Integration by substitution [Textbook: 5.5]

Integration by parts [Textbook: 5.6]

Partial Fractions

Techniques of Integration

There are a number of techniques that can be used to evaluate integrals. Here we cover the most important analytic techniques:

1. Substitution

2. Integration by parts

3. Partial fractions

There are also a number of *numerical techniques*.

Techniques of integration: substitution

We can often convert integrals into a simple form by "swapping variables" between the integration variable *x* and a new variable *u.*

Example: Evaluate $\int_0^1 (6x+3)^5\,dx$

Example: Evaluate $$\int_0^5 \frac{1}{4x+1}\,dx$$

Often, substitution can be use to evaluate “tricky-looking” integrals.

Example: $\int xe^{x^2}\,dx$

Example: Evaluate $\displaystyle\int_{0}^{5} x\sqrt{x+1}\,dx$

$$\int \sin^4 x \cos x \; dx$$

$$\int \tan(2x)dx =$$

$$\int \frac{dx}{\sqrt{2x+1}}$$

$$\int \frac{dx}{(3x+1)^2}$$

$$\int \frac{\cos\sqrt{x}dx}{\sqrt{x}}$$

$$\int \frac{\sin 2x dx}{4+\cos 2x}$$

Integrating powers of sine and cosine

Often we have to integrate integrals that looks like this:

$$\int \cos^2 x dx \qquad \int \sin^3 x dx$$

If the power is *odd* then we can evaluate this by using the identity

$$\cos^2 + \sin^2 x = 1$$

$$\int \sin^3 x dx =$$

If the power is *even* then we use the “half-angle identities”:

$$\cos^2 x = \frac{1}{2}\left(1 + \cos 2x\right)$$

$$\sin^2 x = \frac{1}{2}\left(1 - \cos 2x\right)$$

so

$$\int \cos^2 x dx$$

Trigonometric substitution

We can often evaluate integrals involving a square root, e.g.

$$\int \sqrt{9 - x^2} dx$$

By making a *trigonometric substitution*

$$x = a \cos \theta \qquad x = a \text{sinh} \theta$$

$$x = a \tan \theta \qquad x = a \sec \theta$$

Note that we have to ensure that x lies in the *range* of the trig function.

The trick is to use pick the right substitution to get rid of the square root.

$$\sqrt{a^2 - x^2}$$

$$\sqrt{a^2 + x^2}$$

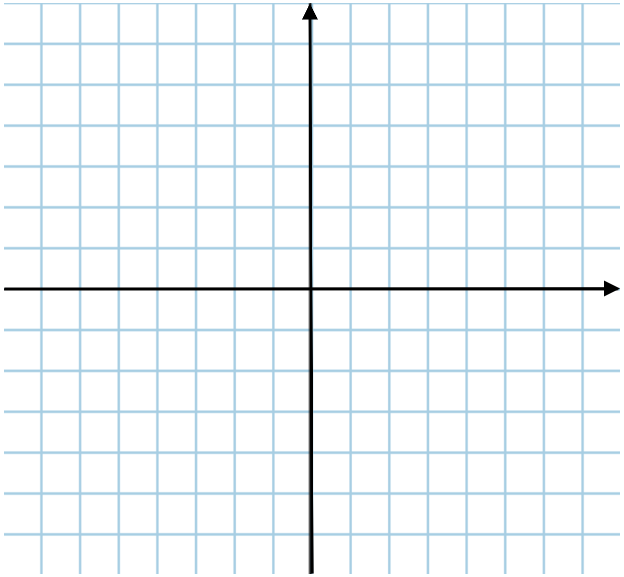
$$\int \frac{1}{x^2 + a^2} dx$$

Examples:

$$\int \frac{1}{x^2 + a^2} dx$$

$$\int \frac{1}{\sqrt{x^2+4}}dx$$

Example:
Show that the area of a circle is πr^2



Impossible Integrals

It is important to realise that there are some integrals that *cannot be done*, i.e. whose answer cannot be expressed in terms in simple “closed form”.

Examples:

The elliptic integrals:

E.g.

$$\int_0^a \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

The error function $\displaystyle\int e^{-x^2}dx$

These integrals can sometimes be evaluated in special cases.

$$\int \frac{x}{x^2+2}\,dx$$

½ ln |x^2+2| + C

$$\int e^{-2x^2}\,dx$$

<u>No closed form</u>

$$\int \frac{dx}{\sqrt{1-4x^2}}$$

½ $\sin^{-1}(2x) + C$

$$\int x\sqrt{x^2+1}dx$$

$1/3\ (x^2+1)^{3/2} + C$

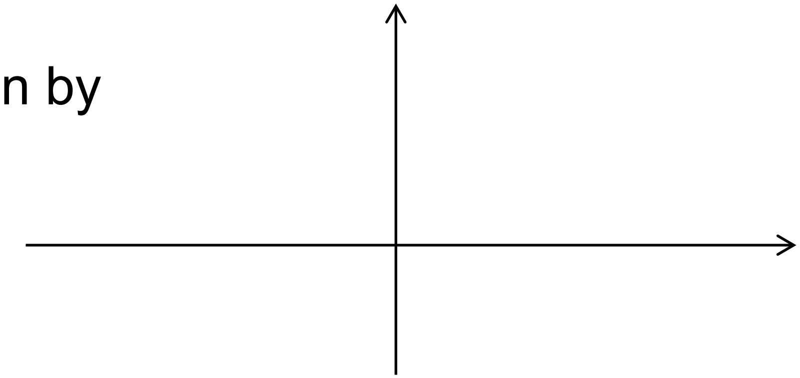
$$\int \theta \sin \theta^2 d\theta$$

$-1/2 \cos(\theta^2) + C$

Example: Arc length

The *arc-length* along a curve y =f(x) is given by

$$l = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



Compute the length along the curve y = $x^{3/2}$ from x = 0 to x = 9.

Techniques of Integration: integration by parts

Products of functions, e.g.

$$x^2 \sin x \qquad xe^{4x}$$

can often be integrated “by parts”.

The product rule for derivatives is

$$\frac{d}{dx}(uv) = uv' + vu'$$

Integrate both sides with respect to x:

So, for any product u(x) v'(x), we can integrate as follows:

$$\int uv'dx = uv - \int vu'dx$$

This only works if the second integral (on the RHS) is simpler that the original one.

When presented with an integral of a product, e.g.

$$\int x \cos x dx$$

1. We want one of the functions to be easy to integrate (choose this one as v')

2. We want the other function to be easy to differentiate (choose this one to be u)

$$\int x \cos x dx$$

$$\int uv'dx = uv - \int vu'dx$$

Important: If we don't pick the *correct* part of the product to integrate, then we end up with a more complicated integral.

Example:

$$\int uv'dx = uv - \int vu'dx$$

$$\int x \sin x \, \mathrm{dx}$$

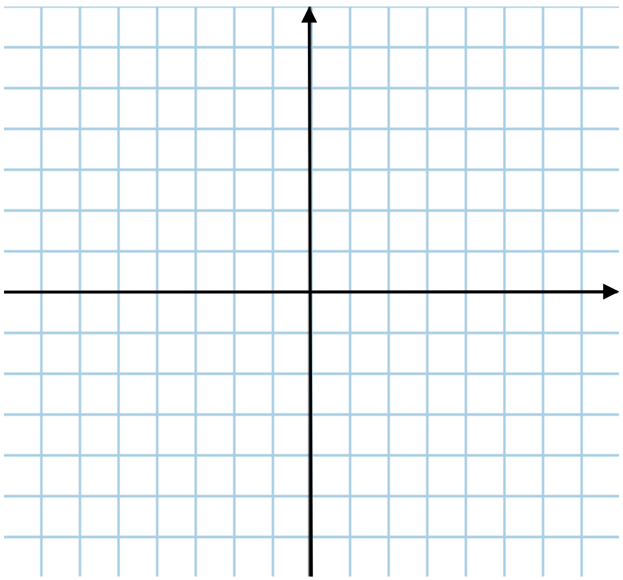
Example: Find $\int x \ln x \, \mathrm{dx}$

$$\int uv'dx = uv - \int vu'dx$$

$$\int uv'dx = uv - \int vu'dx$$

Example:

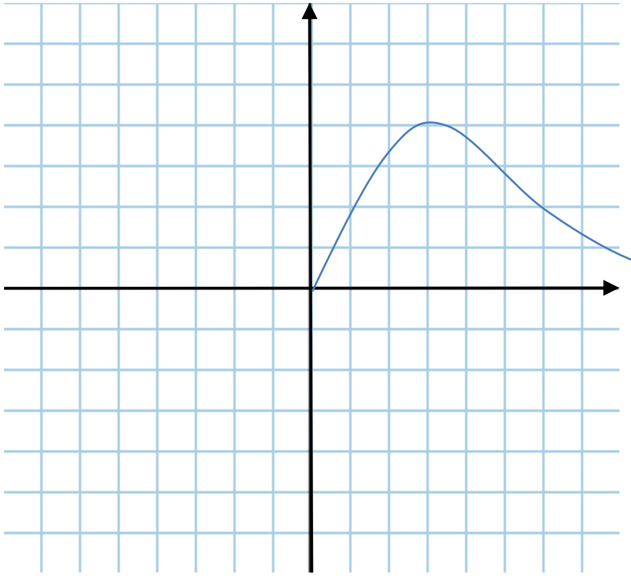
$$\int (2x-1)\ e^x dx$$



Example: (with integration limits)

$$\int uv'dx = uv - \int vu'dx$$

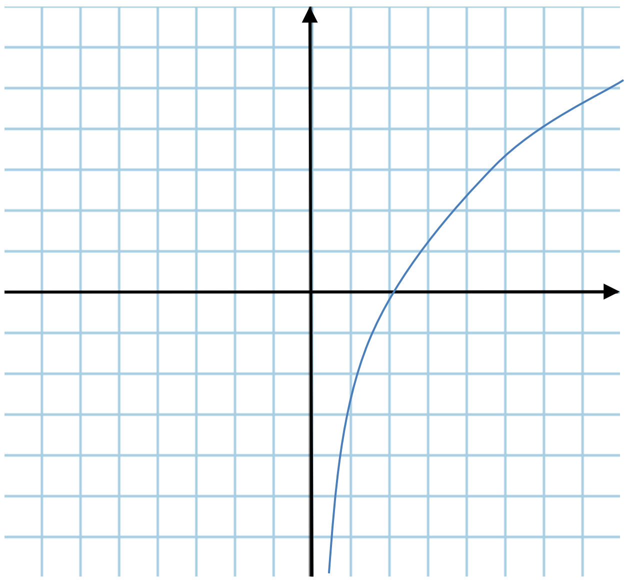
$$\int_0^2 xe^{-x}dx$$



$$\int uv'dx = uv - \int vu'dx$$

Example:

$$\int_1^2 \ln x \, dx$$



$$\int e^x \cos x \, dx$$

How to integrate functions in the form

$$\int \frac{p(x)}{q(x)}\,dx$$

where *p(x)* and *q(x)* are polynomials. Examples:

$$\int \frac{x^3+6x}{x^2+6}\,dx$$

$$\int \frac{10}{x^3+3x^2+3x+1}\,dx$$

$$\int \frac{9x^2+3}{x^3+x+7}\,dx$$

Finding integrals using partial fractions

We can use the method of partial fractions to find integrals of the form

$$\int \frac{p(x)}{q(x)}\,dx$$

where *p(x)* and *q(x)* are polynomials. For example, for the integral

$$\int \frac{5x+8}{(x-4)(x-2)}\,dx$$

We write the integrand as a sum of two fractions:

We can then do the integral:

$$\int \frac{5x+8}{(x-4)(x-2)}\,dx =$$

To find the constants A and B, we use the following trick:

$$\frac{5x+8}{(x-4)(x-2)} = \frac{A}{x-4} + \frac{B}{x-2}$$

Example:

$$\int \frac{1}{x^2 + 2x - 3}\,dx$$

Integrate:

$$\int \frac{2x + 1}{(x - 2)^2} dx$$

What about integrals that look like this?

$$\int \frac{1}{x(x-1)^2}\,dx =$$

$$\int \frac{3}{x^2 + 2x + 5} dx =$$

Examples of integrals

$$\int x^2 e^{x^3}\, dx$$

[1/3 e^{x^3} + C]

$$\int \frac{1}{(x+1)(x+2)} dx$$

ln|(x+2)/(x+1)| + C

$$\int \frac{2x^3 + 3x^2 + 2x + 18}{x^4 + 7x^2 + 6} dx = \int \left[\frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{x^2 + 6} \right] dx$$