The Integral Calculus

Integration has two seemingly unrelated definitions:

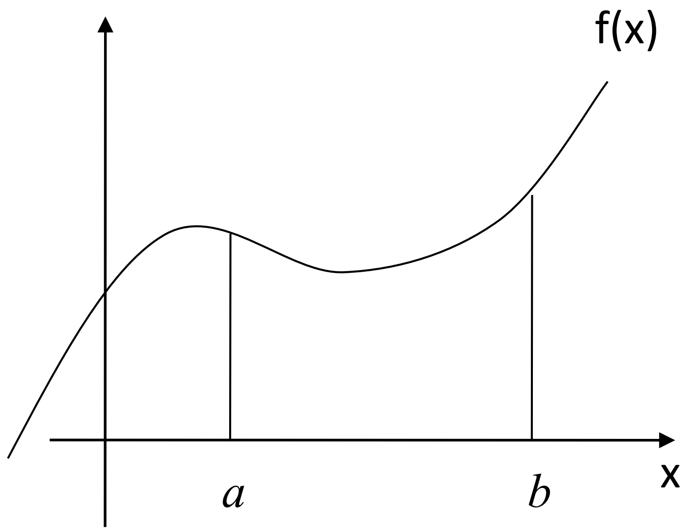
These are useful for:

1. Integration is the *reverse operation to differentiation*

1. Solving Differential equations

2. Integration is *the limit when we sum a large number of small quantities*.

2. Computing *aggregate* quantities



Integrals as anti-derivatives

An *indefinite integral F(x)* of a function *f(x)* is written

$$F(x) = \int f(x)dx$$

and is the *Antiderivative* of *f(x),* i.e. It is the function for which

$$\frac{dF}{dx} = f(x)$$

F(x) is often known as the *primitive function* of *f(x)*

E.g.

Let f(x) = 2x + 1. A function that has f(x) as its derivative is

Let g(x) = k=const. A function that has g(x) as its derivative is

If $F(x)$ *is indefinite integral* of a function $f(x)$ then *G(x)=F(x) + C, where C is a constant is also an indefinite integral of f(x)*

$$G'(x) = \left(F(x) + C\right)' = F'(x) + C' = f(x)$$

$$\int \left(f(x) + g(x)\right) dx = \int f(x)dx + \int g(x)dx$$

$$\int \left(f(x) - g(x)\right) dx = \int f(x)dx - \int g(x)dx$$

$$\int kf(x)dx = k\int f(x)dx$$

$$\int \left(f(x)g(x)\right) dx \neq \int f(x)dx \int g(x)dx$$

$$\int \frac{f(x)}{g(x)} dx \neq \frac{\int f(x)dx}{\int g(x)dx}$$

We find the primitive function F(x) by *reverse differentiation.*

$$\frac{dF}{dx} = f(x)$$

E.g.

$$f(x) = \cos x$$

$$f(x) = \sinh kx$$

$$f(x) = \frac{1}{x}$$

$$\int x \, dx =$$

$$\int x^k \, dx =$$

$$\int \sin ax \, dx =$$

$$\int \cos ax \, \mathrm{d}x =$$

$$\int e^{ax}\, dx =$$

$$\int \cosh ax\, dx =$$

$$\int \sinh ax\, dx =$$

More complicated integrals:

Because

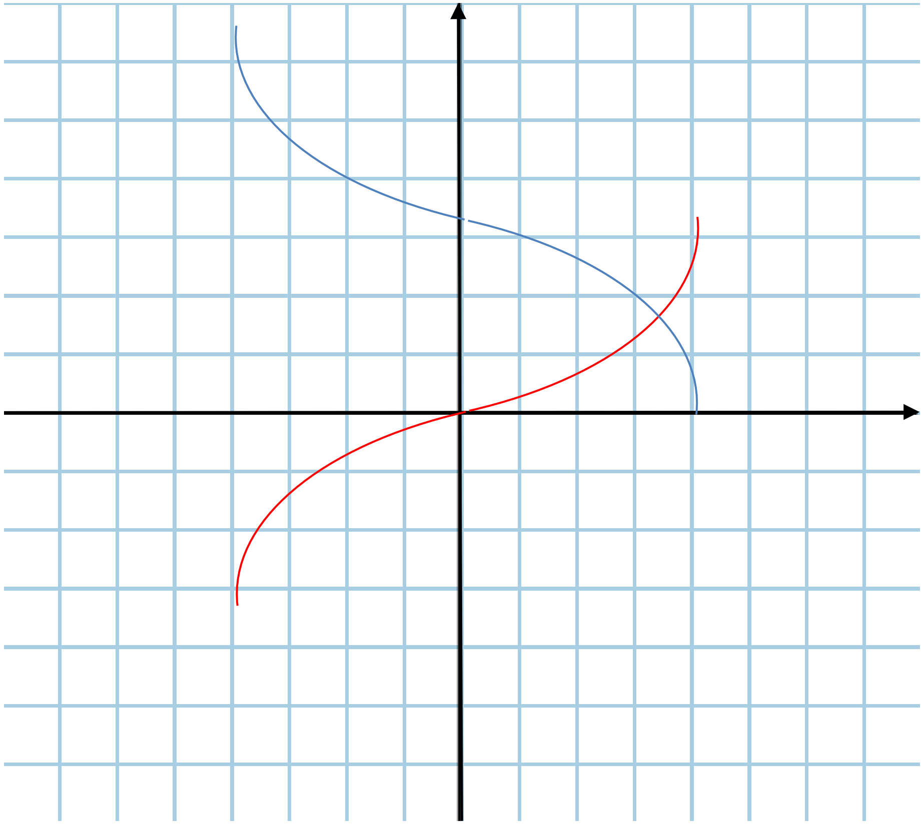
$$\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$$

We then have

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = -\cos^{-1}x + C$$

$$\frac{d}{dx}\operatorname{arcsin}^{-1}x = \frac{1}{\sqrt{1-x^2}}$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$



We also know

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$$

therefore

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

Often the more complicated integrals are given using *Tables of Integrals.*
e.g.

Table of integrals

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \text{arcosh}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \text{arsinh}\frac{x}{a} + C_1 = \log\left(x + \sqrt{a^2 + x^2}\right) + C_2$$

$$\int \frac{dx}{1 - x^2} = \text{artanh}x + C_1 = \frac{1}{2}\log\left|\frac{1+x}{1-x}\right| + C_2$$

$$\int \cosh x \, dx = \sinh x + C$$

$$\int \sinh x \, dx = \cosh x + C$$

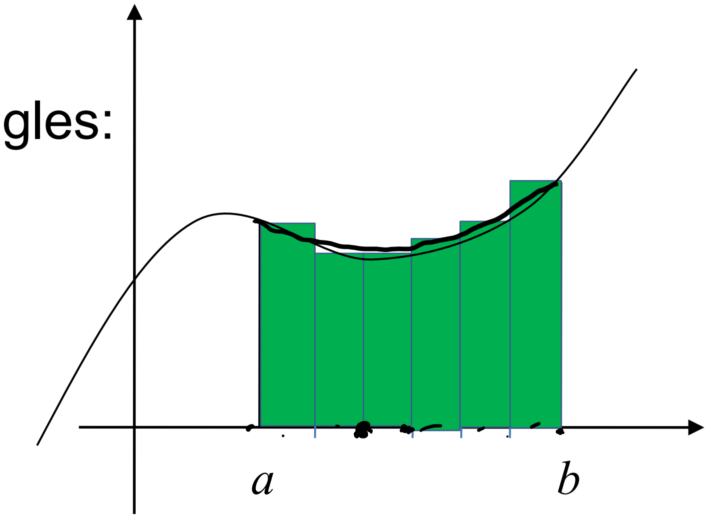
$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + C$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\arctan\frac{x}{a} + C$$

2. **An *integral* is also the sum of a large number of very small quantities**

1D definite integrals

We think of a one-dimensional definite integral as the sum of areas of infinite number of rectangles:



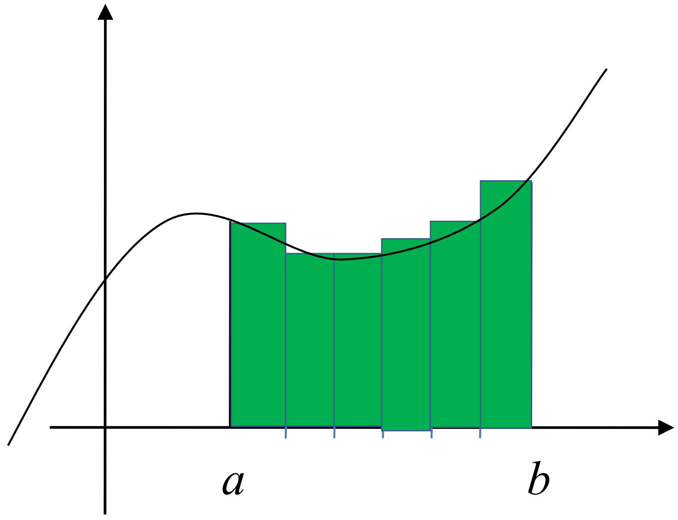
$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$ $\qquad x_i = a + i\Delta x$

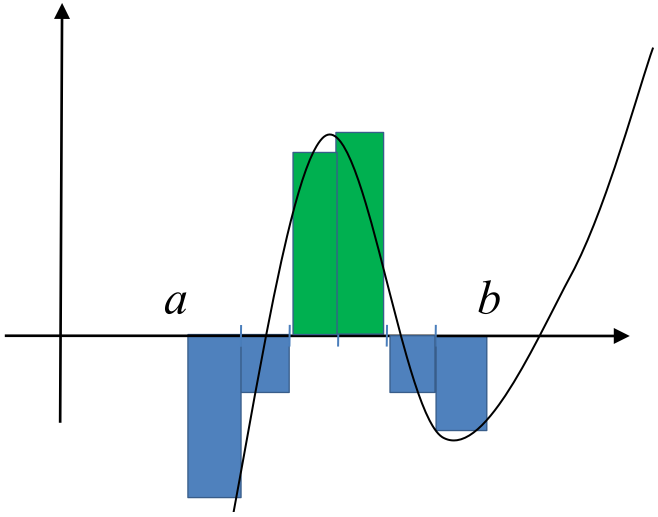
This is known as the *Riemann sum* of the integral.

As the number of rectangles increases, a better and better approximation for the area under the curve is obtained.

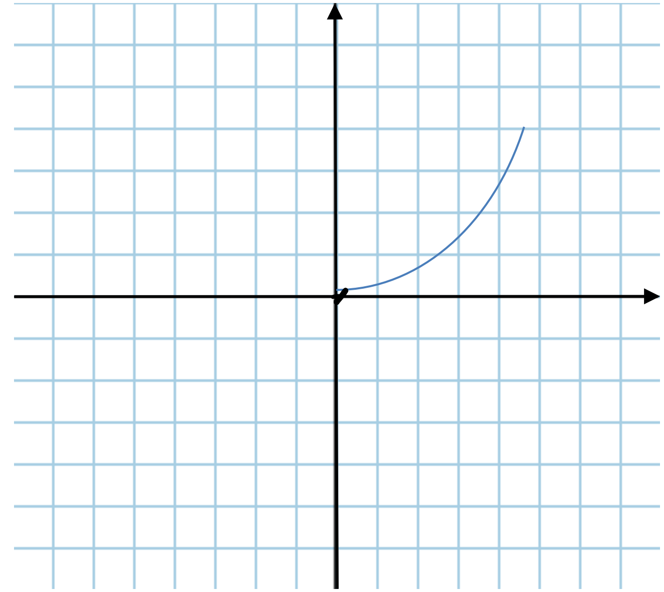
NB: The integral is often thought of as the *area* under a graph.



However, integrals can also be *negative* or *zero* (unlike areas).



$$\int_0^2 3x^2 dx \qquad \int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$



$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

$$x_i = a + i\Delta x = 0 + i\frac{2}{n} = \frac{2i}{n}$$

$$f(x_i) = 3x_i^2 = 3\frac{4i^2}{n^2} = \frac{12i^2}{n^2}$$

$$\int_0^2 3x^2 dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{12i^2}{n^2}\frac{2}{n} = \lim_{n\to\infty} \frac{24}{n^3} \sum_{i=1}^{n} i^2 =$$

$$= \lim_{n\to\infty} \frac{4(n+1)(2n+1)}{n^2} = 8, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\int_0^2 3x^2 dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{12i^2}{n^2}\frac{2}{n} = \lim_{n\to\infty} \frac{24}{n^3} \sum_{i=1}^{n} i^2 =$$

$$= \lim_{n\to\infty} \frac{4(n+1)(2n+1)}{n^2} = 8,$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

The fundamental theorem of calculus

These two ideas of integration (anti-derivatives, Riemann sums) are *the same.*

This is expressed by the following formal statements:

1. If *f(x)* is continuous on an interval [*a*,*b*] and *a* <x < *b* then

$$\int_a^b f(x)dx = F(b) - F(a)$$

2. If *f(x)* is continuous on an interval [*a*,*b*] and *F'(x)* = *f(x)*, then

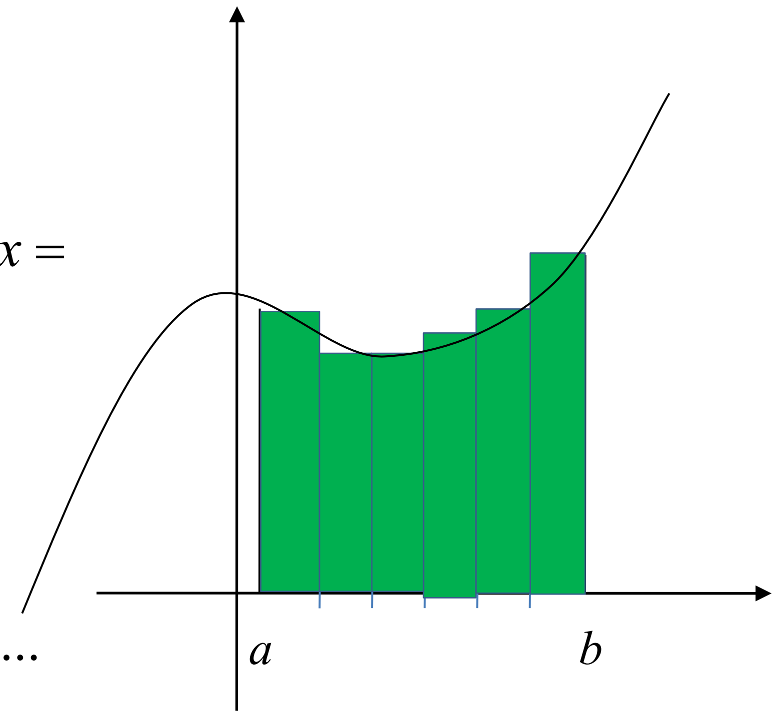
$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

$$\frac{d}{dx}\int_a^x f(x)dx = \frac{d}{dx}(F(x) - F(a)) = \frac{dF(x)}{dx} - \frac{dF(a)}{dx} = \frac{dF(x)}{dx} = f(x)$$

$$\int_a^b f(x)dx = F(b) - F(a)$$

Proof:

Divide [a,b] into n equal sections. Then



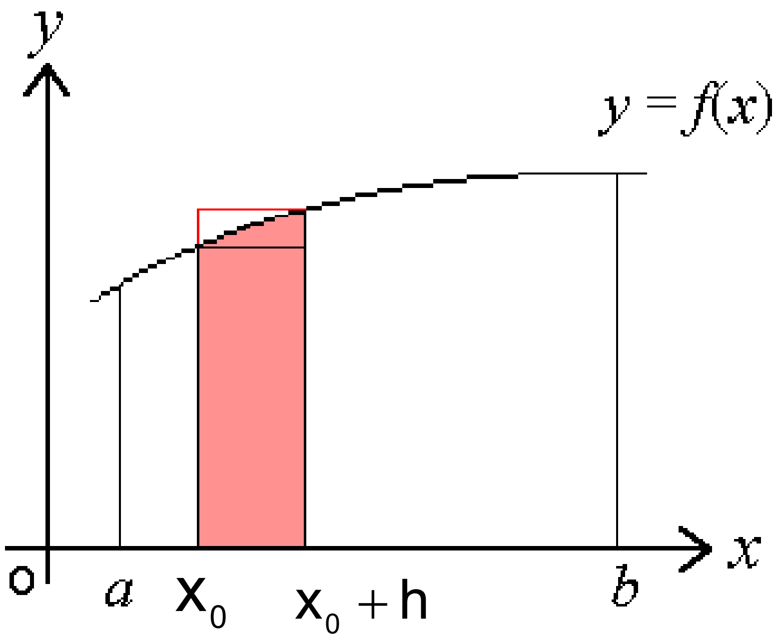
$$\int_a^b \frac{dF}{dx}dx = \lim_{n\to\infty}\sum_{i=1}^n \frac{dF(x_i)}{dx}\Delta x = \lim_{\Delta x\to 0}\sum_{i=1}^n \frac{dF(x_i)}{dx}\Delta x =$$

$$\sum_{i=1}^n \lim_{\Delta x\to 0}\frac{dF(x_i)}{dx}\Delta x = \sum_{i=1}^n \lim_{\Delta x\to 0}\frac{\Delta F(x_i)}{\Delta x}\Delta x =$$

$$\sum_{i=1}^n \lim_{\Delta x\to 0}\Delta F(x_i) = \lim_{\Delta x\to 0}\sum_{i=1}^n \Delta F(x_i) =$$

$$\lim_{\Delta x\to 0}(F(a+\Delta x) - F(a) + F(a+2\Delta x) - F(a+\Delta x) + \ldots$$

$$+ F(b) - F(a+(n-1)\Delta x) = F(b) - F(a)$$



• If we define the area function for *f*(*x*) between *a* and x to be A(x), then we have

$$h \times f(x_0) \le A(x_0 + h) - A(x_0) \le h \times f(x_0 + h)$$

• Assuming the function is continuous and sufficiently smooth then dividing through by the distance *h*, and letting this value tend towards zero, we find

$$f(x_0) \le \frac{A(x_0 + h) - A(x_0)}{h} \le f(x_0 + h)$$

• Hence $A'(x_0) = f(x_0)$. In other words, the area function is the antiderivative of *f*(*x*), so

$$A(x) = \int f(x)dx$$

General properties of definite Integrals

$$\int_a^b \left(f(x) + g(x) \right) dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\int_a^b \left(f(x) - g(x) \right) dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$

$$\int_a^b \left(f(x)g(x) \right) dx \neq \int_a^b f(x)dx \int_a^b g(x)dx$$

$$\int_a^b \frac{f(x)}{g(x)} dx \neq \frac{\int_a^b f(x)dx}{\int_a^b g(x)dx}$$

General properties of definite Integrals

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

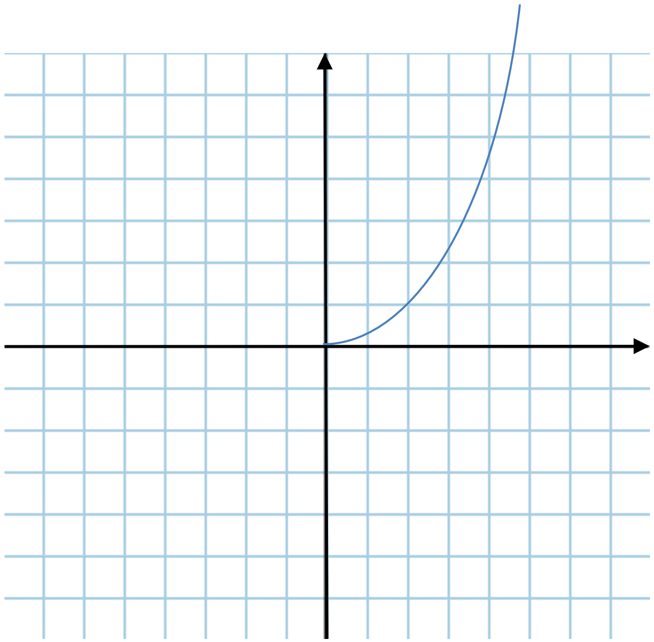
$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^b dx = b - a$$

$$\int_a^b f'(x)dx = f(x)\Big|_a^b = f(b) - f(a)$$

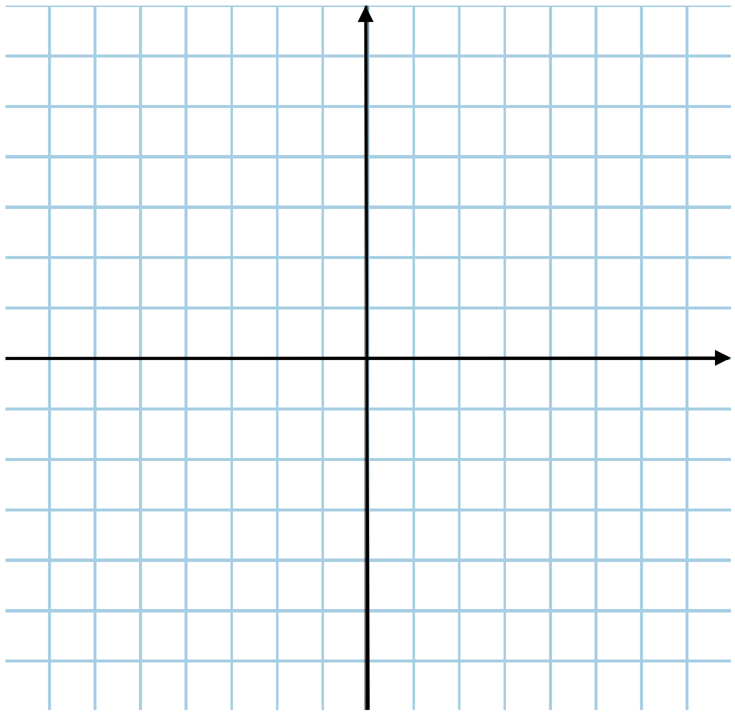
Example: Evaluate

$$\int_0^2 3x^2\,dx$$



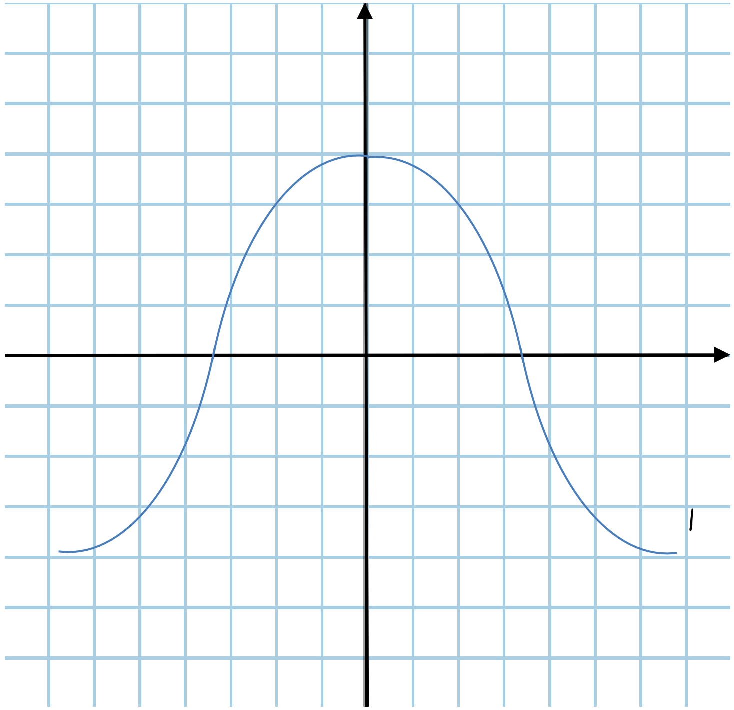
Example: Evaluate

$$\int_0^4 f(x)\, dx =$$

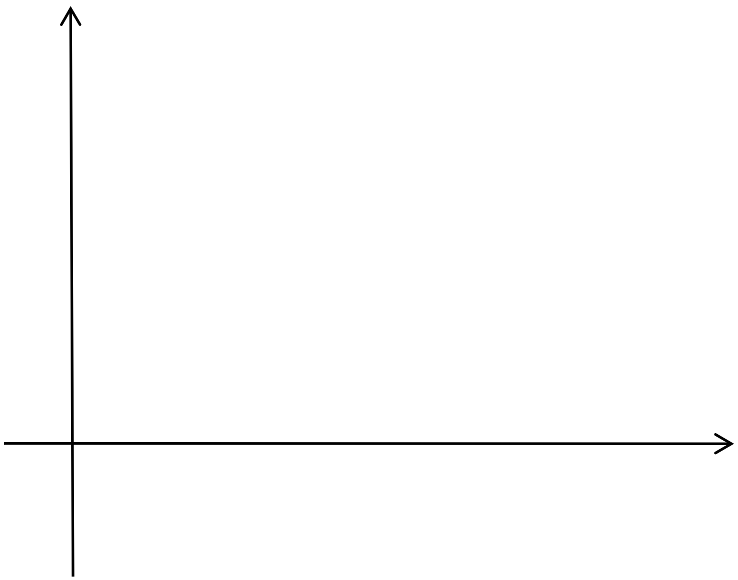


Example: Evaluate

$$\int_0^{\pi/2} \cos x \, dx =$$



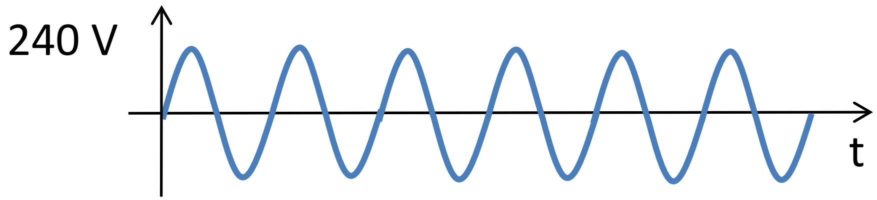
Example: Compute the average value of the function f(x) = 3 + sin 2x over the interval 0 < x < ¼2



Formula for average of f(x) over an interval [a,b]:

$$\overline{f(x)} = \frac{1}{b-a}\int_a^b f(x)dx$$

Example: Compute the root-mean-square (RMS) of the following function:



Formula for RMS of f(x) over an interval [a,b]:

$$RMS = \sqrt{\frac{1}{b-a}\int_a^b (f(x))^2 dx}$$

$$\overline{f}^2 = \frac{1}{T}\int_0^T f^2(t)dt$$

$$V(t) = V_0 \sin \omega t,$$

$$T = \frac{2\pi}{\omega}$$

Example: Evaluate

$$\int_0^4 2^x \, dx$$