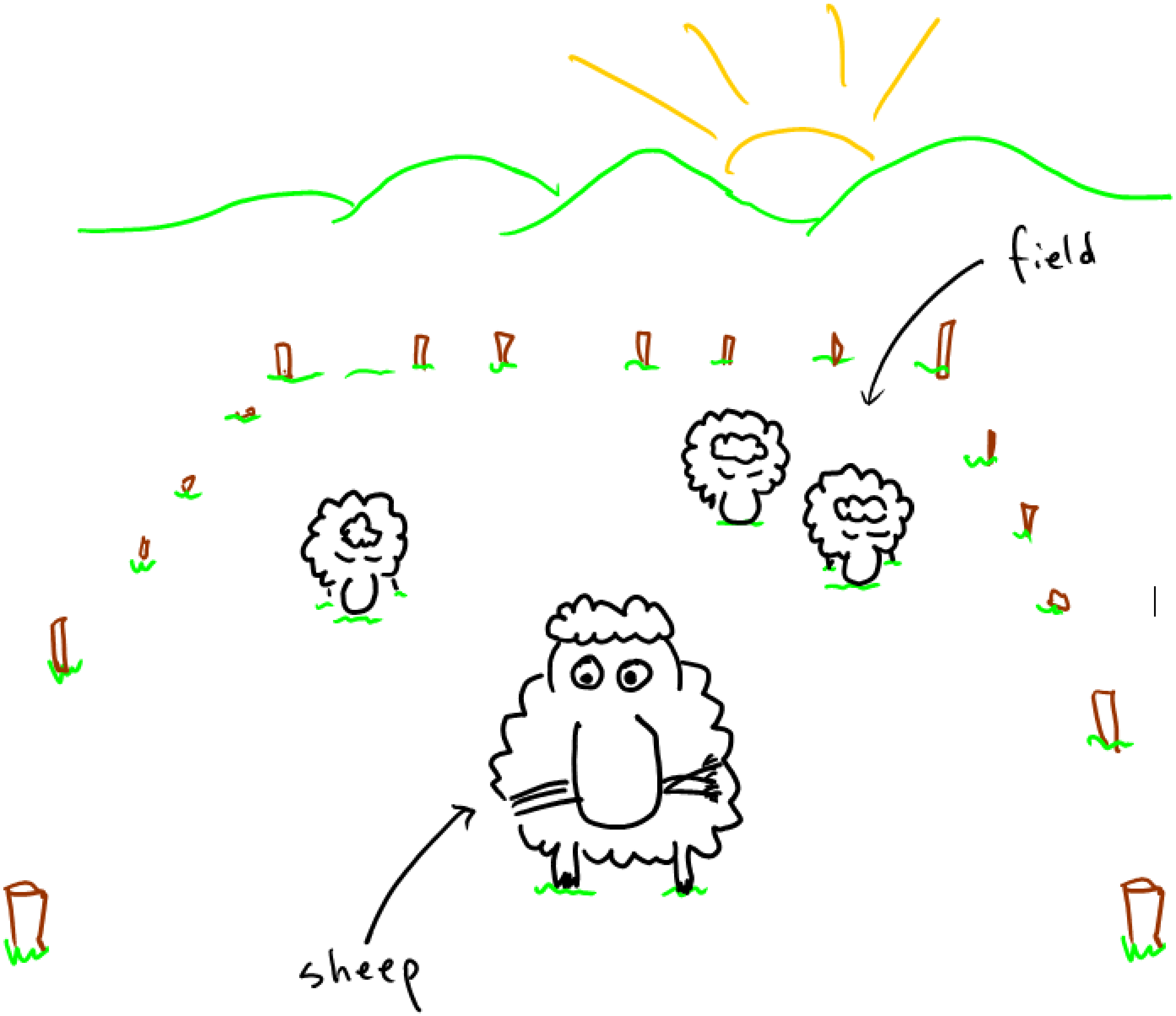
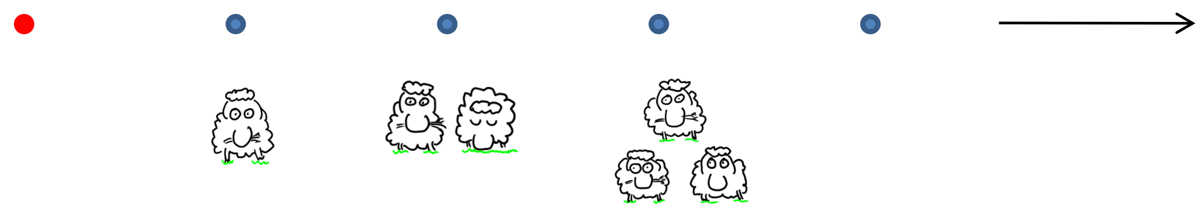
Complex Numbers



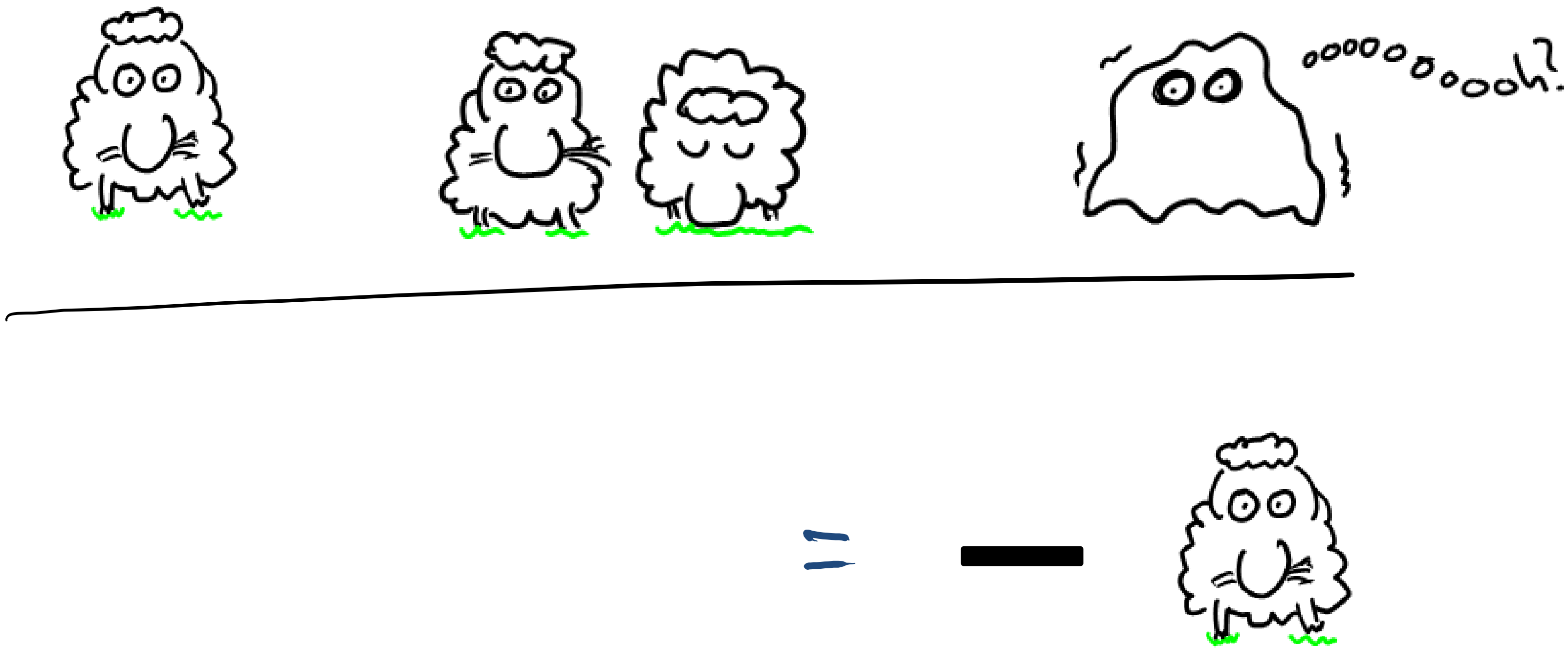
field
sheep

Having an idea of *numbers of things,* together with the operation of *addition*, leads to the set of numbers known as The Positive Integers.

We denote this set Z^{+}

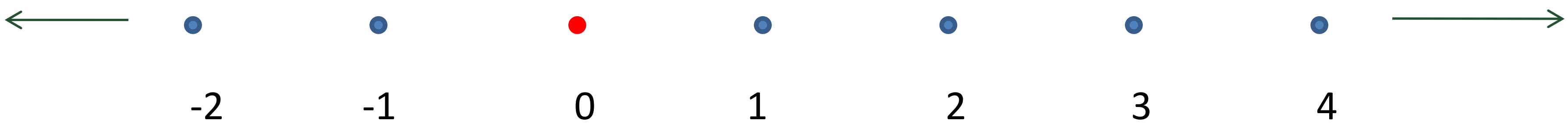


What happens if we want to solve the following problem?

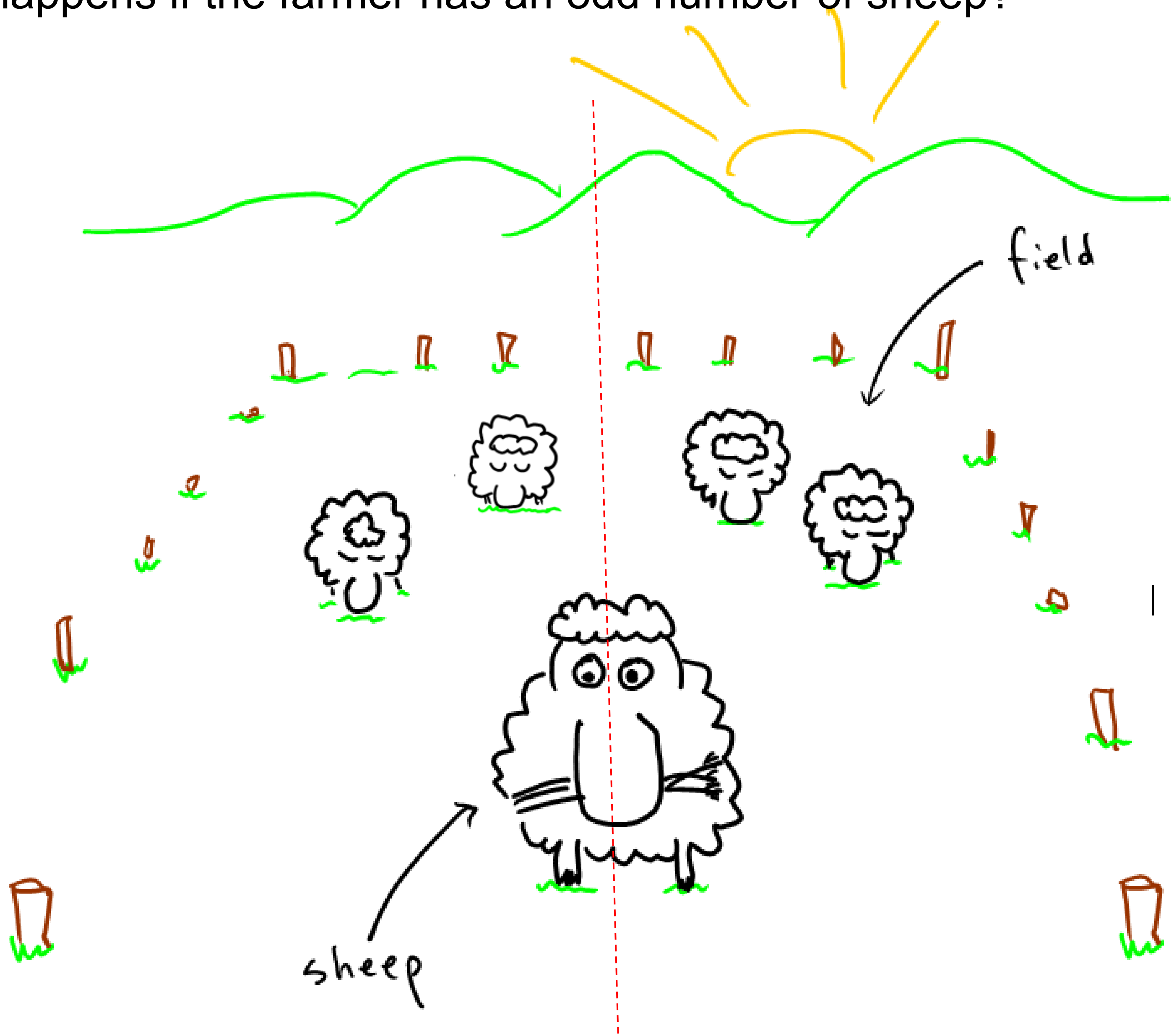


The operation of subtraction leads to the (full) set of Integers, including negative numbers.

We call this set Z

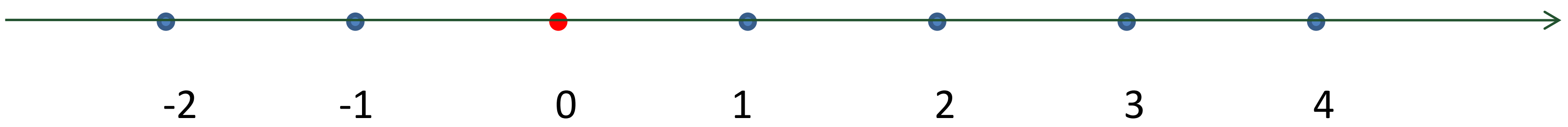


What happens if the farmer has an odd number of sheep?

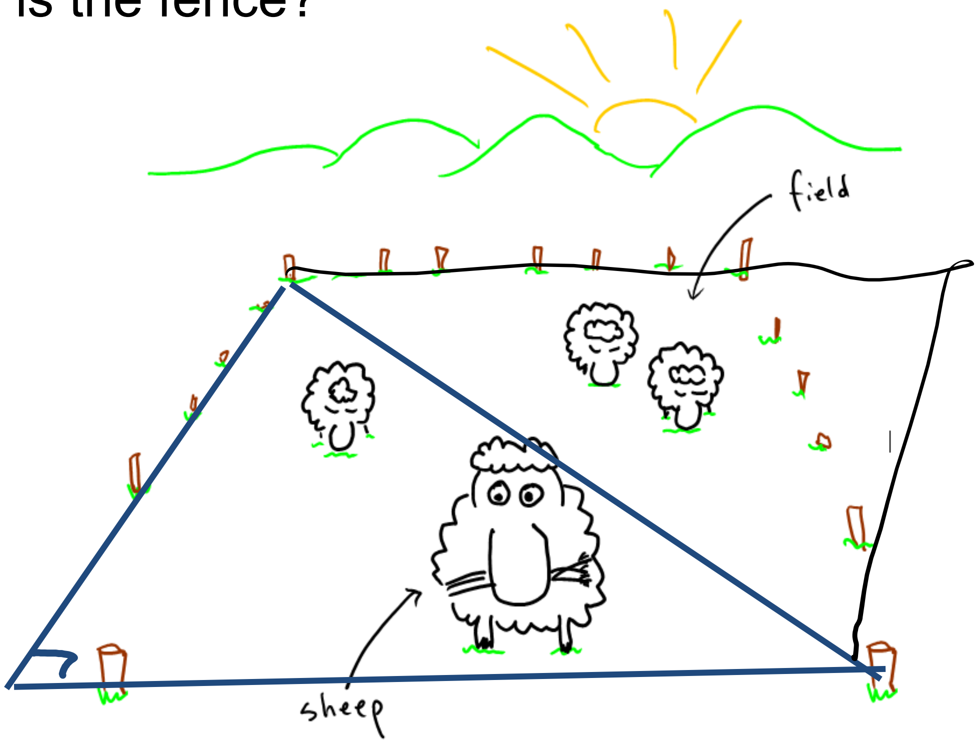


The operation of division leads to the set of Rational Numbers

We call this set Q

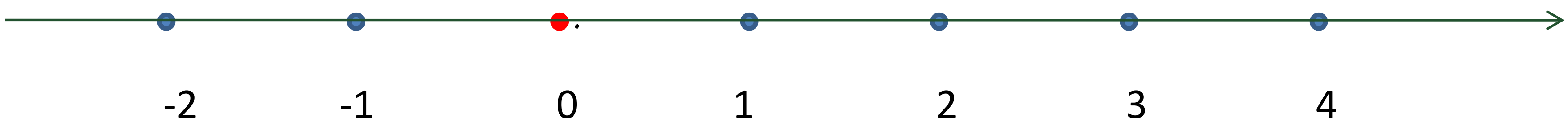


What happens if we want to solve the following problem:
A farmer wants to build a fence diagonally across a square field.
How long is the fence?

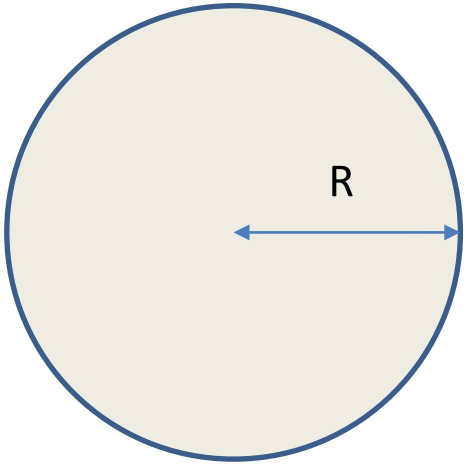


Hippasus of Metapontum,
Credited with discovering irrational numbers

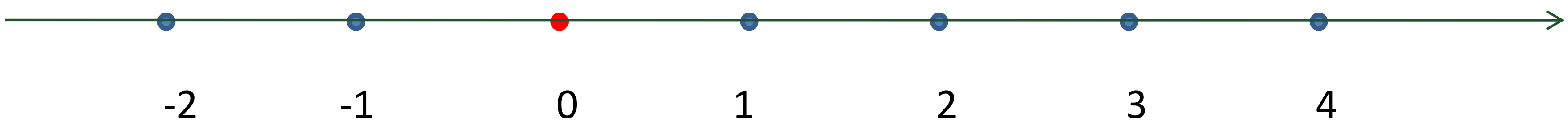
The operation of taking *fractional powers of positive rational numbers* (i.e. square roots etc.) leads to the set of Irrational Numbers:



A circle has radius R – what is the circumference?



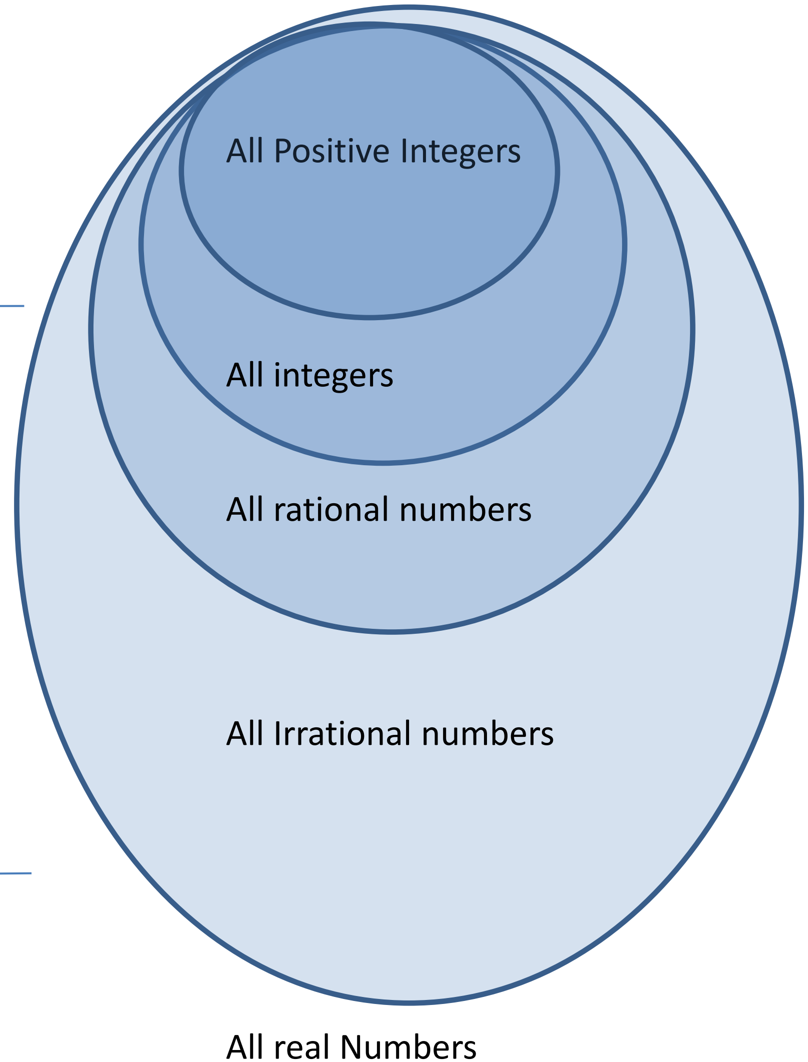
Putting in all the positive and negative integers, the rational and irrational numbers, and making sure the Transcendental numbers are included leads to the set of Real Numbers:



We call this set R

Each time we extend our number system, we are able to solve a new range of mathematical (and practical) problems.

To solve	**we need**	**new symbols**
$x+1=5$	Positive integers	
$x+1=0$	Negative numbers	
$x\times 3=2$	Fractions	
$x^2=2$	Irrational numbers	
$x^2=-2$		



We introduce the symbol *i,* which satisfies the equation

$$i^2 = -1$$

That is,

$$i = \sqrt{-1}$$

This is called an *imaginary* number.

We can use **i** to solve equations like:

$$x^2 = -4$$

We can combine imaginary numbers and *real* numbers to form *complex numbers*

Definition:

A complex number is

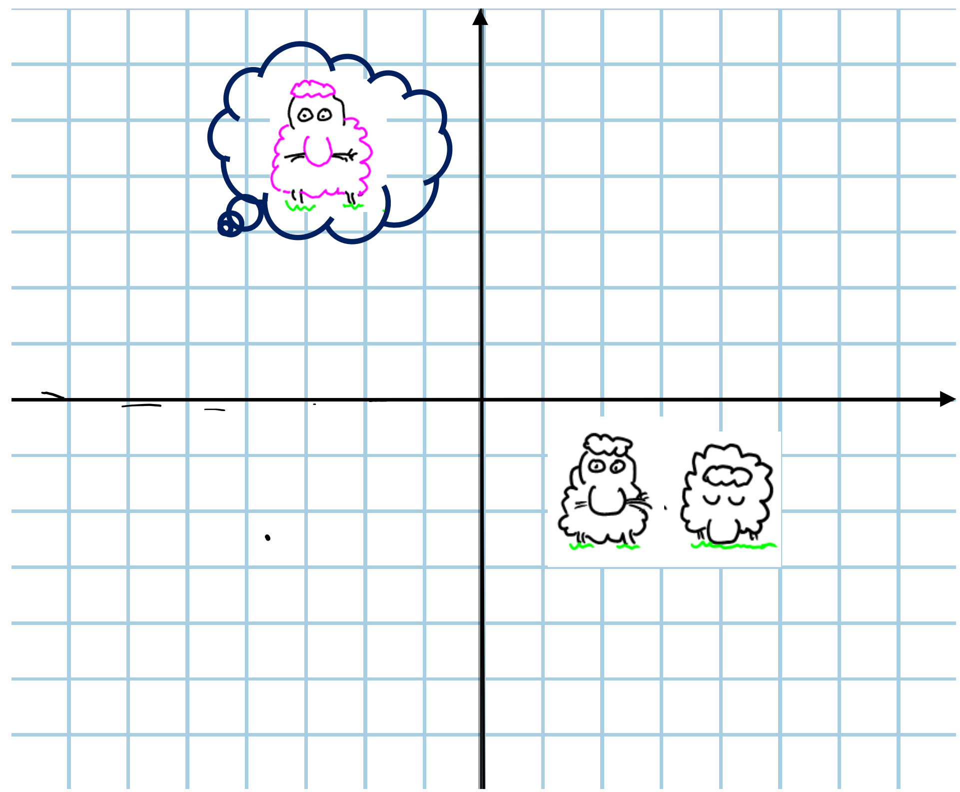
Real part

$$z = a + i\,b$$

Imaginary part

where a and b are real numbers.

We call the set of all complex numbers $\mathbb{C}$



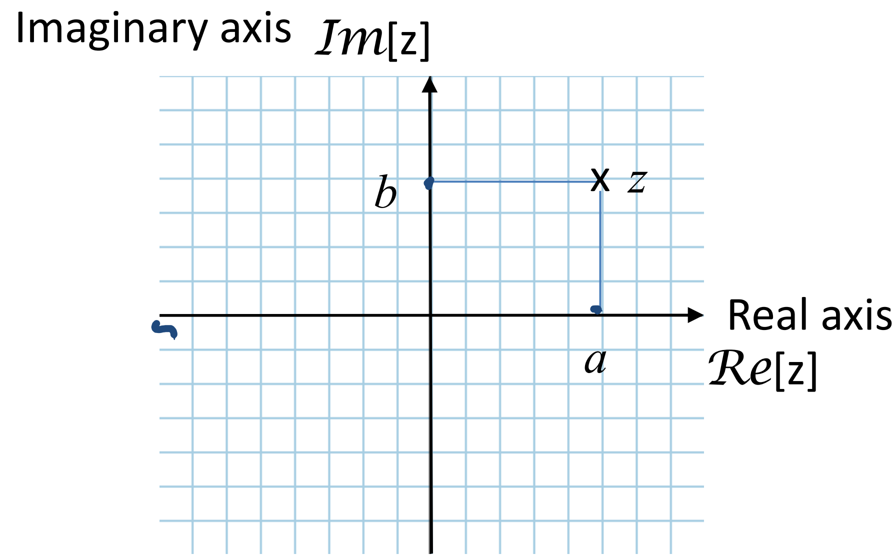
Vizualising complex numbers

We can represent complex numbers in two dimensions on an *Argand diagram* (also called the *complex plane*):

Real part

$$z = a + i\,b$$

Imaginary part



For the complex number

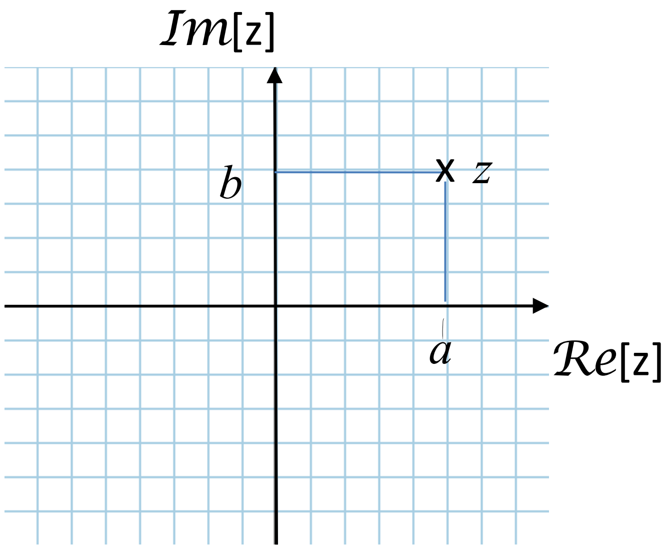
$$z = a + i\,b$$

The *complex conjugate* is

$$\bar{z} = a - ib$$

The *modulus* is

$$|z| = \sqrt{a^2 + b^2}$$



If two complex numbers are equal, then their real and imaginary parts are equal:

1. If

$$a + i\,b = 3 + 2\,i$$

Then we know that

a = 3 b = 2

2. If

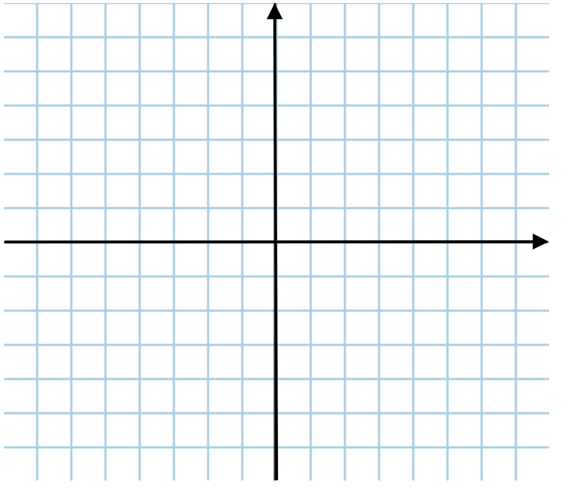
$$c+2+ 3d\,i = 7 - 6\,i$$

then

Complex numbers can be added and subtracted,
so long as we keep the real and imaginary parts separate. E.g.

$(3 + 2\,i) + (-1 + i) =$

$(2 - i) - (1 + i) =$



We can multiply two complex numbers by remembering that $i^2 = -1$

e.g.

If $z_1 = (3 + 2\,i)$ and $z_2 = (-1 + i)$, then $z_1\, z_2 =$

<u>Rule of thumb:</u>

You can add, subtract, and multiply complex numbers just like real numbers, so long as you remember that $i^2 = -1$

We can use the complex conjugate to *divide* two complex numbers.

e.g.

$$\frac{5-6i}{2+i} =$$

The modulus can be calculated by multiplying a complex number with its conjugate:

$$z\,\bar{z} = |z|^2$$

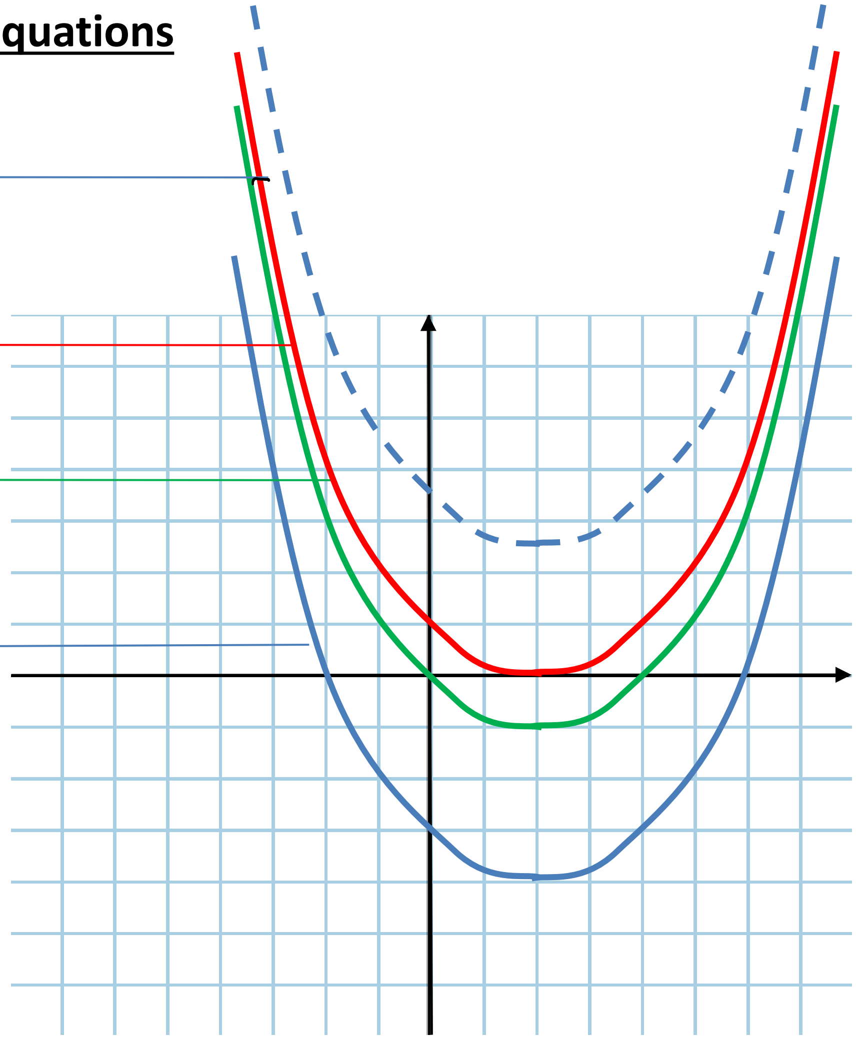
Using complex numbers to solve equations

$$y = x^2 - 2x + 5$$

$$y = x^2 - 2x + 1$$

$$y = x^2 - 2x + 0$$

$$y = x^2 - 2x - 3$$

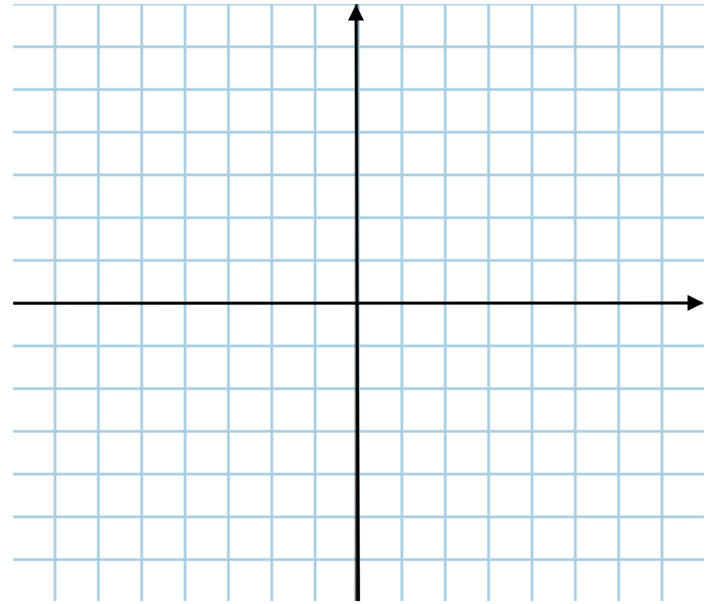


Solving quadratic equations

Using complex numbers, all equations of the form

$$az^2 + bz + c = 0$$

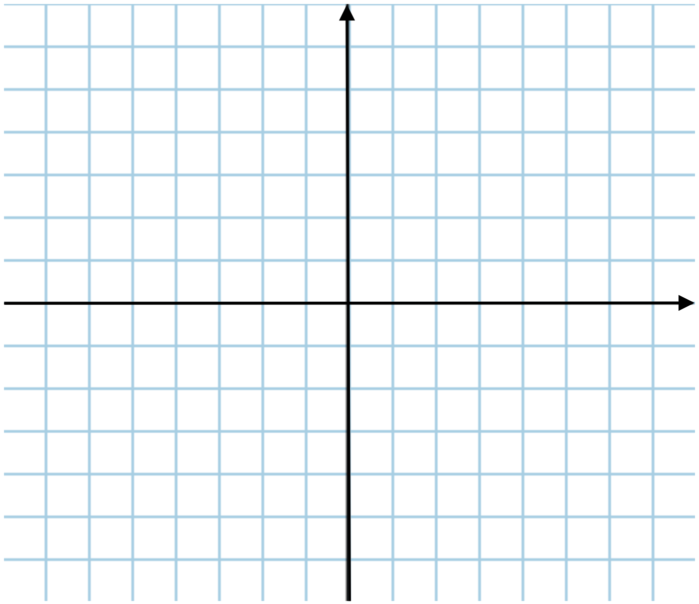
have two solutions, and can be solved using the quadratic formula.



Eg. Solve the following equations:

1. $z^2 + 0z + 16 = 0$

2. $z^2 - 4z + 5 = 0$

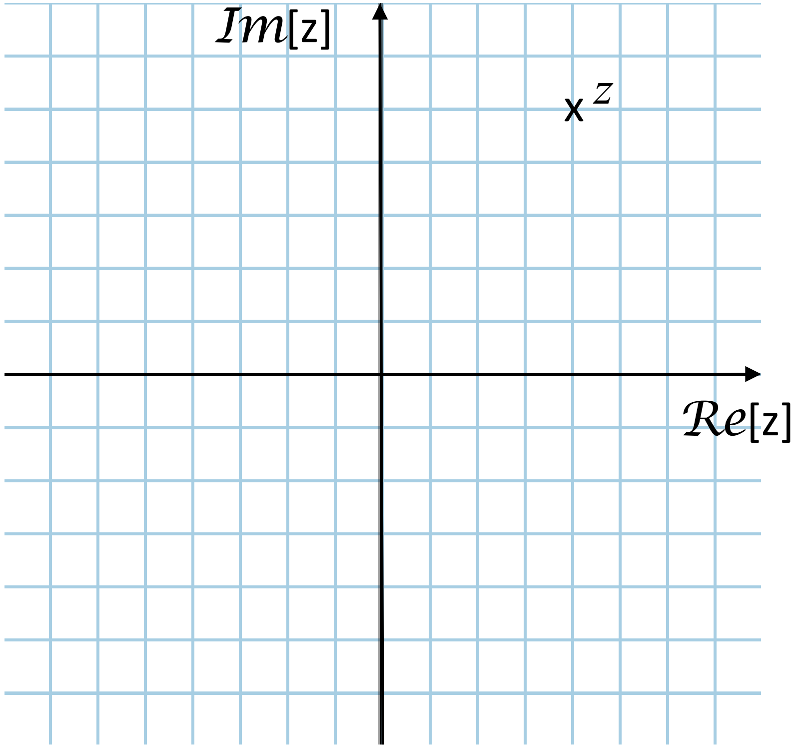


3. $z^2 - 2z + 7 = 0$

<u>Polar form representation:</u>

A complex number can also be written

$$z = r(\cos\theta + i\sin\theta)$$



Where

$$r = \sqrt{a^2 + b^2} = |z|$$

is the modulus, and

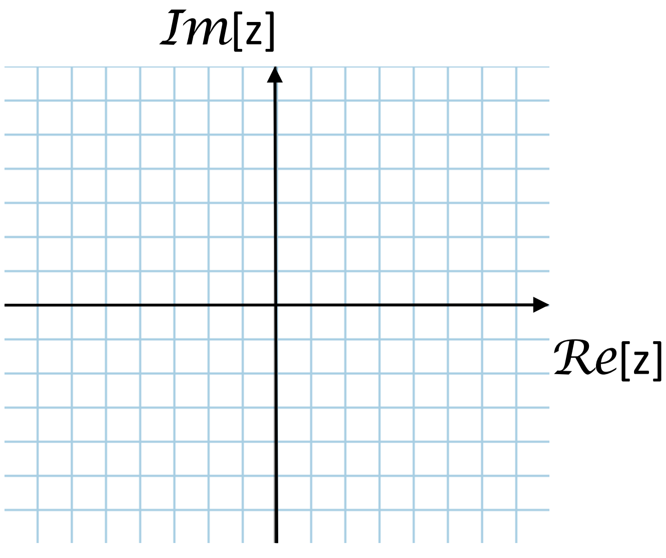
$$\theta = \cos^{-1}\frac{a}{r} = \sin^{-1}\frac{b}{r}$$

is the *argument*. If θ lies in the range $[0,2\pi)$ then it is the *principal argument,* sometimes written

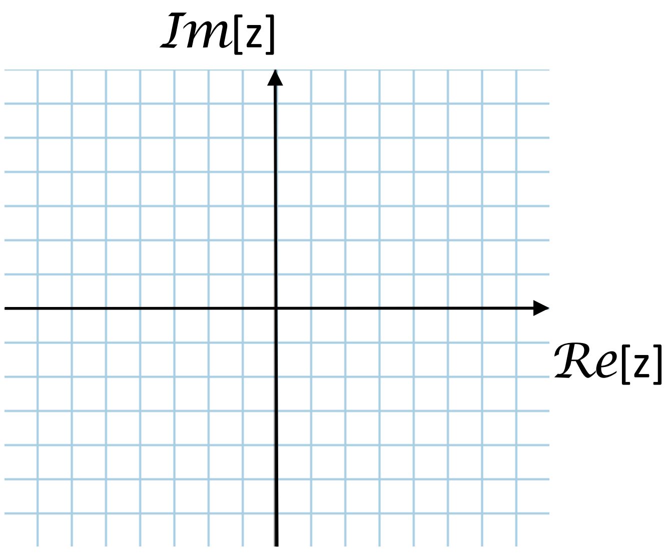
$$\mathrm{Arg}(z)$$

E.g.

1. Write z = 1 + i in polar form.



2. Write $1+i\sqrt{3}$ in polar form.



E.g.

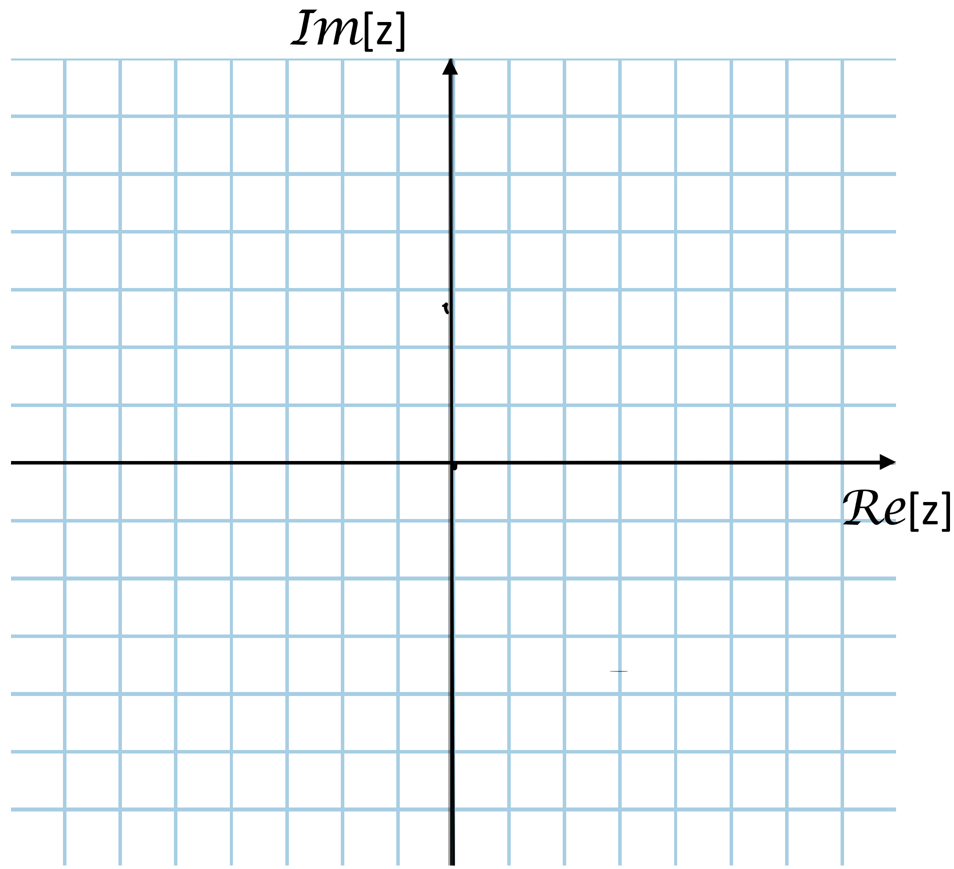
1. Find the modulus and argument of the following complex numbers:

a) i

b) 1

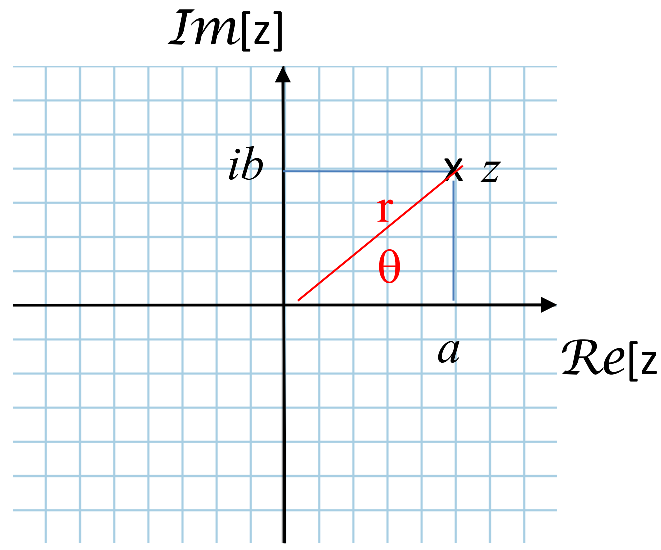
c) -1

$d) -i$



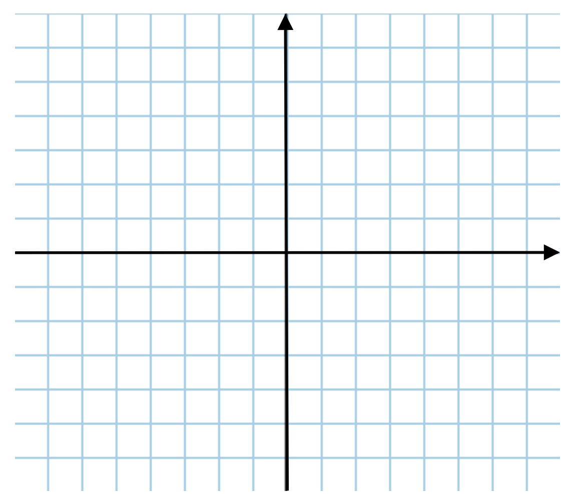
Finding the argument

The argument is the number θ for which both the following are true:



$$\cos\theta = \frac{a}{r} \quad \text{and} \quad \sin\theta = \frac{b}{r}$$

A quick and "dirty" way to find the argument:

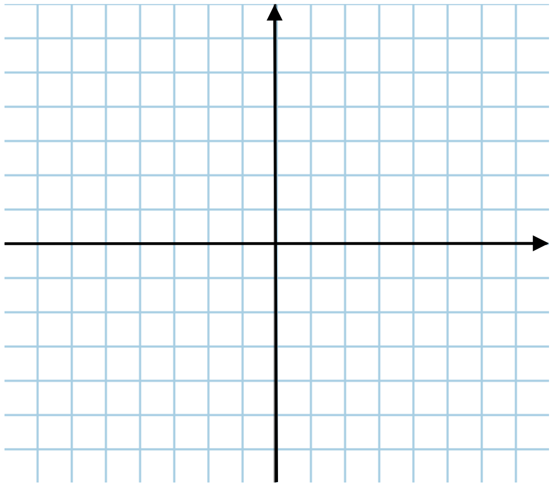


1. Compute the inverse cosine, sine or inverse of tan for b/a

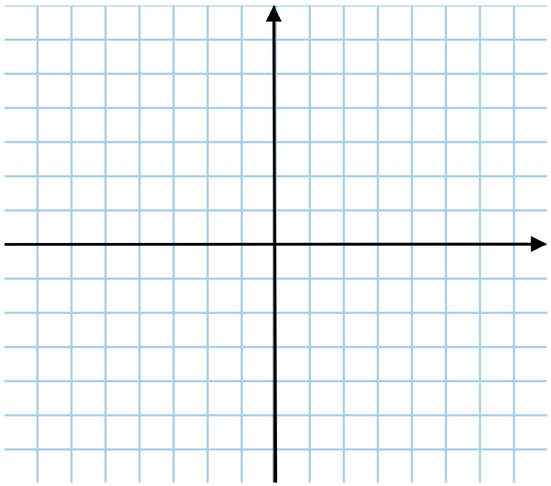
2. Check if it is in the correct quadrant

Example: Find the argument of

a) $z = 1 - i$

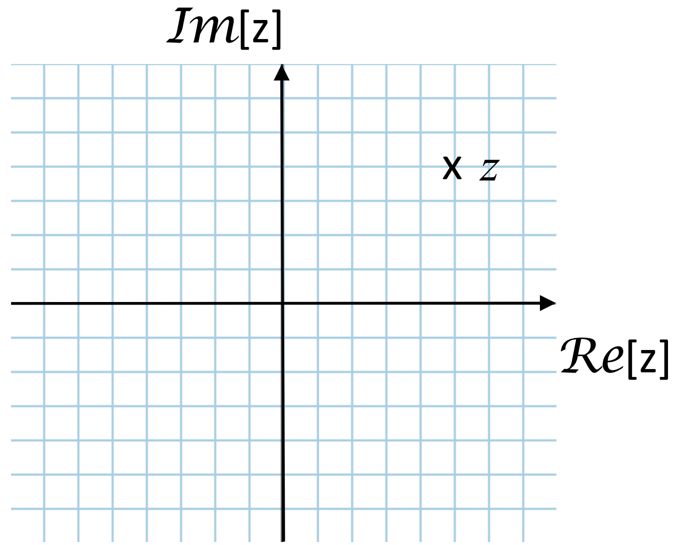


b) $z = -1 + i$



Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$



Any complex number can be written as

$$z = re^{i\theta}$$

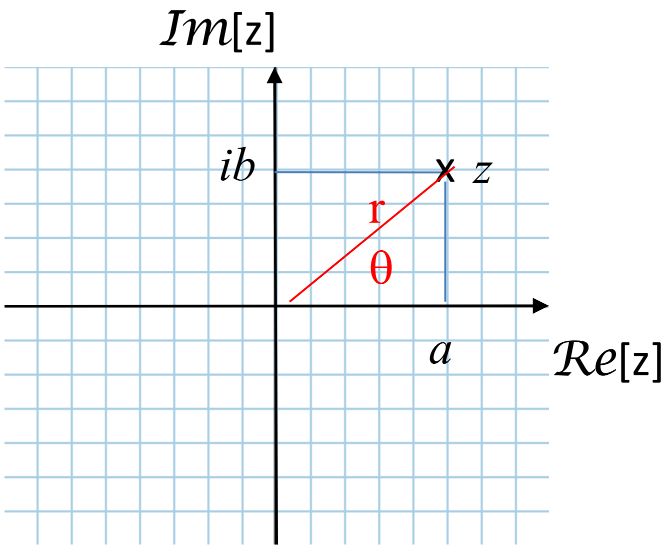
So we can write a complex number in three ways:

Cartesian form

Real part

$$z = a + i\,b$$

Imaginary part



with i^2=-1.

Polar form

$$z = r(\cos\theta + i\sin\theta)$$

r = |z|

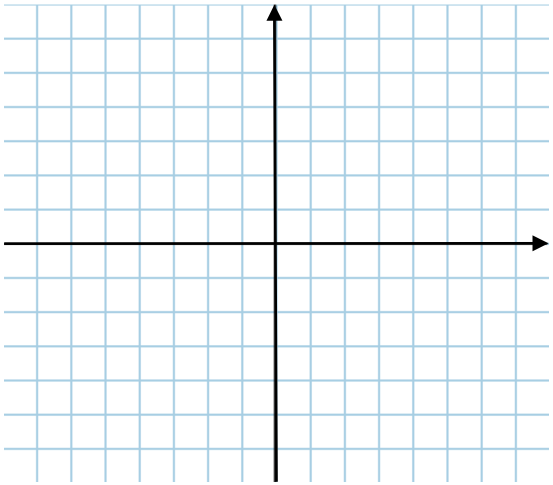
μ = Arg z

Exponential (polar) form

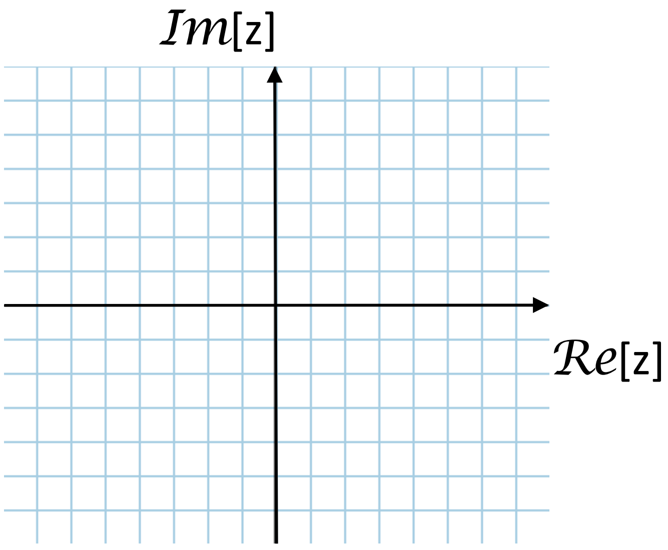
$$z = re^{i\theta}$$

Using exponential form, we obtain the very interesting and unusual formula

$$e^{i\pi} = -1$$

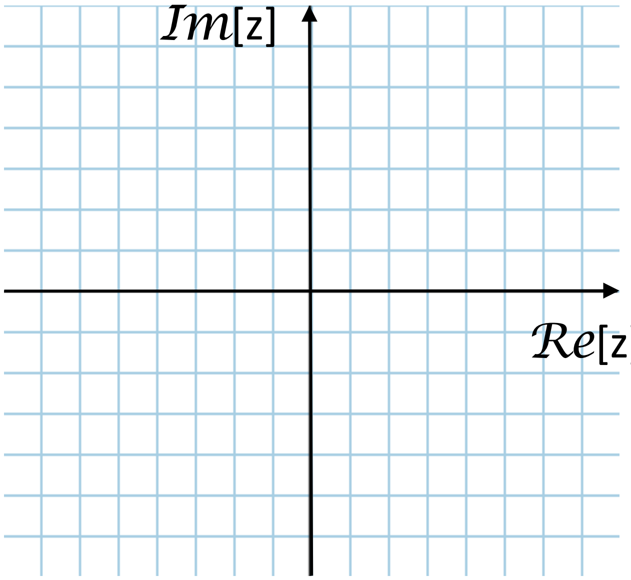


Multiplying by a complex number is like a rotation in the complex plane.



Examples:

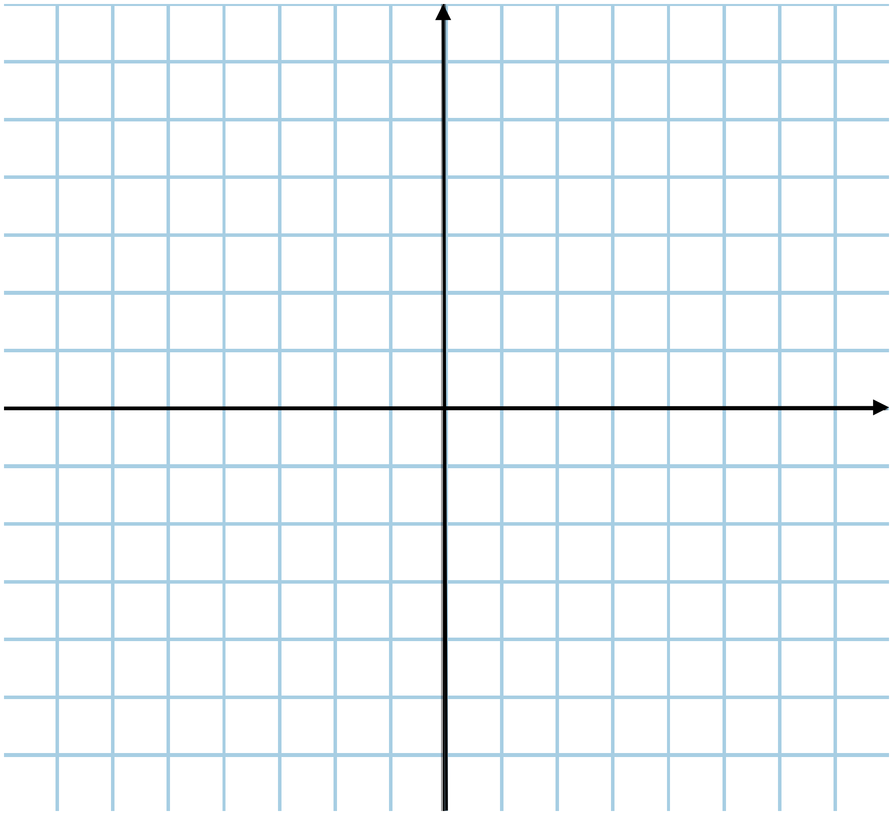
Real and Imaginary	Exponential
$1-i$	
	$e^{i\pi/6}$
	$2e^{i3\pi/2}$
-2	
	$e^{i13\pi/6}$



We can also use polar form to compute *the power* of any number:

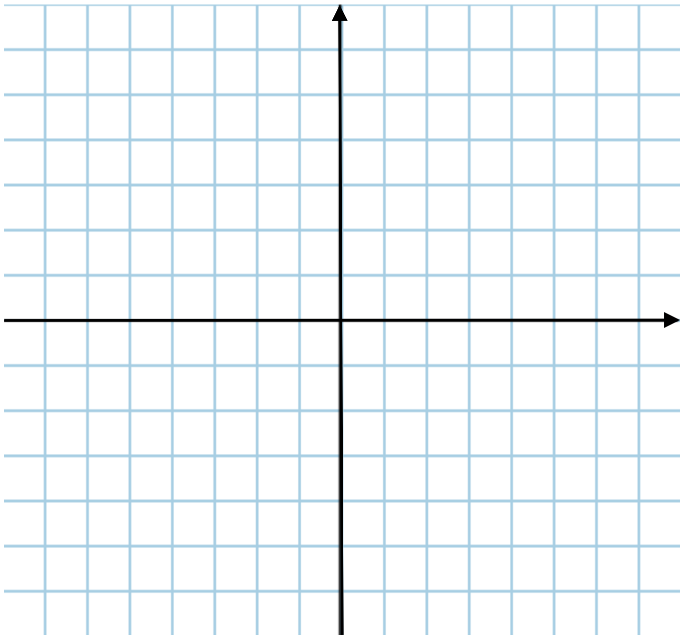
$$z_1 = re^{i\theta}$$

$$z_1^2 =$$



Note that if we add 2 π to the argument, we obtain the same complex number:

$$z = 2e^{i(\frac{\pi}{3}+2\pi)}$$

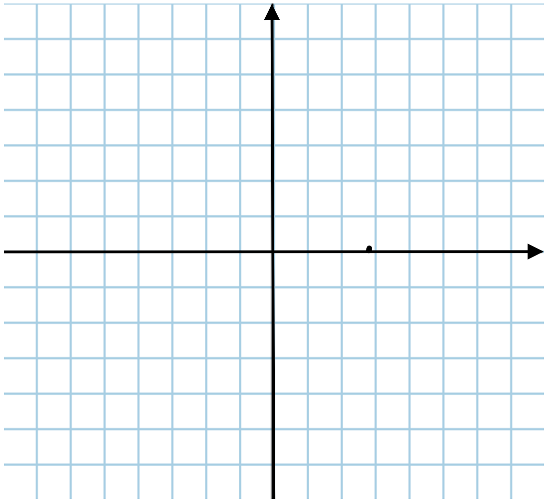


Recall: if the argument lies in the range 0 < Arg z < 2 π then we say that it is the *principle argument.*

Roots of complex numbers

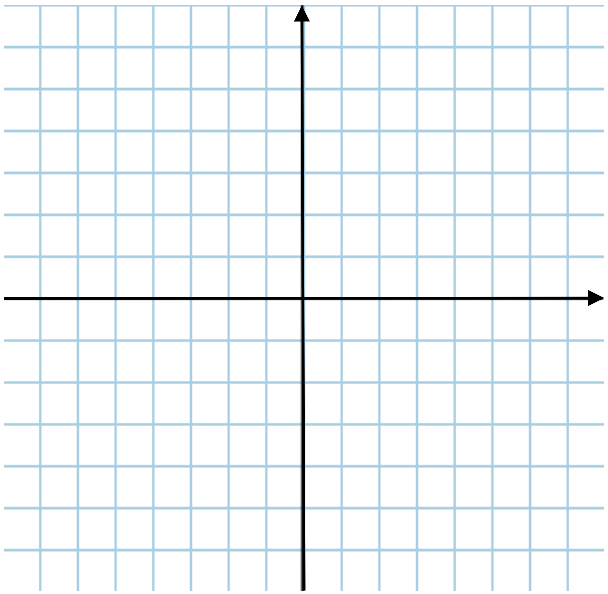
Example:

Find the cube root of -1.



<u>Procedure:</u> To find the nth root of a complex number w,
i.e. to solve

$$z^n = w$$



1. Write w in polar form

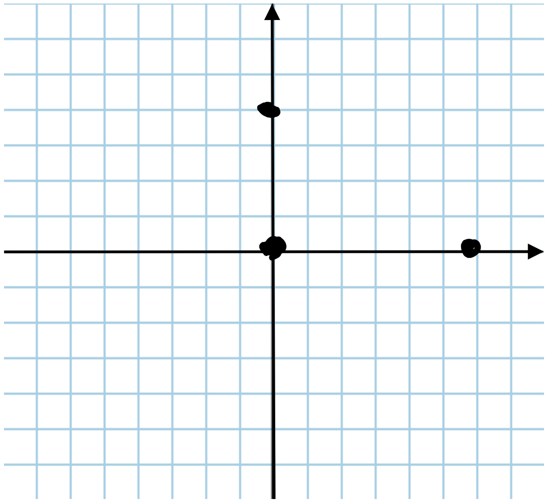
2. Add $2\pi m$ to the argument

3. Take the nth root on both sides

4. There will be n solutions

Example:

Find all fourth roots of 16.



Write 16 in polar form:

Add $2\pi m$ to the argument:

Then the m^{th} solution is:

So the roots are:

Complex numbers and trigonometry

We can express sine and cosine in terms of complex numbers in exponential form:

$$\begin{aligned}\cos\theta &= \frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right)\\ \sin\theta &= \frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right)\end{aligned}$$