Mathematics 2

Lecture 4: Poisson Distribution
Continuous Random Variables
Exponential Distribution

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Poisson Distribution

Reference: Devore § 3.6

- **Poisson random variables** are used to model the number events that occur over time, over a distance, or an area. The number of events may be any integer greater or equal to 0.
  - We may observe a period of time where no event occurs.

- Characteristics of Poisson random variables
  - The random variable X is the number of times that a specified event occurs over some unit of measurement (time/distance/area).
  - The probability that an event occurs in a given unit of time/distance/area etc is constant.
  - The number of events that occur in one unit of time etc is independent of what happens in other units of time.

Poisson Distribution

- To describe the Poisson distribution, we define a parameter $\lambda > 0$ to be the mean number of times that the event occurs over some specified interval of time/distance/area etc.
  - The value of λ is always specific to the interval
  - If we change intervals, we must change λ.
- The probability mass function of X is

$$\boxed{f(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad \text{for } x = 0, 1, 2, \ldots}$$

- The mean and variance of X are

$$\boxed{\mathrm{E}(X) = \lambda \quad \mathrm{Var}(X) = \lambda}$$

Poisson Distribution: Example - Emails

- In a large corporate computer network, user logons to the system can be modelled by a Poisson process with a mean of 25 logons per hour.
  a. What is the probability that there are exactly 30 logons in a particular hour?

Poisson Distribution: Example - Emails

b. What is the probability that there is no logon in an interval of 6 minutes?

Poisson Distribution: Example – Emails-EXCEL

c. What is the probability that there are between 20 and 30 logons in an hour given the output below?

Cumulative Distribution Function

```
Poisson with mean = 25

 x   P( X <= x )
19      0.133575
20      0.185492
29      0.817896
30      0.863309
31      0.899932
```

Poisson Distribution: Example - Disks

Contamination is a problem in the manufacture of optical storage disks. The number of contaminated particles that occur on an optical disk follows a Poisson distribution with a mean of 0.1 per cm^2 of media surface. The area of the disk under study is 100 cm^2. Find the probability that we observe 12 contaminated particles.

```
Probability Density Function
Poisson with mean = 10
x    P( X = x )
12   0.0947803
```

Poisson Distribution: Problems - Disks

1. Find the probability that we observe less than 12 contaminated particles.

2. Find the probability that we find more than 6, but less than 15, contaminated particles.

3. Suppose that there exists a smaller disk with area $40cm^2$. Find the probability that we observe fewer than two contaminated particles on this disk.

Poisson Distribution: Problems

Cumulative Distribution Function

Poisson with mean = 10

x	P(X <= x)
5	0.067086
6	0.130141
11	0.696776
12	0.791556
14	0.916542
15	0.951260
16	0.972958

Cumulative Distribution Function

Poisson with mean = 4

x	P(X <= x)
1	0.091578
2	0.238103

Poisson Distribution: Problems

1. Find the probability that we observe less than 12 contaminated particles.

2. Find the probability that we find more than 6, but less than 15, contaminated particles.

3. Suppose that there exists a smaller disk with area 40cm^2. Find the probability that we observe fewer than two contaminated particles on this disk.

Poisson Distribution: Challenging Example

Suppose that cars that wish to pass straight through an intersection from north to south arrive at an intersection at an average rate of 1 car every 10 seconds. Suppose that the cycle of lights are timed in such a way that these cars must wait 2 minutes at the intersection before being given a green light. Suppose that 15 cars can clear the intersection in this direction in each cycle. What is the probability that at least one car that had stopped at the red light will not pass through the intersection at the next green light if there were no cars waiting after the previous green light.

Continuous Random Variables

Reference: Devore §4.1-4.2

- Recall that the set of values that a **continuous random variable** can take is the set of real numbers ($\mathbb{R}$).

- A probability distribution for a continuous random variable X is specified by a function $f(x)$ which is called the **probability density function (pdf)**.

- The following requirements must be met for a function to be a pdf:
  - $f(x) \geq 0$
  - The total area under the density curve is equal to 1.

- The probability that a random variable takes a value within a range is the area under the density curve over that range.
  - So we can think of probability as an integral of $f(x)$

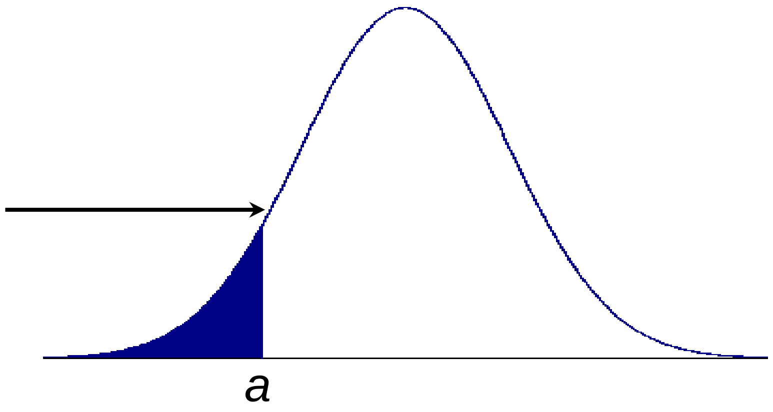
Probability Density Functions

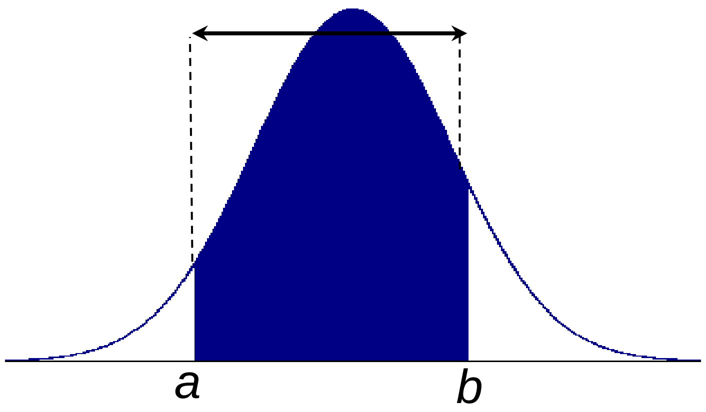
- Notice that for a continuous random variable X,

$$P(X = a) = 0$$

for any specific value *a*, because the “area above a point” under the curve has no area. (Note that this is not true for discrete random variables). Then

$$P(X \le a) = P(X < a) = \int_{-\infty}^{a} f(x)dx \qquad P(a \le X \le b) = P(a < X < b) = \int_a^b f(x)dx$$





Example: Copper Current

- Let X be the current measured in a thin copper wire in milliamperes (mA). Assume that X can take any value between 0 and 20 with equal probability. What is the probability that a current measurement is less than 10 milliamperes?

Example: Probability Density Functions

Show that the following is a valid probability density function

$$f(y) = \begin{cases} \frac{3}{2}(1 - y^2), & 0 \le y \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

Example: Probability Density Functions

- Find $P(0.3 < Y < 0.7)$

- Find $P(Y > 0.5)$

Cumulative Distribution Function

- The **cumulative distribution function** of a continuous random variable *X* with probability density function *f(x)* is

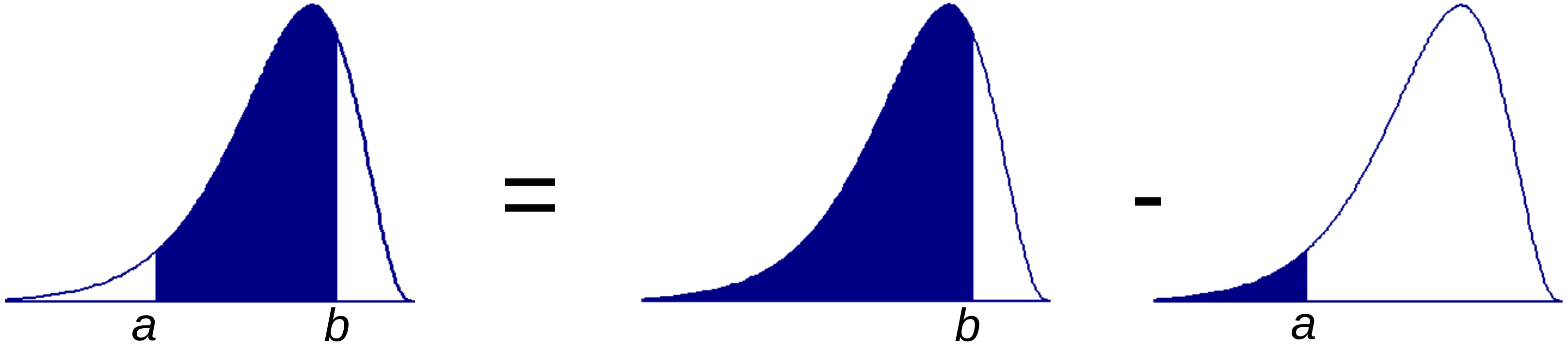
$$\boxed{F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx}$$

  for $-\infty < x < \infty$.
- The cumulative density function *f(x)* of a continuous function can be used to determine probabilities.

Calculating Probabilities

$$P(a < X < b) = \int_a^b f(x)dx$$
$$= F(b) - F(a)$$

$$P(a < X < b) \qquad = F(b) \qquad - F(a)$$



Calculating Probabilities

- *X* is a Uniform random variable with probability density function $f(x) = 1/20$ for $0 \le x \le 20$. Find $P(X<10)$

Cumulative Distribution Function

```
Continuous uniform on 0 to 20
x       P( X <= x )
10          0.5
```

Mean and Variance of a Distribution

- Suppose that X is a continuous random variable with a probability density function of $f(x)$
  - Then the **mean** (or expected value) of X, denoted μ, or $E(X)$ is given by

$$\boxed{\mu = \mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x) dx}$$

  - The **variance** of X, denoted $Var(X)$ or σ^2 is given by

$$\boxed{\sigma^2 = \mathrm{Var}(X) = \int_{-\infty}^{\infty} x^2 f(x) dx - (\mathrm{E}(X))^2}$$

  - The **standard deviation** of X is the square root of the variance.

Example: Trains

- The amount of time Y (in minutes) that a train is late is a continuous random variable with pdf

$$f(y) = \begin{cases} \frac{3}{500}(25 - y^2), & \text{if } -5 < y < 5; \\ 0, & \text{otherwise.} \end{cases}$$

- Find the mean and standard deviation for the amount of time that the train is late (in minutes).

Example: Trains

Example

- Find the expected value and standard deviation of:

$$f(y) = \begin{cases} \frac{3}{2}(1-y^2), & 0 \le y \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

Example

Exponential Distribution

Reference: Devore § 4.4

- If the random variable *X* equals the distance between successive events in a Poisson process with mean *λ* > 0, then it has an **exponential distribution** with density function

$$\boxed{f(x) = \lambda e^{-\lambda x} \text{ for } 0 \leq x < \infty}$$

- The mean and variance of this process are

$$\boxed{\mathrm{E}(X) = \frac{1}{\lambda} \text{ and } \mathrm{Var}(X) = \frac{1}{\lambda^2}}$$

- Used to model **lifetime or time to failure**
- Used to model **inter-arrival time**

Exponential Distribution: Example - Emails

The time between the arrival of emails at your computer is exponentially distributed with a mean of 2 hours. What is the probability that you do not receive a message during a 2-hour period?

- That is, what is the probability that the inter-arrival time is greater than 2 hours?

Exponential Distribution: Example - Emails

What is the probability that you do not receive a message during a 2-hour period?

```
Cumulative Distribution Function
Exponential with mean = 2
x  P( X <= x )
2      0.632121
```

Poisson and Exponential Distributions

- The Poisson and Exponential distributions are related
  - If $X \sim \text{Poi}(\lambda)$ is the number of times that an event occurs in a fixed interval, then $Y \sim \text{Exp}(1/\lambda)$ is the interval between two consecutive events.
  - That is, if the frequency of an event takes a Poisson distribution, then the period takes an Exponential distribution

- What is the probability that we receive three emails in an hour?

Lack of Memory

In an exponential distribution, the probability that an event will not occur for a certain period of time in the future does not depend on the past behaviour of the process.

$$\boxed{P(X > b \mid X > a) = P(X > b - a)}$$

Exponential Distribution: Example - Emails

a) If you have not received a message in the last 4 hours, what is the probability that you do not receive a message in the next 2 hours?

b) (Exercise) If you have not received a message in the last 6 hours, what is the probability that you do not receive a message in the next 2 hours?

Exponential Distribution: Example - Machine

A machine that is part of a production line has a lifetime that follows an Exponential distribution with a mean of 150 hours.

a. What is the probability that the machine has not broken down after 50 hours?

Exponential Distribution: Example - Machine

b. Suppose that the machine has been operational for 60 hours. What is the probability that a single machine will still be operational after 110 hours? Why can we say this?

Lecture 4 Revision Exercises

Use software where appropriate

- Devore §3.6 Q81, 83, 85, 87, 89
- Devore §4.1 Q1, 3, 5, 7
- Devore §4.2 Q13(a,c,d,e), 15b, 21
- Devore §4.4 Q59, 61, 63

N.B. You don’t need to calculate the CDFs

Montgomery, Runger and Hubele (Old Text)

- Q 3.119 - 3.144