Differential Equations

Differential Equations	[Textbook: 7.1]
Separable Equations	[Textbook: 7.3]
Integrating factors (exponential growth and decay)	[Textbook: 7.4]

Differential Equations

Differential equations are one of the tools for modelling the nature.

They occur in physics, chemistry and biology.

All laws of nature are formulated in terms of differential equations

Differential equation is an equation which connects function and its derivatives

Differential equations appear whenever *rates of change* are needed to model some process.

Example: Growth of bacteria

N = Number of bacteria

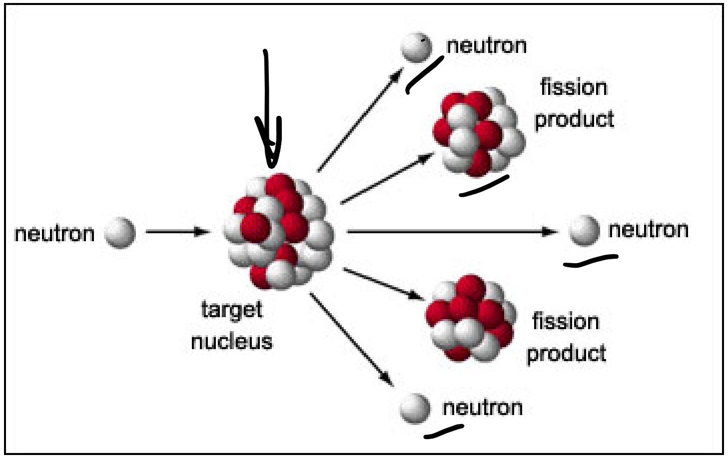
$$\frac{dN}{dt} = kN$$

$\frac{dN}{dt}$ = rate of increase in bacteria

"The mathematics of uncontrolled growth are frightening. A single cell of the bacterium E. coli would, under ideal circumstances, divide every twenty minutes. That is not particularly disturbing until you think about it, but the fact is that bacteria multiply geometrically: one becomes two, two become four, four become eight, and so on. In this way it can be shown that in a single day, one cell of E. coli could produce a super-colony equal in size and weight to the entire planet Earth."

Michael Crichton (1969) *The Andromeda Strain,* Dell, N.Y. p247

Nuclear fission



N = Number of neutrons

$$\frac{dN}{dt}$$ = rate of increase of neutrons

$$\frac{dN}{dt} = kN$$

Newton's law of cooling

The rate of change of the temperature of a hot body is proportional to the temperature difference between the body and its surroundings

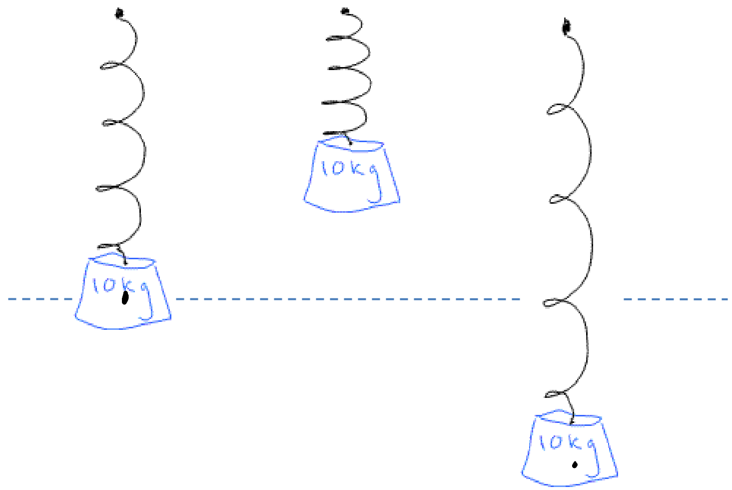
Temperature T

Rate of change of temperature $\frac{dT}{dt}$

Ambient/surrounding temperature T_0

$$\frac{dT}{dt} = -k(T - T_0)$$

Mass on a spring



Newton's second law: $$m\frac{d^2x}{dx^2} = F$$

Hooke's law: $$F = -kx$$

$$\frac{d^2x}{dx^2} + \frac{k}{m}x = 0$$

Notation

Many of these equations have the same form.

$$\frac{dT}{dt} = -k(T - T_0) \qquad \frac{dN}{dt} = +r(N - N_0)$$

Throughout the next few lectures we will use *x* as the independent variable and *y(x)* as the *solution* to the DE.

$$\frac{dy}{dx} = -k(y - y_0)$$

Derivatives with respect to x can also be written using “dash” notation:

$$y' \equiv \frac{dy}{dx}$$

Terminology

The *order* of a differential equation is the highest order of the derivative in the equation.

The *degree* of a differential equation is the power of the highest derivative.

A *linear DE* is any linear relation between the function and its derivatives. That is, anything of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

Examples:

Equation	Order	Degree	Linear?
$\frac{d^2 y}{dx^2} = -ky$	2	1	Yes
$\left(\frac{dy}{dt}\right)^2 + ky = \sin 2t$	1	2	No
$1 + \left(\frac{d^3 y}{dx^3}\right)^4 = 2\frac{d^2 y}{dx^2}$	3	4	No

Once the DE is specified the task is to find a *solution* to the DE.

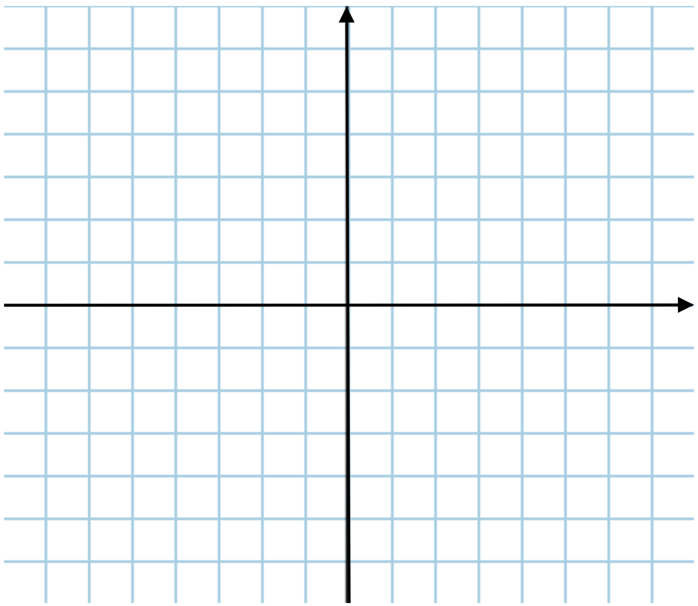
E.g. A solution to

$$\frac{dy}{dx} = y$$

is

$$y(x) =$$

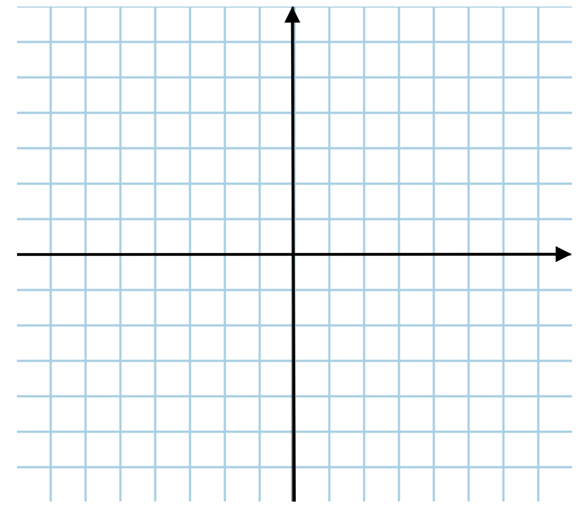
Check that this is the solution:



Once the solution has been found the DE can be graphed as a curve on x-y axes

We can *redefine* familiar functions as being solutions to particular DEs.
E.g. A solution to the DE

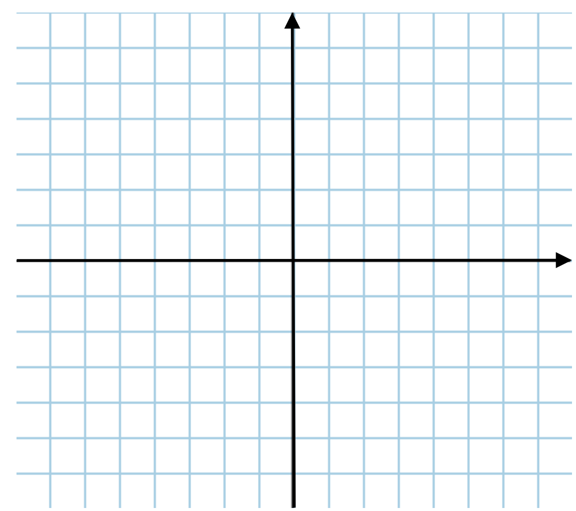
$$\frac{d^2y}{dx^2} = -y$$



is $y(x) = \cos x$

So we can *define* cos x as
"The *even* solution to

$$y'' = -y$$



(**sin x is the *odd solution*)**

Often we can find more than one solution that satisfies the DE.
e.g.

		solutions	general solution
1st order	$\frac{dy}{dx} = 2x$	$y(x) = x^2$	
2nd order	$\frac{d^2y}{dx^2} = 1$	$y(x) = \frac{1}{2}x^2 \quad ; \frac{1}{2}x^2 + 2x + 3$	
2nd order	$\frac{d^2y}{dx^2} = -y$	$y(x) = \sin x \quad ; \quad \cos x$	
3rd order	$\frac{d^3y}{dx^3} = 6$	$y(x) = x^3 + 2x$	

For linear DEs, we can add any multiples of two solutions and the result is also a solution.

Find the general solution for the following DE.

$$\frac{d^3y}{dx^3} = 6$$

In this course we will examine three classes of differential equations

1. Directly integrable differential equations

2. First order differential equations

3. Linear second order differential equations

Directly Integrable differential equations

Any differential equation of the form

$$\frac{d^n y}{dx^n} = f(x)$$

can be solved by *integrating.*

Example:
Find the general solution to the equation

$$\frac{d^2 y}{dx^2} = 2x + 1$$

In this course we consider three classes of differential equations

1. Directly integrable differential equations

2. First order differential equations

3. Linear second order differential equations

First order differential equations
Two types:

1. Separable

$$\frac{dy}{dx} = F(x)G(y)$$

2. Linear

$$\frac{dy}{dx} + p(x)y = q(x)$$

Separable first order DEs:

We can solve anything that looks like:

$$\frac{dy}{dx} = F(x)G(y)$$

Example: Find the general solution to

$$\frac{dy}{dx} = x^2 y$$

Example: Find the general solution to

$$(x^2+3)\frac{dy}{dx}=xy$$

Example: Find the solution to

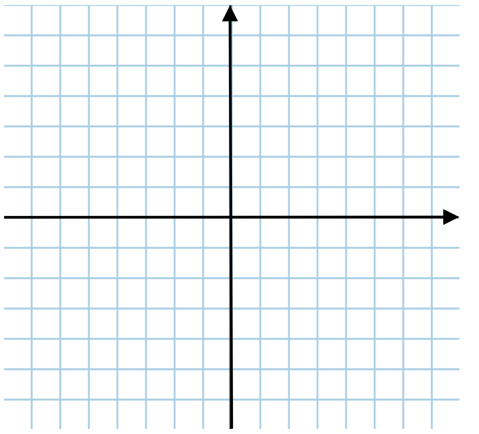
$$\frac{dy}{dx} = \frac{x}{y}$$

Subject to the *initial condition* y(0) = 3.

Example: Find the general solution to

$$\frac{dy}{dx} = \cos^2 y \cos x$$

Final (important) example:
The equation for exponential growth/decay is *separable:*



$$\frac{dy}{dx} = ky$$

A bacteria culture initially contains 200 cells and grows at a rate proportional to its size, *N*. After an hour the population has increased to 600.

(a) Find an expression for the number of bacteria after *t* hours.

(b) Find the number of bacteria after 5 hours.

$$\frac{dN}{dt} = kN$$

Example: Find the general solution to

$$x\frac{dy}{dx}+y=3x^2$$

Example: Find the general solution to

$$x\frac{dy}{dx}+2y=x$$

Linear first order Des

The *standard form* of a linear DE is

$$\frac{dy}{dx} + p(x)y = q(x)$$

Strategy: multiply by a special function that simplifies the DE.
This function is called an *integrating factor*.

$$I(x)\frac{dy}{dx} + I(x)p(x)y = I(x)q(x)$$

$$\frac{dI(x)y}{dx} = I(x)q(x) \Rightarrow I(x)y = \int I(x)q(x)dx$$

$$I(x)\frac{dy}{dx} + \frac{dI(x)}{dx}y = I(x)q(x)$$

$$\frac{dI(x)}{dx} = I(x)p(x) \Rightarrow \frac{dI(x)}{I(x)} = p(x)dx \Rightarrow \ln I(x) = \int p(x)dx \quad I(x) = e^{\int p(x)dx}$$

Steps to solve a linear DE in the form : $$\frac{dy}{dx} + p(x)y = q(x)$$

1. Find the integrating factor $I(x) = e^{\int p(x)dx}$

2. Multiply the DE by I(x)

3. Check that the equation reduces to $\frac{d}{dx}(Iy) = I(x)q(x)$

4. Integrate this to find y(x):

$$y(x) = \frac{1}{I(x)}\left(\int I(x)q(x)dx + C\right)$$

Example: Find the general solution to

$$\frac{dy}{dx} + p(x)y = q(x)$$

$$\frac{dy}{dx} + \frac{y}{x} = 2x^2$$

Example: Find the general solution to

$$x\frac{dy}{dx} - 2y = x^2$$

Example: Newton's law of cooling

The rate of change in temperature of a body is proportional to the difference between the temperature of the body and that of the environment. A cup of coffee has an initial temperature of 70^{o}C. If the ambient temperature is 25 ^{o}C, and after 5 minutes the temperature is 50 ^{o}C, at what time will the coffee become exactly body temperature?