Mathematics 2 33230

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Assessments

Assessments are based on six Skills *Tests hosted on Canvas:*
The Skills tests cover and overlap four Mastery test areas weighted as

Mastery Testing area 1:	5%	
Mastery Testing area 2:	15%	
Mastery Testing area 3:	15%	50%
Mastery Testing area 4:	15%	
Final Exam:		50%

If you achieved at least 40% in the final exam and your overall mark is at least 50% then congratulations! You have passed the subject.

Content of the Mathematical Component

- Linear Algebra

✓ General solution of linear systems
✓ Determinants
✓ Eigenvalues and eigenvectors
✓ Linear transformations

- Multivariable Calculus

✓ Partial derivatives
✓ Maximum & Minimum
✓ Directional Derivative
✓ Optimization
✓ 2D Integration
✓ 3D Integration

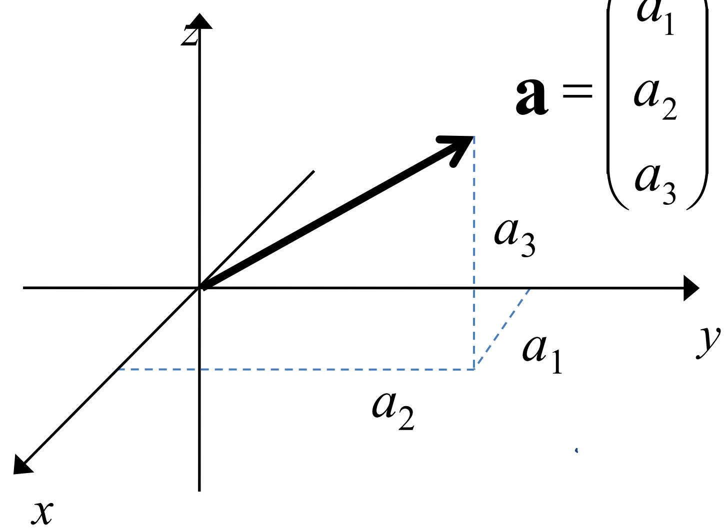
Matrices
(A Brief Review)

Matrices Review

A matrix is a rectangular array of numbers.

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 0 \\ 9 & 1 & 2 \\ 1 & 3 & 4 \end{pmatrix} \qquad \mathbf{M} = \begin{pmatrix} 2 & 6 & -2 & 3 \\ -1 & 5 & 0 & 7 \end{pmatrix} \qquad \mathbf{M} = \begin{pmatrix} 1 & -2 & 4 \\ 5 & 7 & -1 \\ 2 & 8 & 0 \\ -2 & 5 & -2 \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 5 \\ 2 \\ -3 \end{pmatrix}$$

$$\mathbf{v} = \begin{pmatrix} -1 & 4 & -2 \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$



$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

Matrices Review

The *elements* of a matrix $\mathbf{A}$ are often written

$$a_{ij}$$

i^{th} row

j^{th} column

$$[A]_{ij} = a_{ij}$$

A matrix has order m x n if it has m rows and n columns.

Eg:

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 0 \\ 9 & 1 & 2 \\ 1 & 3 & 4 \end{pmatrix} \qquad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

- (1 x n) matrix is known as a *row vector;* $\mathbf{v} = \begin{pmatrix} -1 & 4 & -2 \end{pmatrix}$
- (m x 1) matrix is called a *column vector*. $\mathbf{v} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

Two matrices are said to be ***equal*** if they have the same sizes and all their elements are equal.

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 0 \\ 9 & 1 & 2 \\ 1 & 3 & 4 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 1 & -2 & 0 \\ 9 & 1 & 2 \\ 1 & 3 & 4 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & -2 & 0 \\ 9 & 1 & 2 \\ 1 & 5 & 4 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 1 & -2 & 0 & 2 \\ 9 & 1 & 2 & 1 \\ 1 & 3 & 4 & 3 \end{bmatrix}$$

$$\mathbf{E} = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 9 & 1 & 2 & 0 \\ 1 & 3 & 4 & 0 \end{bmatrix}$$

Types of square matrices:

An upper triangular matrix has the form $$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

A lower triangular matrix has the form $$\begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ b_{21} & b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

A diagonal matrix has the form $$\begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix}$$

Matrices Review

If two matrices have *the same sizes*, they may be added or subtracted.

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 2 & -3 & 6 & 1 \\ -1 & 1 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 3 & 5 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix} =$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 3 & 6 & 1 \\ -1 & 1 & 2 & 1 \end{pmatrix} =$$

Matrices Review

It follows that matrices are *commutative* and *associative* under addition:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$$

A zero matrix is any matrix with all elements equal to zero, and is usually written **0:**

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A} = \mathbf{A}$$

Matrices Review

It is straightforward to multiply matrices by a scalar quantity *k*:

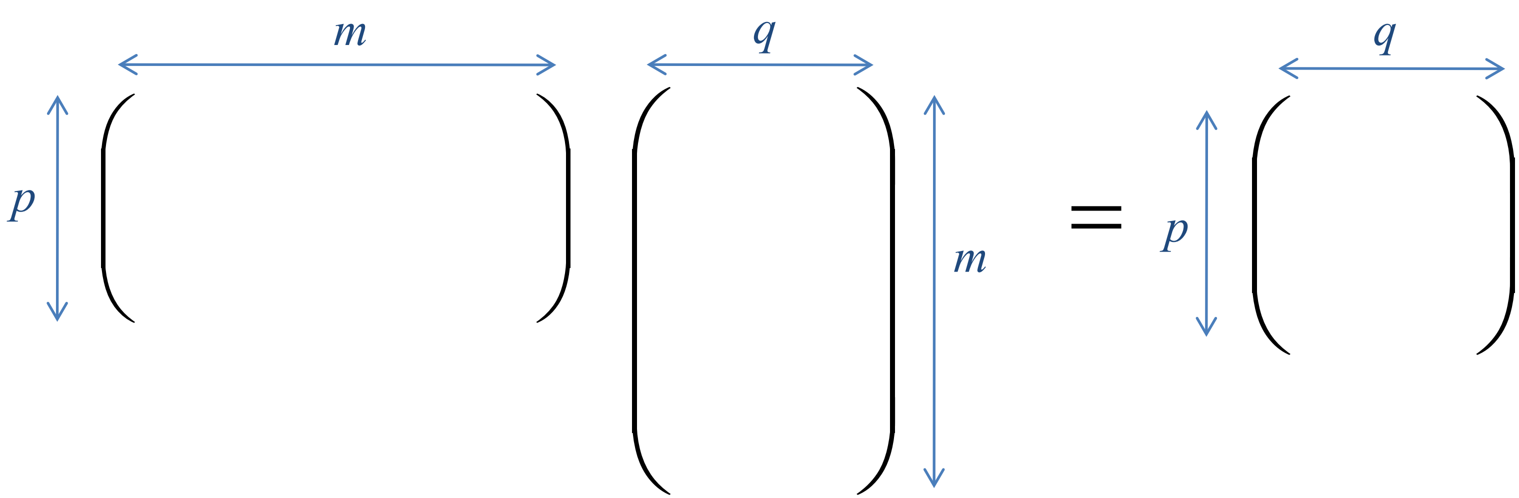
$$k\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2k & k & 0 \\ k & k & k \\ 0 & k & 2k \end{pmatrix}$$

Multiplying two matrices together is more complicated.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

Multiplication of Matrices

Rule: the number of columns of the first matrix must equal to the number of rows of the second.



Multiplication of two matrices can be written as a sum:

$$[\mathbf{AB}]_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

Multiplication of Matrices

$$\begin{pmatrix} 1 & -2 & 0 \\ \boxed{0 \quad 1 \quad 2} \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 2 & 0 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -2 & -5 \\ 5 & & \\ 8 & & \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 2 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 2 & 0 & 2 \end{pmatrix}$$

Matrices Review

If the matrices have the “wrong” dimensions for multiplication, then the product *does not exist*. E.g:

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 3 & 2 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 6 & 1 \\ -1 & 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 3 \end{pmatrix}$$

Matrices Review

Matrix multiplication is *associative*

$$\mathbf{A(BC) = (AB)C}$$

but is *non-commutative*. Usually,

$$\mathbf{AB \neq BA}$$

$$\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} =$$

Because **AB ≠ BA**, we must always specify on *which side* we are doing the matrix multiplication

Exercise:

$$\mathbf{X} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$$

Multiplying **X** on the left by **A**:

$$\mathbf{AX} = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} =$$

Multiplying **X** on the right by **A**:

$$\mathbf{XA} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix} =$$

Matrices Review

Matrix multiplication has some other "strange" properties:

1. $\mathbf{AB} = \mathbf{0}$ does not necessarily mean that $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$

$$\mathbf{AB} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

2. $\mathbf{AB} = \mathbf{AD}$ does not necessarily mean that $\mathbf{B} = \mathbf{D}$

$$\mathbf{AD} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ -3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Matrices Review

The identity matrix $\mathbf{I}$ is the matrix with the property: $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

The Inverse of a matrix

The inverse of a square matrix $\mathbf{A}$ (written $\mathbf{A}^{-1}$) has the property that

$$\mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I} \qquad \text{and} \qquad \mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}$$

A matrix has *a unique* inverse.

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad \det \mathbf{A} = |\mathbf{A}| = ad - bc$$

Find the inverse of matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

The *transpose* of a matrix is obtained by interchanging the rows and the columns:

$$\mathbf{A} = \begin{pmatrix} 2 & 6 & -2 & 3 \\ -1 & 5 & 0 & 7 \end{pmatrix}$$

$$\mathbf{A}^T = \begin{pmatrix} 2 & -1 \\ 6 & 5 \\ -2 & 0 \\ 3 & 7 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 2 & 1 \\ 5 & 0 \\ -1 & 3 \end{pmatrix}$$

If A and B have the same order,

$$(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$$

$$(\mathbf{A}^T)^T = \mathbf{A}$$

$$(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$$

Using the rules for addition, subtraction, multiplication and inverses, as well as the special matrices **I** and **0**, we can re-arrange matrix equations.

Example: Suppose **A**, **B**, **C** and **X** are all 2 x 2 matrices, where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Re-arrange the equation

$$\mathbf{A} + 2\mathbf{BX} = \mathbf{C}$$

to find **X**.

Example: $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{X}$ are all 2 x 2 matrices, with

$$\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 4 & 2 \\ 0 & -1 \end{bmatrix}$$

Re-arrange the equation

$$\mathbf{X} + \mathbf{XB} = \mathbf{C}$$

to find **X**.

Summary: Rules of matrix algebra

Equality:

If $\mathbf{A} = \mathbf{B}$ and $\mathbf{A} = \mathbf{C}$ then $\mathbf{B} = \mathrm{C}$

Addition:

$\mathbf{A+B = B + A}$
$\mathbf{(A + B) + C = A + (B + C)}$

Multiplication:

$\mathbf{(A + B)\ C = AC + BC}$
$\mathrm{k}\ \mathbf{(A + B)} = \mathrm{k}\mathbf{A} + \mathrm{k}\ \mathbf{B}$
$\mathrm{k}\mathbf{(AB)} = (\mathrm{k}\mathbf{A})\ \mathbf{B}$
$\mathbf{(A\ B)\ C = A(BC)}$

Special matrices:

$\mathbf{AI = IA = A}$
$\mathbf{A + 0 = A}$

If A has an inverse,

$\mathbf{A^{-1}\ A = A\ A^{-1} = I}$

Transposes:

$\mathbf{(A+B)^T = A^T + B^T}$
$\mathbf{(AB)^T = B^T A^T}$

Remember that in general:

$\mathbf{AB \neq BA}$
$\mathbf{AB} = \mathbf{0}$ does not necessarily mean that $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$
$\mathbf{AD} = \mathbf{AC}$ does not necessarily mean that $\mathbf{D} = \mathbf{C}$

Linear Equations

- Linear algebra is one of the essential parts of mathematics.
  In short it is the study of the linear equations.

- Linear equation in the variables $x_1, x_2, \ldots, x_n$ is an equation that can be written as

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

- Here a_1, a_2, …a_n, b are real or complex numbers and n is any positive integer number

- The following equations are linear

$$4x_1 - 5x_2 + \sqrt{2} = x_3 \qquad or \qquad 4x_1 - 5(x_2 - 2) = 4x_3$$

- While the following equations are not

$$4x_1x_2 - 5x_2 + x_3 = 2 \qquad or \qquad 4\sqrt{x_1} - 5x_2 - 4x_3 = 2$$

Linear Equations

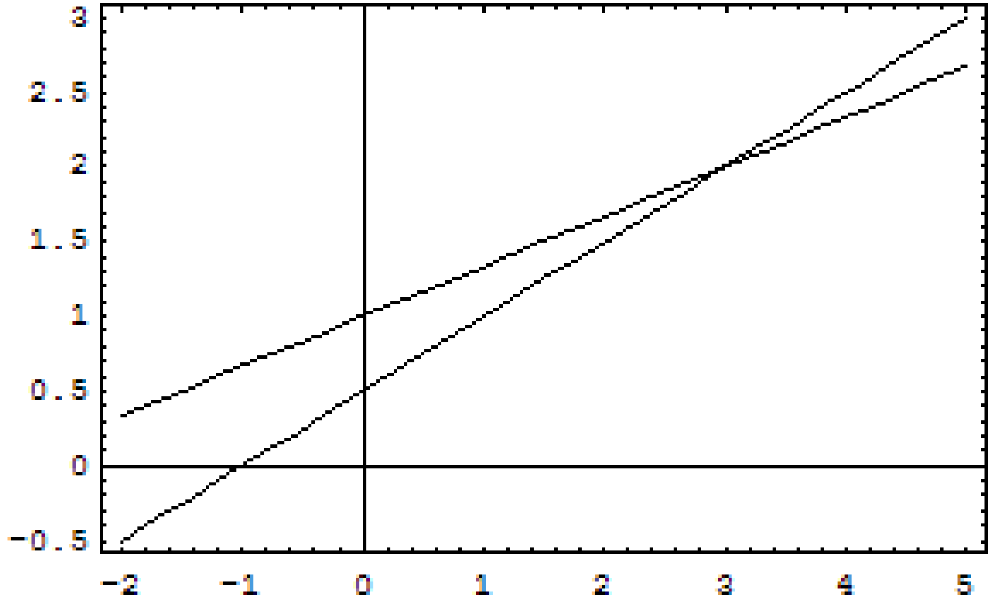
Consider the system of linear equations

$$x_1 - 2x_2 = -1$$
$$-x_1 + 3x_2 = 3$$

The plot of these lines are straight lines in the (x_1, x_2) plane

$$x_2 = \frac{1}{2} + \frac{x_1}{2}$$

$$x_2 = 1 + \frac{x_1}{3}$$



The solution is

$$x_1 = 3$$
$$x_2 = 2$$

Linear Equations

The linear equation can be written as

$$x_1 - 2x_2 = -1 \qquad \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$-x_1 + 3x_2 = 3$$

The matrix form of the linear system

$$\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\mathbf{Ax} = \mathbf{b} \qquad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

Linear Equations

If the inverse of **A** exist then the solution can be written as

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 3 \end{pmatrix} =$$

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$$

$$= \frac{1}{3-2} \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Linear Equations

The system of linear equations can be written in a matrix form

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3 \end{cases} \qquad \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

The Augmented matrix of the linear system is

$$\begin{pmatrix} 1 & -2 & -1 \\ -1 & 3 & 3 \end{pmatrix} \quad R_2 \to R_2 + R_1$$

$$\begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \end{pmatrix} \quad R_1 \to R_1 + 2R_2$$

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$x_1 = 3$$

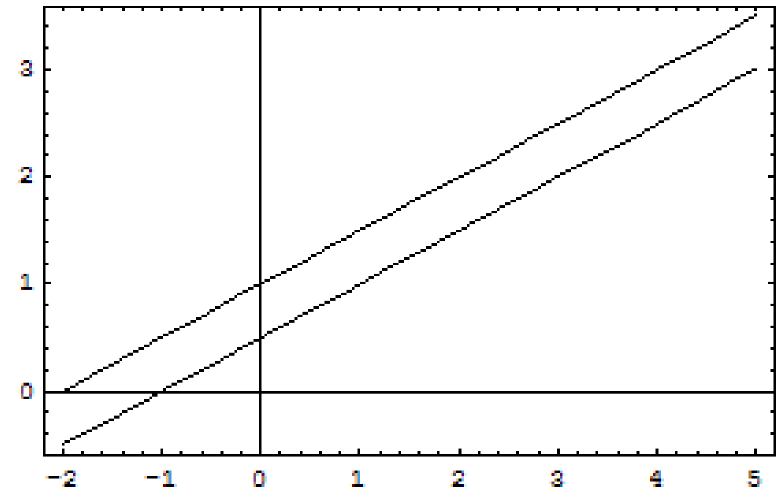
$$x_2 = 2$$

Linear Equations

An other Example

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 2x_2 = 3 \end{cases} \qquad \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & -1 \\ -1 & 2 & 3 \end{pmatrix}$$



$$\begin{pmatrix} 1 & -2 & -1 \\ -1 & 2 & 3 \end{pmatrix} \quad R_2 \to R_2 + R_1$$

$$\begin{pmatrix} 1 & -2 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$x_1 - 2x_2 = 3$

$0 = 2$ Inconsistency

No solutions

Linear Equations

An other Example

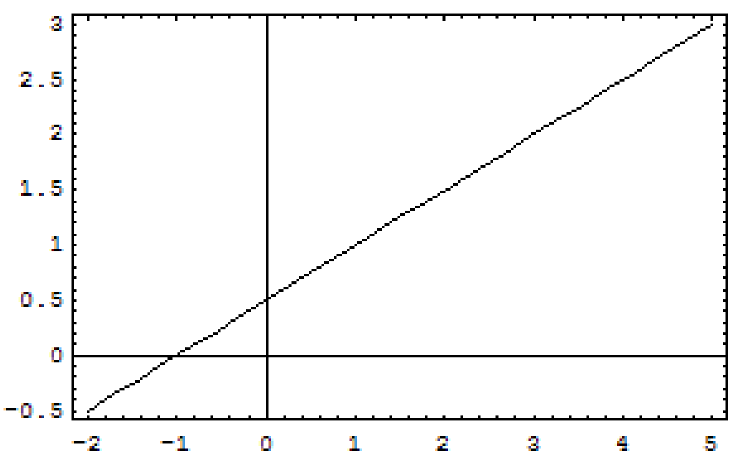
$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 2x_2 = 1 \end{cases}$$

$$\begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & -1 \\ -1 & 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & -1 \\ -1 & 2 & 1 \end{pmatrix} \quad R_2 \to R_2 + R_1$$

$$\begin{pmatrix} 1 & -2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$



$$\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1 - 2x_2 = -1 \\ 0 = 0 \end{cases}$$

$$\begin{cases} x_1 = 2x_2 - 1 \\ x_2 \quad \textit{free parameter} \end{cases}$$

$$\begin{cases} x_1 = -1 + 2t \\ x_2 = t \end{cases}$$

Linear Equations

- A number of applications in science, business, economy are linear in nature. This naturally leads to linear systems.

- A number of nonlinear problems can be approximated to be linear This will also lead to linear systems.

- A number of problems in Statistics, Operational Analysis, Optimization, leads to linear systems.

- Solutions of partial differential equations, ordinary differential equations, finite-difference equations can lead to the system of linear equations.

- In short Linear Algebra has vast applications in many branches of Sciences and Mathematics

- Digital Image processing

Linear Equations

- A system of linear equations (or simply a linear system) is a collection of one or more linear equations involving the same variables

$$
\begin{aligned}
2x_1 - 3x_2 + 5x_3 &= -2 \\
4x_1 + 7x_2 - 9x_3 &= 8
\end{aligned}
$$

- In general

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

- A solution of the system is a list $x_1, x_2, \ldots, x_n$ of numbers that makes each equation a true statement.

Linear Equations

- A system of linear equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots & \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

- The set of coefficients a_{ij} form the matrix of the linear system

$$
\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}
$$

- The element a_{ij} located in the ith row and the jth column of matrix **A**. The size of the matrix is $m \times n$

Matrix Form of Linear Equations

- Denoting

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

- The system can be written in the matrix form

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

- Two fundamental questions about a linear system

1. Does at least one solution exist?
2. Is the solution unique?