# Finding Solutions of Linear Systems

## Gaussian Ellimination

## Inverse of a Square Matrix

# Linear Equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

$$\mathbf{Ax} = \mathbf{b}$$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad Det\ \mathbf{A} = a_{11}a_{22} - a_{12}a_{21}$$

$$Det\ \mathbf{A} \neq 0 \Rightarrow unique\ \ solution$$

$$Det\ \mathbf{A} = 0 \Rightarrow no\,solution\ \ or\ \ \infty\ \ solutions$$

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3 \end{cases} \qquad \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 2x_2 = 3 \end{cases} \qquad \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 2x_2 = 1 \end{cases} \qquad \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

# Solution of linear systems

- Lets consider 3 by 3 linear system

$$\begin{cases} a_{11}x_1 & +a_{12}x_2 & +a_{13}x_3 & = b_1 \\ a_{21}x_1 & +a_{22}x_2 & +a_{23}x_3 & = b_2 \\ a_{31}x_1 & +a_{32}x_2 & +a_{33}x_3 & = b_3 \end{cases}$$

# Solution of linear systems

- Example: 3 by 3 linear system

$$\begin{bmatrix} 1 & -2 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 4 \end{bmatrix} \Leftrightarrow \begin{cases} x & -2y & +z & =2 \\ & y & -z & =0 \\ & & 2z & =4 \end{cases} \Leftrightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$$

$$z = 2;$$

$$y = z = 2;$$

$$x = 2 + 2y - z = 2 + 4 - 2 = 4;$$

$$\mathbf{x} = \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}$$

# Solution of linear systems

- Example: Find the solution of the linear system

$$\begin{cases} 4x & -2y & +5z & =16 \\ x & +y & & =0 \\ x & +3y & -2z & =-6 \end{cases} \qquad \begin{bmatrix} 4 & -2 & 5 & 16 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & -2 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 3 & -2 & -6 \\ 4 & -2 & 5 & 16 \end{bmatrix} \qquad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix} \qquad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & -2 & -6 \\ 0 & -6 & 5 & 16 \end{bmatrix}$$

$$\begin{matrix} R_2 \to R_2 / 2 \\ R_3 \to R_3 + 6R_2 \end{matrix} \qquad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -3 \\ 0 & -6 & 5 & 16 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & -1 & -2 \end{bmatrix}$$

$$-z = -2 \qquad y = -3 + z = -3 + 2 = -1$$

$$z = 2 \qquad x + y = 0$$

$$y - z = -3 \qquad x = -y = 1$$

# Solving Linear Systems

- Augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

$$\mathbf{Ax} = \mathbf{b}$$

- Elementary row operations

1. Add to one row a multiple of another row
2. Interchange two rows
3. Multiply all elements of a row by a nonzero constant

- A nonzero row/column is a row/column which contains at least one nonzero entry
- A **leading entry** of a row refers to the leftmost nonzero entry

- A **leading entry** of a column refers to the uppermost nonzero entry

# Row Reduction and Echelon Form

- A rectangular matrix is in echelon form (**EF**) if it has the following properties

1. All nonzero rows are above any rows of all zeros
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it
3. All entries in a column below a leading entry are zeros

- A matrix in echelon form with the following additional conditions is called reduced echelon form (**REF**) if

$\bullet - nonzero\ number$

$* - any\ number$

1. The leading entry in each non zero row is 1
2. Each leading 1 is the only nonzero entry in its column

$$\begin{bmatrix} 2 & -4 & -6 & 1 \\ 0 & 1 & 8 & -3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad \begin{bmatrix} \bullet & * & * & * \\ 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

# Row Reduction and Echelon Form

$$\begin{bmatrix} \bullet & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \bullet & * & * & * & * \\ 0 & 0 & 0 & 0 & \bullet & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \bullet & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & * & * & 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

- Echelon form — Reduced echelon form **REF** or **(RREF)**

- A pivot position in a matrix **A** is a location in **A** that corresponds to the leading 1 in the reduced echelon form

- The dots identify the pivots positions

- Pivots are leading elements in a row for a matrix in echelon form

# Row Reduction

An Example: Reduce given matrix into **REF** form

- Step 1: Start from the leftmost nonzero column,
  The pivot position is at the top

pivot column
$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

- Step 2: Select a nonzero entry in the pivot column as a pivot. If necessary interchange rows to move this entry into the pivot position

pivot
$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \qquad R_1 \leftrightarrow R_3$$

# Row Reduction

An Example:

- Step 3: Use row operations to create zeroes in all positions below the pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \quad R_2 \to R_2 - R_1 \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

- Step 4: Cover the row containing the pivot position and cover all rows if any above it. Apply the Steps 1-3 to the remaining matrix.
- Repeat the process until there are no more nonzero rows to modify

- Pivot $\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$

Pivot column: c2

# Row Reduction

An Example:

- Step 4: Use row operations to create zeroes in all positions below the pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} R_2 \to R_2 / 2 \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} R_3 \to R_3 - 3R_2$$

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

This is in echelon form

# Reduced Echelon Form

- Step 5: Start with the rightmost pivot create zeros above each pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_3 \\ R_1 \to R_1 - 6R_3 \\ \phantom{x} \end{matrix} \begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} R_1 \to R_1 + 9R_2 \begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad R_1 \to R_1 / 3$$

This is reduced echelon form

- Steps 1-4 are called the **forward phase**;
- Step 5 is called **backward phase**

# Row Reduction

- An Example: Find **EF** and **RREF** form of the matrix

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \quad R_1 \leftrightarrow R_4 \quad \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

- All elements below the pivot must be zero

$$R_2 \rightarrow R_2 + R_1$$
$$R_3 \rightarrow R_3 + 2R_1$$

$$R_3 \rightarrow R_3 - 5/2\, R_2$$
$$R_4 \rightarrow R_4 + 3/2\, R_2$$

$$R_3 \leftrightarrow R_4$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# Row Reduction

- An Example

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} \bullet & * & * & * & * \\ 0 & \bullet & * & * & * \\ 0 & 0 & 0 & \bullet & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- A rectangular matrix is in echelon form if it has the following properties

1. All nonzero rows are above any rows of all zeros
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it
3. All entries in a column below a leading entry are zeros

- A matrix in echelon form with the following additional conditions is called reduced echelon form if

1. The leading entry in each non zero row is 1
2. Each leading 1 is the only nonzero entry in its column

# Row Reduction

- An Example: Find **EF** and **RREF** form of the matrix

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3/(-5) \quad \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- All elements above pivots must be zero

$$\begin{matrix} R_2 \to R_2 + 6R_3 \\ R_1 \to R_1 + 9R_3 \end{matrix} \begin{bmatrix} 1 & 4 & 5 & 0 & -7 \\ 0 & 2 & 4 & 0 & -6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 \to R_1 - 2R_2 \quad \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 2 & 4 & 0 & -6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \to R_2/2 \quad \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# Solution of linear systems

- Example: An augmented matrix reduced into echelon form (EF) . Find all solutions of corresponding linear system

$$\begin{bmatrix} 1 & 1 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Associate linear system has the form

$$\begin{bmatrix} x_1 & +x_2 & -5x_3 & =1 \\ & x_2 & +x_3 & =4 \\ & & 0 & =0 \end{bmatrix}$$

- Variables $x_1$ and $x_2$ : **basic or leading variables**; $x_3$ is a **free variable**
- To write the solution express basic variables $x_1$ and $x_2$ in terms of free variable $x_3$.

$$x_2 = 4 - x_3, \; x_1 = 1 - x_2 + 5x_3$$

$$x_1 = 1 - x_2 + 5x_3 = 1 - (4 - x_3) + 5x_3 = -3 + 6x_3$$

# Solution of linear systems

- Example: An augmented matrix reduced into reduced echelon form (REF) . Find all solutions of corresponding linear system

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Associate linear system has the form

$$\begin{bmatrix} x_1 & & -5x_3 & =1 \\ & x_2 & +x_3 & =4 \\ & & 0 & =0 \end{bmatrix}$$

- Variables $x_1$ and $x_2$ corresponding the pivot columns are called the **basic or leading variables**; $x_3$ is called a **free variable**
- To write the solution express basic variables $x_1$ and $x_2$ in terms of free variable $x_3$.

$$x_1 = 1 + 5x_3, \quad x_2 = 4 - x_3$$

$$x_3 \quad is \ free$$

# Solution of linear systems

- Example: For which value of h the system is inconsistent ?

$$\begin{cases} x \quad +5y \quad -3z \quad = -4 \\ -x \quad -4y \quad +z \quad = 3 \\ -2 \quad -7y \quad = h \end{cases} \qquad \begin{bmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{bmatrix}$$

$$\begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 + 2R_1 \end{matrix} \quad \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{bmatrix}$$

$$R_3 \to R_3 - 3R_2 \quad \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{bmatrix} \quad h \neq 5$$

$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \neq 0 \end{bmatrix} \quad 0 = h - 5 \neq 0$$

Inconsistency, no solution

# Solution of linear systems

- Example: Find all solutions of linear system for h=5

$$R_3 \to R_3 - 3R_2 \quad \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{bmatrix} \quad h=5$$

$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad z - is\ free \text{ var} \qquad x + 5y - 3z = -4$$

$$y - 2z = -1 \qquad x = -4 - 5y + 3z$$

$$y = -1 + 2z \qquad x = -4 - 5(-1 + 2z) + 3z$$

$$x, y - are\ basic \text{ var} \qquad x = -4 + 5 - 10z + 3z = 1 - 7z$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-7z \\ -1+2z \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -7z \\ 2z \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -7 \\ 2 \\ 1 \end{pmatrix}$$

# Solution of linear systems

- Example:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-7z \\ -1+2z \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -7z \\ 2z \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -7 \\ 2 \\ 1 \end{pmatrix}$$

$$z = t \in R$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -7 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} -7 \\ 2 \\ 1 \end{pmatrix}$$

$$\mathbf{r} = \mathbf{a} + \mathbf{v}t$$

Equation of a line though the point **a** along the direction **v**

# For 3 x 3 systems

$$\begin{aligned} x_1 &\quad -2x_2 &\quad +2x_3 &\quad = &\quad 1 \\ x_1 &\quad -2x_2 &\quad +x_3 &\quad = &\quad 1 \\ 3x_1 &\quad -2x_2 &\quad +x_3 &\quad = &\quad 1 \end{aligned}$$

Put the equations in matrix form:

$$\begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 1 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 1 \\ 3 & -2 & 1 \end{bmatrix}$$

If we can find an inverse of this matrix, we can solve this system:

Finding the inverse of a 3 x 3 matrix

The inverse can be found by using a series of row_operations:

1. Multiplying a row by a scalar

2. Adding or subtracting two rows.

3. Swapping any two rows

$$\begin{array}{rrrcl} x_1 & -2x_2 & +2x_3 & = & 1 \\ x_1 & -2x_2 & +x_3 & = & 1 \\ 3x_1 & -2x_2 & +x_3 & = & 1 \end{array}$$

To find the inverse:

1. First write the matrix of the linear system:

$$\begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 1 \\ 3 & -2 & 1 \end{bmatrix}$$

2. "Augment" the Identity matrix:

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 3 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

3. Use row operations to transform the left half into the identity matrix **I**. The right half will then be the inverse.

Example: Solve the system of equations

$$\begin{aligned} x_1 - 2x_2 + 2x_3 &= 1 \\ x_1 - 2x_2 + x_3 &= 1 \\ 3x_1 - 2x_2 + x_3 &= 1 \end{aligned}$$

Augmented matrix:

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 3 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 3R_1 \end{array} \Leftrightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 4 & -5 & -3 & 0 & 1 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 4 & -5 & -3 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right] \quad R_3 \to (-1)\times R_3 \quad \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 4 & -5 & -3 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right] \begin{array}{c} R_2 \to R_2 + 5R_3 \\ R_1 \to R_1 - 2R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & -1 & 2 & 0 \\ 0 & 4 & 0 & 2 & -5 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right] \quad R_1 \to 2R_1 - R_2 \quad \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 0 & -1 & 1 \\ 0 & 4 & 0 & 2 & -5 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right] \begin{array}{c} R_1 \to R_1 / 2 \\ R_2 \to R_2 / 4 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1/2 & 1/2 \\ 0 & 1 & 0 & 1/2 & -5/4 & 1/4 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right] \Rightarrow \mathbf{A}^{-1} = \left[\begin{array}{ccc} 0 & -1/2 & 1/2 \\ 1/2 & -5/4 & 1/4 \\ 1 & -1 & 0 \end{array}\right]$$

So the solution to

$$\begin{aligned} x_1 &\quad -2x_2 &\quad +2x_3 &\quad = \quad 1 \\ x_1 &\quad -2x_2 &\quad +x_3 &\quad = \quad 1 \\ 3x_1 &\quad -2x_2 &\quad +x_3 &\quad = \quad 1 \end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 1 \\ 3 & -2 & 1 \end{bmatrix}$$

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} 0 & -1/2 & 1/2 \\ 1/2 & -5/4 & 1/4 \\ 1 & -1 & 0 \end{bmatrix}$$

is:

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \begin{bmatrix} 0 & -1/2 & 1/2 \\ 1/2 & -5/4 & 1/4 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/2 \\ 0 \end{bmatrix}$$

Example: Find the inverse of a matrix **A** by a series of *row-operations:*

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2\ R_1$
$R_3 \rightarrow R_3 - R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 3 & -1 & 0 & 1 \end{array}\right]$$

$R_2 \rightarrow R_3$
$R_3 \rightarrow R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}\right]$$

$R_2 \rightarrow R_2 - 3\ R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 5 & -3 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}\right]$$

$$\mathbf{A}^{-1} = \begin{bmatrix} -4 & 3 & -1 \\ 5 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix}$$

$R_1 \rightarrow R_1 - R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 3 & -1 \\ 0 & 1 & 0 & 5 & -3 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}\right]$$

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$