Determinants

Determinants

$$\det \mathbf{A} = \begin{vmatrix} 1 & -1 \\ -3 & 1 \end{vmatrix} = 1 - (-1) \times (-3) = -2$$

$$\det \mathbf{A} = \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 1 - (-1) \times 2 = 3$$

$$\det \mathbf{A} = \begin{vmatrix} 4 & 3 \\ 4k & 3k \end{vmatrix} = 12k - 12k = 0$$

$$\det \mathbf{A} = \begin{vmatrix} 4 & 4k \\ -5 & -5k \end{vmatrix} = -20k + 20k = 0$$

Determinants

If $\det \mathbf{A} = |\mathbf{A}| = ad - bc \neq 0$

Then **Ax=b** has a unique solution **x=A^{-1}b** $\quad \mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

If **b=0** and **A^{-1}** exist then **Ax=0** has only trivial solution **x=0**

A 2×2 matrix is invertible if and only if its determinant is nonzero

$$\det \mathbf{A} = |\mathbf{A}| \neq 0$$

If the determinant is zero: $\det \mathbf{A} = |\mathbf{A}| = 0$ then

a) the system **Ax=b** either **does not** have a solution or

b) the system **Ax=b** has **infinitely** many solutions

Determinants

Let consider now 3×3 invertible matrix (it is assumed that $a_{11}\neq 0$)

$$\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

- Lets do the following row operations

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \begin{matrix} R_2 \to a_{11}\times R_2 \\ R_3 \to a_{11}\times R_3 \end{matrix} \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{pmatrix}$$

$$\begin{matrix} R_2 \to R_2 - a_{21}\times R_1 \\ R_3 \to R_3 - a_{31}\times R_1 \end{matrix} \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22}-a_{12}a_{21} & a_{11}a_{23}-a_{13}a_{21} \\ 0 & a_{11}a_{32}-a_{12}a_{31} & a_{11}a_{33}-a_{13}a_{31} \end{pmatrix}$$

Determinants

Lets eliminates the element at a_{32} position

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22}-a_{12}a_{21} & a_{11}a_{23}-a_{13}a_{21} \\ 0 & a_{11}a_{32}-a_{12}a_{31} & a_{11}a_{33}-a_{13}a_{31} \end{pmatrix}$$

$$R_3 \to (a_{11}a_{22}-a_{12}a_{21})R_3-(a_{11}a_{22}-a_{12}a_{21})R_2$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22}-a_{12}a_{21} & a_{11}a_{23}-a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{pmatrix}$$

$$\Delta = a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-$$
$$a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}-a_{13}a_{22}a_{31}$$

Determinants

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{pmatrix}$$

$$\Delta = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} -$$
$$a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

If **A** is invertible therefore Δ must be non-zero $\Delta \neq 0$

$$\Delta = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) +$$
$$a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Δ - is the determinant of a 3×3 matrix

Determinants

$$\Delta = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$\Delta = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

The determinant of 3×3 matrix is written in terms of determinants of 2×2 matrices

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Determinants

$$\Delta = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\Delta = a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13}$$

Here A_{ij} denote the submatrix formed by deleting the ith row and jth column of **A**

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13}$$

Determinants

The (i, j) cofactor of **A** is the number C_{ij} given by $C_{ij} = (-1)^{i+j} \det A_{ij}$

$$\det A = \Delta = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$$

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Determinants

The (i, j) cofactor of **A** is the number C_{ij} given by $C_{ij} = (-1)^{i+j} \det A_{ij}$

$$\det A = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

$$\det A = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$

In general determinant of $n \times n$ matrix can be computed by a cofactor expansion across any row or down any column

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

$$\det A = a_{1k}C_{1k} + a_{2k}C_{2k} + \cdots + a_{nk}C_{nk}$$

Determinants

Calculate **Det A**:

$$\mathbf{A} = \begin{pmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{pmatrix}$$

Expansion along the first row

$$\det \mathbf{A} = 1\begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 5\begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0\begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} = -2 - 5\times 0 + 0 = -2$$

Expansion along the third row

$$det\mathbf{A} = 0\begin{vmatrix} 5 & 0 \\ 4 & -1 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} + 0\begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} = -(-2)\times(-1) = -2$$

Expansion along the third column

$$\det \mathbf{A} = 0\begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 5 \\ 0 & -2 \end{vmatrix} + 0\begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} = -(-1)\times(-2) = -2$$

Determinants

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Expansion along the first row

$$Det\ \mathbf{A} = |\mathbf{A}| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} =$$

$$a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

Expansion along the second column

$$Det\ \mathbf{A} = |\mathbf{A}| = -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} =$$

$$a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$$

Determinants

Calculate **Det A**:

$$\mathbf{A} = \begin{pmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{pmatrix}$$

Expansion along the first column

$$\mathbf{A} = 3\begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix} + 0\times C_{21} + 0\times C_{31} + 0\times C_{41} + 0\times C_{51}$$

Determinants

$$\mathbf{A} = 3\begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix} + 0\times C_{21} + 0\times C_{31} + 0\times C_{41} + 0\times C_{51}$$

Expansion along the first column

$$\det \mathbf{A} = 3\times 2\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix}$$

Expansion along the last row

$$\det \mathbf{A} = 3\times 2\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} = -6\times(-2)\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} = -12$$

Properties of Determinants

A common factor in any row can be taken out of the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

A common factor in any column can be taken out of the determinant

$$\begin{vmatrix} a_{11} & a_{12} & ka_{13} \\ a_{21} & a_{22} & ka_{23} \\ a_{31} & a_{32} & ka_{33} \end{vmatrix} = k \begin{vmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{vmatrix}$$

Properties of Determinants

- Determinant of a matrix with a zero row or column is zero

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 \quad \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix} = 0$$

- Determinant of a matrix with a same row or column is zero

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 \quad \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & a_{12} \\ a_{21} & a_{22} & a_{22} \\ a_{31} & a_{32} & a_{32} \end{vmatrix} = 0$$

Properties of Determinants

- If any two rows of a matrix **A** are interchanged then the determinant swaps its sign

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{vmatrix}$$

- If any two columns of a matrix **A** are interchanged then the determinant swaps its sign

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{vmatrix}$$

Properties of Determinants

- Determinant of a matrix in echelon form is given by the product of its diagonal elements

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33}$$

- If one of the pivots of a matrix is missing then the determinant of its matrix is zero $a_{ii} = 0$

- Transposing a matrix does not change its determinant

$$Det\ \mathbf{A}^T = Det\ \mathbf{A}$$

Determinants Properties

Let **U** be a square matrix in echelon form then:

- **det U=** product of pivots in **U**

- If **A** is invertible and **U** is in the echelon form of **A** produced without row scaling with r row exchanges then

  det A= $(-1)^r$ product of pivots **in U**

- **Det (AB)= Det A Det B**

- **Det A^T= Det A**

- The determinant of an inverse $$Det\ \mathbf{A}^{-1} = \frac{1}{Det\ \mathbf{A}}$$

Properties of Determinants

- Linear property of determinants

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+b_{12} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{12} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- Therefore

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+ka_{11} & a_{22}+ka_{12} & a_{23}+ka_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ ka_{11} & ka_{12} & ka_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Properties of Determinants

Let **A** be a square matrix:

- If a multiple of one row of **A** is added to another row to produce a matrix **B** then **det A=det B**.

$$R_2 \to R_2 + 2R_1$$
$$R_3 \to R_3 + R_1$$

$$\det \mathbf{A} = \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix}$$

$$\det \mathbf{A} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = (-3)\times(-5) = 15$$

- An Example: Calculate det **A**

$$\begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{vmatrix}$$

$R_4 \rightarrow R_4 + R_2$

$$= -2\begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{vmatrix} = -2\begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & -3 & 1 \end{vmatrix} =$$

$R_2 \rightarrow R_2 - 3R_1; \quad R_3 \leftrightarrow R_2$

$$= (-2)(-1)\begin{vmatrix} 1 & 2 & -1 \\ 0 & -3 & 1 \\ 0 & 0 & 5 \end{vmatrix} = 2 \times 1 \times (-3) \times 5 = -30$$

Cramer's Rule

- Determinant of a matrix with a zero row or column is zero

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

- Then $\det \mathbf{A} \neq 0$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}, \quad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}.$$

Cramer‘s Rule

- Use the Cramer’s rule to solve the system

$$\begin{cases} 3x_1 - 2x_2 = 6 \\ -5x_1 + 4x_2 = 8 \end{cases} \qquad \begin{pmatrix} 3 & -2 \\ -5 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$$

- Then $x_1 = \dfrac{\begin{vmatrix} 6 & -2 \\ 8 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ -5 & 4 \end{vmatrix}} = \dfrac{24+16}{12-10} = 20 \quad x_2 = \dfrac{\begin{vmatrix} 3 & 6 \\ -5 & 8 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ -5 & 4 \end{vmatrix}} = \dfrac{24+30}{2} = 27$

Invertible matrix theorem

Let **A** be a square invertible matrix then

- **A** is row equivalent to an identity matrix (REF~**I)**
- **A** has n pivots
- The equation **Ax=b** has only the unique solution
- The equation **Ax=0** has only trivial solution **x=0**.
- The determinant of **A** is not zero

Inverse of a Matrix

- The inverse of a matrix can be found from the relation

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{Det\mathbf{A}} \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^{T},$$

- The transpose of a cofactor matrix

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

is called the adjoint of matrix **A.**

Find the inverse of matrix **A**

$$\mathbf{A} = \begin{pmatrix} 2 & -3 & 4 \\ 0 & -1 & 5 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{Det\mathbf{A}}\left(\mathbf{A}^{c}\right)^{T}$$

$$\mathbf{A}^{-1} = -\frac{1}{6}\begin{pmatrix} \begin{vmatrix} -1 & 5 \\ 0 & 3 \end{vmatrix} & -\begin{vmatrix} 0 & 5 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 0 & -1 \\ 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} -3 & 4 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 0 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & -3 \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} -3 & 4 \\ -1 & 5 \end{vmatrix} & -\begin{vmatrix} 2 & 4 \\ 0 & 5 \end{vmatrix} & \begin{vmatrix} 2 & -3 \\ 0 & -1 \end{vmatrix} \end{pmatrix}^{T} = -\frac{1}{6}\begin{pmatrix} -3 & 0 & 0 \\ 9 & 6 & 0 \\ -11 & -10 & -2 \end{pmatrix}^{T} = \frac{1}{6}\begin{pmatrix} 3 & -9 & 11 \\ 0 & -6 & 10 \\ 0 & 0 & 2 \end{pmatrix}$$