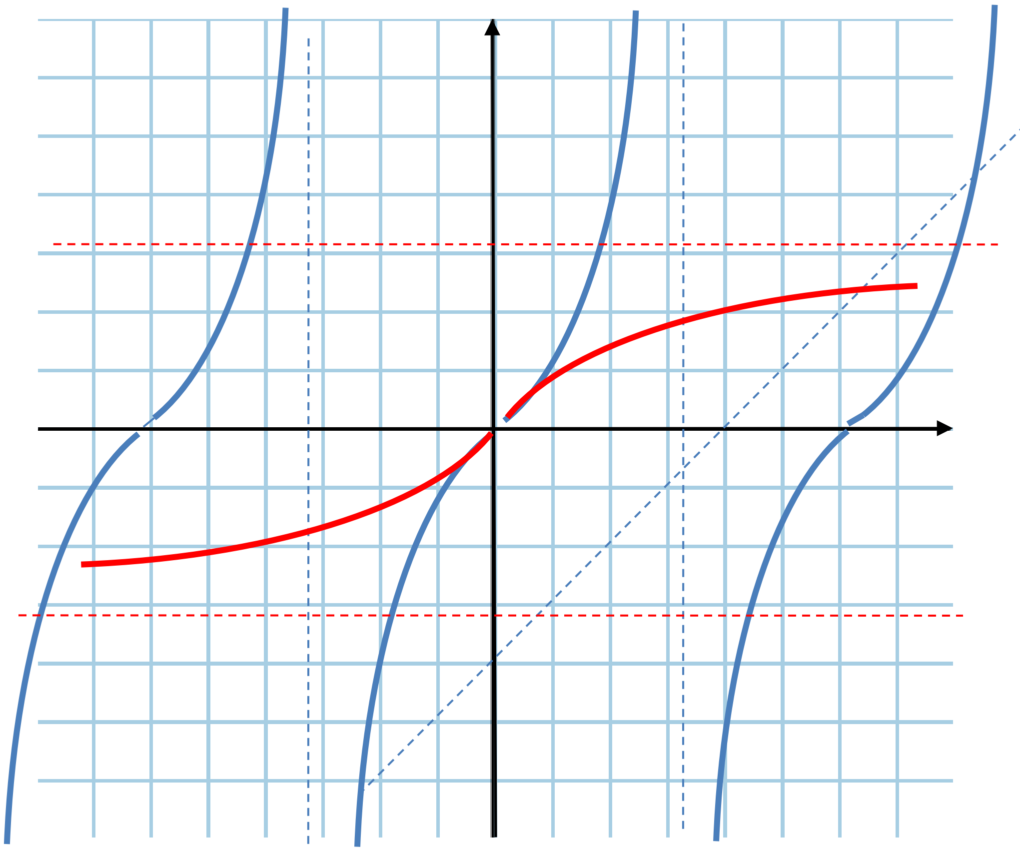
The tangent function is one-to-one in the range $-\pi/2$ to $\pi/2$. The inverse function of $f(x)=\tan x$ is written



The *domain* of $\tan^{-1} x$ is $-\infty < x < \infty$

The *range* of $\tan^{-1} x$ is

$$-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$$

$$\tan x = a$$

$$\sin \sin^{-1} x = x \qquad -1 \le x \le 1$$

$$\sin^{-1} \sin x = x \qquad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$\cos \cos^{-1} x = x \qquad -1 \le x \le 1$$

$$\cos^{-1} \cos x = x \qquad 0 \le x \le \pi$$

$$\sin \sin^{-1} 0.5 = 0.5$$

$$\sin^{-1} \sin \frac{3\pi}{4} =$$

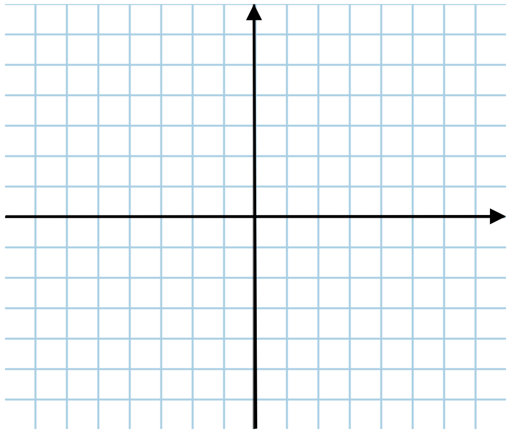
Find

$$\arcsin\left(-\frac{\sqrt{2}}{2}\right) =$$

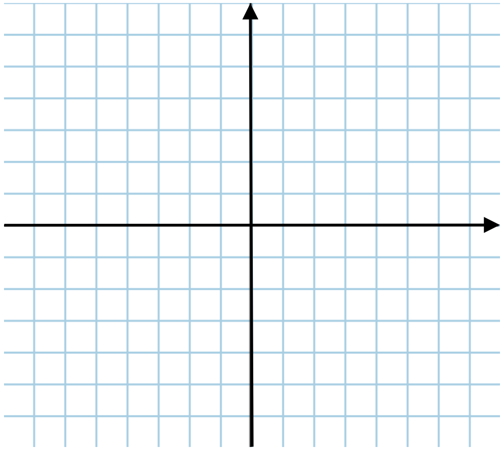
$$\arccos\left(-\frac{1}{2}\right) =$$

We can also define more complicated inverse functions:

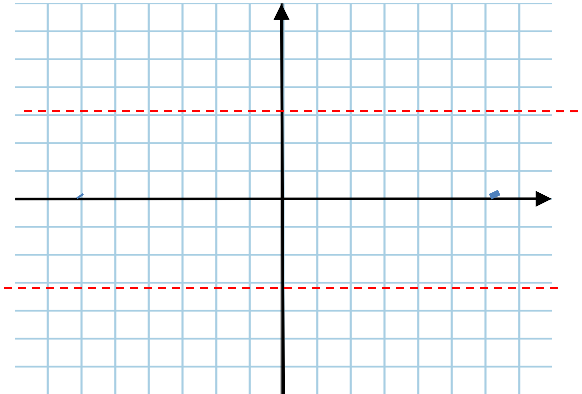
$$f(x) = \sinh^{-1}(x)$$



$$f(x) = \cosh^{-1}(x)$$



$$f(x) = \tanh^{-1}(x)$$



Finding derivatives of the inverse trigonometric functions

$$y = f(x) = \cos^{-1} x$$

Inverse cos x can be defined implicitly by the equation

$$x = \cos y$$

We can then find the derivative using implicit differentiation:

Finding derivatives of the inverse trigonometric function

$$y = \arcsin x$$

Example: compute the derivative of

$$f(x) = \tan^{-1} x$$

The derivative of the logarithmic function is

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

Using the chain rule, we find that

$$\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$$

We can use implicit differentiation to differentiate general exponential functions such as

$$f(x) = 3^x$$

Example:
Differentiate

$$f(x) = 3^{x^2+2}$$

Example: find the derivative of $f(x) = \sinh^{-1}(ax)$

Find the derivative of the function $y = x^x$

Find the derivative of the function $y = x^{\sqrt[3]{x}}$

Find the derivative of the function is $y = f(x)^{g(x)}$