Mathematics 2

Lecture 3: Discrete Random Variables
Binomial Distribution

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Random Variables

- Reference: Devore § 3.1 – 3.3
- Definitions:
  - An **experiment** is any process of obtaining one outcome where the outcome is uncertain.
  - A **random variable** is a numerical variable whose value can change from one replicate of the experiment to another.
- Sample means and sample standard deviations are **random** variables
  - They are different from sample to sample.
  - Population means and standard deviations are not random.

Random Variables - Examples

- Experiment 1: Pick a student at random from the class
    - Let X denote the height of the student

- Experiment 2: Throw a fair dice
    - Let X denote the outcome of the dice.

      $X = 1,2,3,4,5,$ or 6

- Notice that the outcome of both of these events changes every time you take a new sample.

Random Variables

- A random variable can be continuous or discrete.
  - **Continuous random variables** can take any real value, such as measurements.
    - Electrical current, length, pressure, temperature, time voltage, weight etc.
  - **Discrete random variables** are usually counts (whole numbers).
    - Number of scratches on a surface, proportion of defective parts among 1000 tested, number of transmitted bits received in error, number of vehicles on a bridge.

Experiment 1 (Height):

Experiment 2 (Roll of dice):

Random Variables: Exercise

- Decide whether a continuous or a discrete random variable is the best model for each of the following variables.
  - The life time of a biomedical device after implant in a patient.
  - The number of times a transistor in a computer memory changes state in one operation.
  - The strength of a concrete specimen.
  - The number of luxury options selected by an automobile buyer.
  - The proportion of defective solder joints on a circuit board.
  - The weight of an injection-moulded plastic part.
  - The number of particles in a sample of gas.

Discrete Random Variables

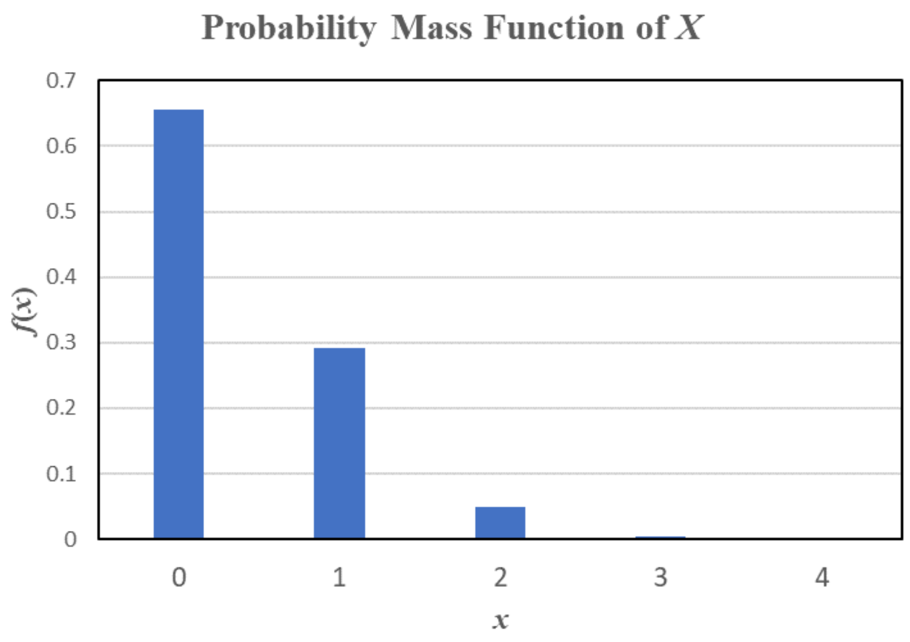
- A random variable *X* is said to be **discrete** if it can only be equal to a finite or countably infinite (e.g. $\mathbb{Z}$) number of distinct values.

- **Example**
  - There is a chance that a bit transmitted through a digital transmission channel is received in error. Let *X* equal the number of bits in the next four bits transmitted in error. Based on past data, we can determine that the probabilities for the possible values of *X* are:

$$P(X=0) = 0.6561$$
$$P(X=1) = 0.2916$$
$$P(X=2) = 0.0486$$
$$P(X=3) = 0.0036$$
$$P(X=4) = 0.0001$$

Discrete Random Variables

- We can plot the **probability mass function** (pmf), the set of probabilities associated with each value that the discrete random variable may take.
  - Sometimes, we have a functional form for the probability mass function.
  - In other cases, we simply have a table of probabilities.
  - For example, the pmf in the example on the last slide is shown below



Discrete Random Variables

- For a discrete random variable *X* with possible values $x_1, x_2, \ldots, x_n$, the probability mass function is

$$f(x_i) = P(X = x_i)$$

where $$0 \leq f(x_i) \leq 1 \qquad \sum_{x_i} f(x_i) = 1$$

- To calculate probabilities using *f(x)*:

$$P(a \leq X \leq b) = \sum_{x=a}^{b} f(x) = \sum_{x=a}^{b} P(X = x)$$

Discrete Random Variables: Example

- For the example on slide 6 what is $P(2 \leq X \leq 3)$? The following probabilities were given:

  $P(X=0) = 0.6561$

  $P(X=1) = 0.2916$

  $P(X=2) = 0.0486$

  $P(X=3) = 0.0036$

  $P(X=4) = 0.0001$

Discrete Random Variables

- The **cumulative distribution function** of a discrete variable *X* is

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$$

- We should note that for discrete distributions

$$P(X \leq x_i) \neq P(X < x_i), \text{ unless } P(X = x_i) = 0$$

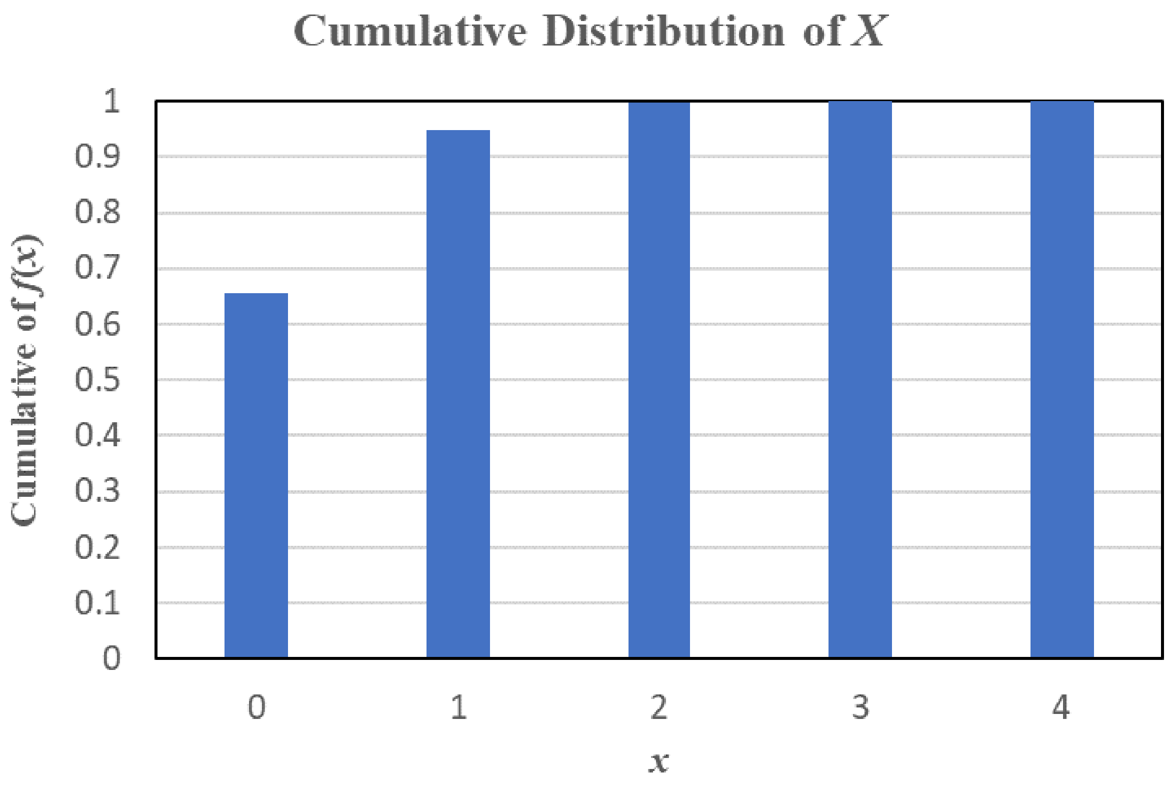
Discrete Random Variables: Example

- For example on slide 6, the cumulative probabilities are found:

Probabilities	Cumulative Probabilities
$P(X=0) = 0.6561$	$P(X \le 0) = 0.6561$
$P(X=1) = 0.2916$	$P(X \le 1) = 0.9477 = (0.6561 + 0.2916)$
$P(X=2) = 0.0486$	$P(X \le 2) = 0.9963 = (0.9477 + 0.0486)$
$P(X=3) = 0.0036$	$P(X \le 3) = 0.9999 = (0.9963 + 0.0036)$
$P(X=4) = 0.0001$	$P(X \le 4) = 1.0000 = (0.9999 + 0.0001)$

Discrete Random Variables: Example

The cumulative distribution function for the example on slide 6 is



Discrete Random Variables: Example

- $P(2 \le X \le 3)$ can be calculated using cumulative distribution functions.

Cumulative Probabilities

$P(X \le 0) = 0.6561$

$P(X \le 1) = 0.9477$

$P(X \le 2) = 0.9963$

$P(X \le 3) = 0.9999$

$P(X \le 4) = 1.0000$

Expected Values

For n = 5000 students let x be the number of courses for which a student is randomly assigned for 5 possible courses. The number of students allocated and the probability mass function is given in the table.

Course	1	2	3	4	5
Allocated	740	2000	960	800	500
Probability p(x)	740/5000 = 0.148	2000/5000 =0.400	960/50000 = 0.192	800/5000 = 0.160	500/5000 = 0.100

What is the expected number of courses assigned?

Expected Values

Course	1	2	3	4	5
p(x)	0.148	0.400	0.192	0.160	0.100

Alternatively, we know ‘x’ and the probability mass function and can find the expected number of courses that will be assigned by:

Expectation and Variance

- Suppose that the possible values of the random variable are x_1, x_2, ..., x_n, and the pmf of X is $f(x)$.
  - Then the **expected value** of X is

$$\mathrm{E}(X) = \sum_{x_i} x_i \times f(x_i)$$

  - The **variance** of X is

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}(X^2) - (\mathrm{E}(X))^2 \\ &= \sum_{x_i} x_i^2 \times f(x_i) - (\mathrm{E}(X))^2\end{aligned}$$

  - Recall that the standard deviation is the **square root of the variance**.

Discrete Random Variables: Example

- For the example on slide 6 what is the expected value, variance and standard deviation?

Note: $P(X=0) = 0.6561$

$P(X=1) = 0.2916$

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$P(X=3) = 0.0036$

$P(X=4) = 0.0001$

Discrete Random Variables: Example

- For the example on slide 6 is the expected value, variance and standard deviation?

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Expectation and Variance: Example

The discrete random variable X has probability mass function

x	8	9	10	11	12
P(X=x)	0.03	0.46	0.22	0.08	0.21

Find the expected value and standard deviation of X.

Expectation and Variance: Example

Binomial Distribution

Reference: Devore § 3.4

- A **binomial experiment** satisfies all of the following
  - There are n independent and identical trials
  - Each trial has only 2 possible outcomes, which we describe as a success or failure
  - P(success) = p and P(failure) = $1 - p = q$, the same for every trial
  - We are interested in X = the number of successes in these n trials
    - $X = 0, 1, \ldots, n-1, n$

- **Example**
  - We toss a fair coin 20 times
    - Let the successful event be that a head appears
    - X = the number of heads in 20 tosses
    - Then $p = 0.5$, $q = 1 - 0.5 = 0.5$, and $n = 20$

Which can be modelled as a Binomial?

- 90% effective poison is given to 10 rats. What is the probability that exactly 9 of them die?

- A laboratory has 8 machines. They break down (independently) 5% of the time. What is the probability that more than one machine brakes down at a particular time?

- A narrow necked bottle contains 3 green and 5 black olives. The bottle is shaken thoroughly and three olives are rolled out, one by one. What is the probability that all three are green?

Binomial Distribution

- We say that X takes a Binomial Distribution
  - This can be shortened to $X \sim \text{Bin}(n, p)$.

- The probability mass function of X is

$$P(X = x) = \binom{n}{x} p^x \times (1-p)^{n-x}$$

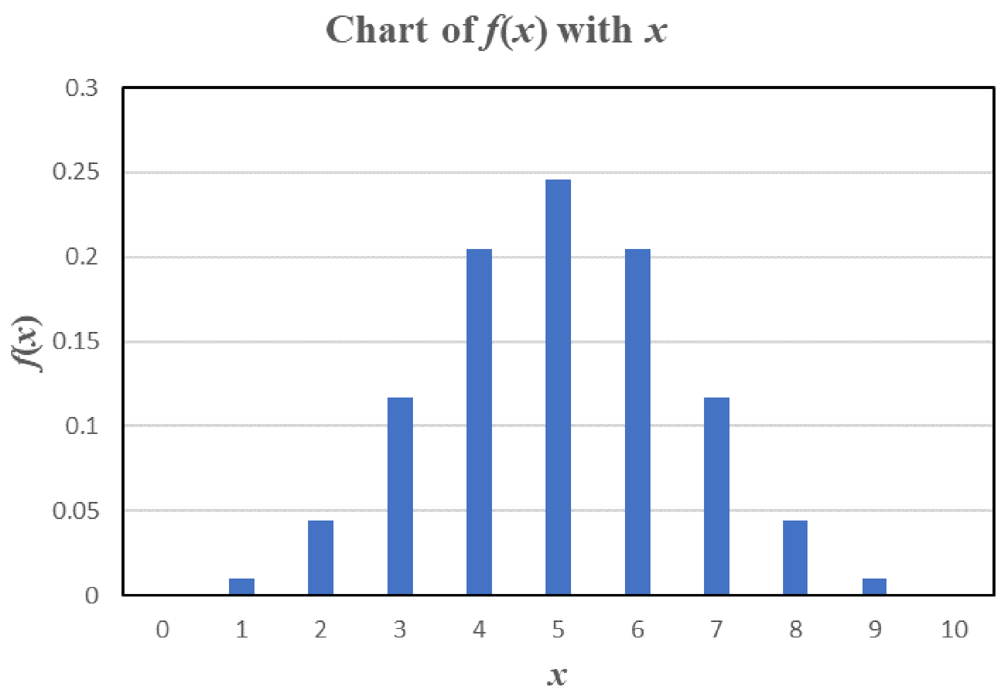
where $x = 0,1,\ldots,n$, $0 < p < 1$, and n is a positive whole number.

- We let $\binom{n}{x} = \frac{n!}{x!(n-x)!}$ and $x! = x \times (x-1) \times \ldots \times 3 \times 2 \times 1$

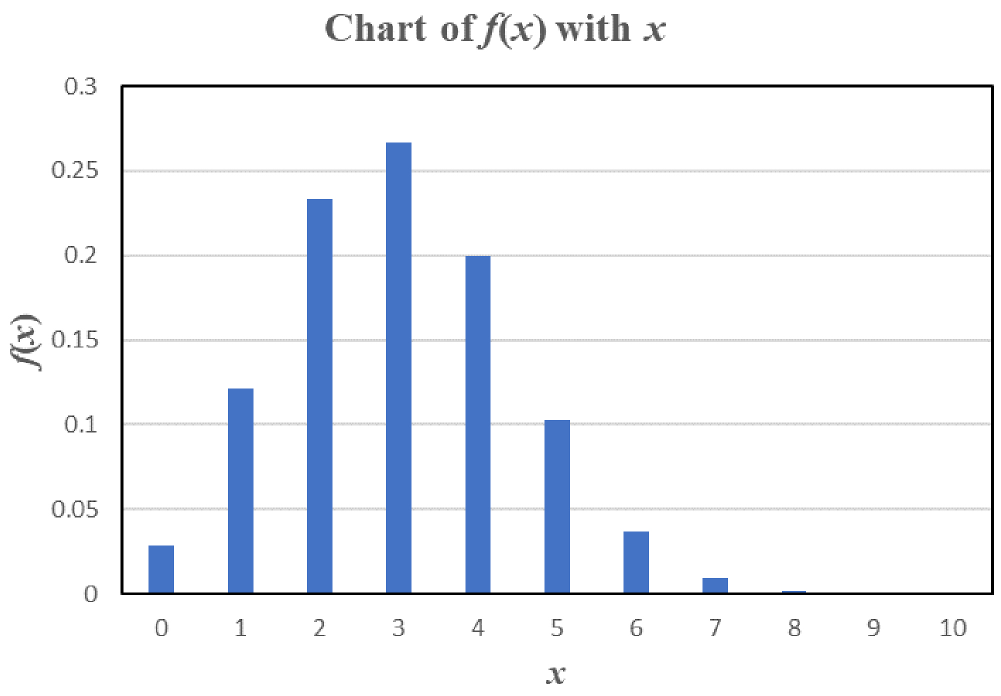
- Most calculators have an ${}^{n}C_{x}$ or nCr or $\binom{n}{x}$ i.e. n ‘choose’ x function.

Binomial Distribution

Example: $n = 10$ and $p = 0.5$



$n = 10$ and $p = 0.3$



Binomial Distribution: Example

(Adapted from MRH 3.113)

The probability of successfully landing a plane using a flight simulator is given as 0.80. Nine randomly and independently chosen student pilots are asked to try to fly the plane using the simulator.

a. What is the probability that all of the student pilots successfully land the plane using the simulator?

Binomial Distribution: Example

b. What is the probability that at least eight student pilots successfully land the plane using the simulator?

c. What is the probability that at most two student pilots fail to successfully land the plane using the simulator?

Binomial Distribution

If X is a binomial random variable with parameters n and p (that is, X ~ Bin(n, p)), then

$$\mathrm{E}(X) = np$$
$$\mathrm{Var}(X) = np(1-p)$$

- Example: If n = 10 and p = 0.7 what are

Binomial Distribution

Example: If n = 10 and p = 0.7 what is the expected value, variance?

$$\mathrm{E}(X) = np$$

$$\mathrm{Var}(X) = np(1-p)$$

Finding Binomial Probabilities: Software Output: EXCEL

Find $f(5) = P(X = 5)$ if $X \sim Bin(10,0.4)$

=BINOM.DIST(5,10, 0.4 ,FALSE)

```
x  P( X = x )
5    0.200658
```

Find $P(X \geq 5)$

=BINOM.DIST(5,10, 0.4 ,TRUE)

```
x  P( X <= x )
4     0.633103
```

Binomial Distribution: Example

An article in the IEEE Computer Applications in Power (Apr 1990) discusses an unmanned watching system that detects intruders using video cameras and microprocessors. Under snowy conditions, the system is detects intruders 50% of the time.

a. What is the probability that in a particular trial of the system with 10 independent intruders, the system will detect at least 7 of the intruders?

Binomial Distribution: Example

b. What is the probability that the system will detect fewer than 4 intruders?

Lecture 3 Revision Exercises

Use software where appropriate

- Devore §3.1 Q7
- Devore §3.2 Q11, 13
- Devore §3.3 Q29, 35, 41
- Devore §3.4 Q49, 51, 55, 57, 65

Montgomery, Runger and Hubele (Old Text)

- Q 3.91- 3.118