Eigenvectors and Eigenvalues

Linear Transformations

Eigenvalues and Eigenvectors

- Applications in Mathematics
- Physics
- Chemistry
- Biology
- Statistics
- Finances

Eigenvalues and Eigenvectors

Let **A** be a square 2×2 matrix

$$\mathbf{Ax} = \mathbf{b}$$

Lets consider the transformation: $\mathbf{Ax} \to \mathbf{y}$

$$\mathbf{y} = \mathbf{Ax}$$

$$\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\mathbf{y} = \mathbf{Au} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$$

$$\mathbf{y} = \mathbf{Av} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

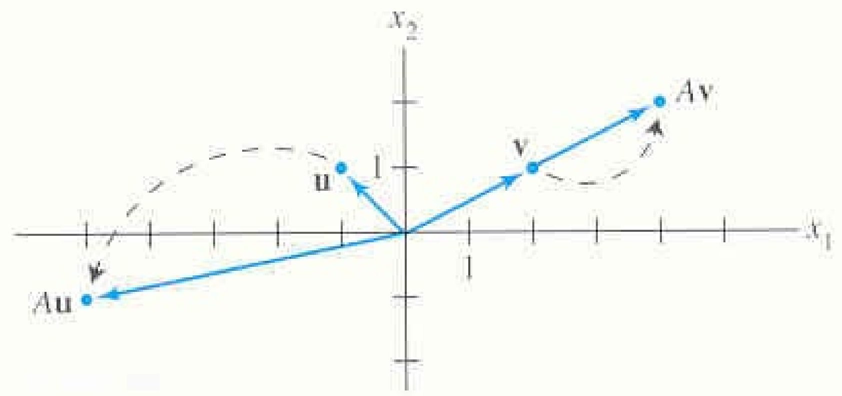


FIGURE 1 Effects of multiplication by A.

$$\mathbf{Av} = 2\mathbf{v}$$

Eigenvalues and Eigenvectors

- Definition: An eigenvector of $n \times n$ matrix **A** is a nonzero vector x such that

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$$

for some scalar λ

- A scalar λ is called an eigenvalue of matrix A if there is a nontrivial solution x of $\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$

- Such x is called an eigenvector corresponding to λ

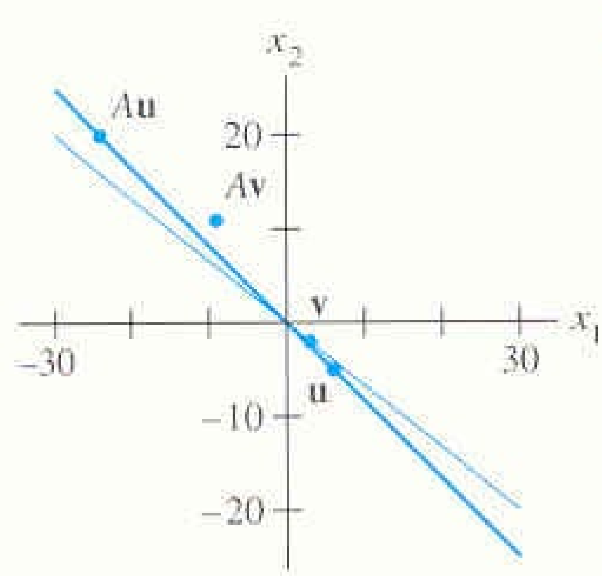
Eigenvalues and Eigenvectors

It is easy to determine if a given vector is an eigenvector of the given matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 6 \\ -5 \end{pmatrix}, \quad v = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$\mathbf{Au} = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 6 \\ -5 \end{pmatrix} = \begin{pmatrix} -24 \\ 20 \end{pmatrix} = -4\begin{pmatrix} 6 \\ -5 \end{pmatrix} = -4\mathbf{u}$$

$$\mathbf{Av} = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -9 \\ 11 \end{pmatrix} \neq \lambda\begin{pmatrix} 3 \\ -2 \end{pmatrix}$$



$A\mathbf{u} = -4\mathbf{u}$, but $A\mathbf{v} \neq \lambda\mathbf{v}$.

Eigenvalues and Eigenvectors

Show that 7 is an eigenvalue of matrix **A.**

$$\mathbf{Ax} = 7\mathbf{x} \qquad \mathbf{Ax} = \lambda\mathbf{x}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 7\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - 7\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - 7\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\left(\begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix} - 7\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Eigenvalues and Eigenvectors

Show that 7 is an eigenvalue of matrix **A.**

$$\mathbf{Ax} = 7\mathbf{x} \qquad \mathbf{Ax} = \lambda\mathbf{x}$$

$$\left(\begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix} - 7\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix} - \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{pmatrix} \quad \begin{matrix} R_1 \to R_1/(-6) \\ R_2 \to R_2/5 \end{matrix} \quad \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \quad R_2 \to R_2 - R_1$$

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} x_1 - x_2 = 0 \\ x_2 \ is \ free \text{ var} \end{matrix} \quad x_1 = x_2 \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix} = 7\begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7x_2 \\ 7x_2 \end{pmatrix} = 7\begin{pmatrix} x_2 \\ x_2 \end{pmatrix}$$

Eigenvectors are not unique.

Eigenvalues and Eigenvectors

Find eigenvalues and eigenvectors of matrix **A**

- $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ $\quad\quad\quad$ $\mathbf{A}\mathbf{x} = 7\mathbf{x}$

$$\mathbf{A} = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \lambda\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\left(\begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Eigenvalues and Eigenvectors

$$\left(\begin{pmatrix}1 & 6\\5 & 2\end{pmatrix} - \begin{pmatrix}\lambda & 0\\0 & \lambda\end{pmatrix}\right)\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}1-\lambda & 6\\5 & 2-\lambda\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\det\left|\begin{pmatrix}1-\lambda & 6\\5 & 2-\lambda\end{pmatrix}\right| = 0$$

$$(1-\lambda)(2-\lambda) - 30 = 0$$

$$2 - \lambda - 2\lambda + \lambda^2 - 30 = 0$$

$$\lambda^2 - 3\lambda - 28 = 0 \quad \textit{characteristic} \quad \textit{equation}$$

$$\lambda_1 = 7, \quad \lambda_2 = -4$$

Eigenvalues and Eigenvectors

The eigenvalues are 7 and -4 $\mathbf{Ax} = 7\mathbf{x}$ $\mathbf{Ax} = -4\mathbf{x}$

To find corresponding eigenvectors $\lambda_1 = 7$ $\lambda_2 = -4$

$$\begin{pmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1-\lambda_1 & 6 & 0 \\ 5 & 2-\lambda_1 & 0 \end{array}\right) \qquad \left(\begin{array}{cc|c} 1-7 & 6 & 0 \\ 5 & 2-7 & 0 \end{array}\right)$$

$$R_1 \to R_1/(-6) \quad R_2 \to R_2/5 \qquad \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right) \qquad R_2 \to R_2 - R_1$$

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \qquad x_1 - x_2 = 0 \quad x_2 \; is \; free \text{ var} \qquad x_1 = x_2 \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix} = 7\begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7x_2 \\ 7x_2 \end{pmatrix} = 7\begin{pmatrix} x_2 \\ x_2 \end{pmatrix}$$

Eigenvalues and Eigenvectors

The eigenvalues are 7 and -4 $\mathbf{A}\mathbf{x} = 7\mathbf{x}$ $\mathbf{A}\mathbf{x} = -4\mathbf{x}$

$\lambda_1 = 7$ $\lambda_2 = -4$

To find corresponding eigenvectors

$$\begin{pmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1-\lambda_2 & 6 & 0 \\ 5 & 2-\lambda_2 & 0 \end{pmatrix} \quad \begin{pmatrix} 1-(-4) & 6 & 0 \\ 5 & 2-(-4) & 0 \end{pmatrix} \quad \begin{pmatrix} 5 & 6 & 0 \\ 5 & 6 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 6 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad R_2 \to R_2 - R_1 \quad \begin{matrix} 5x_1 + 6x_2 = 0 \\ x_2 \; is \; free \text{ var} \end{matrix} \quad x_1 = -\frac{6}{5}x_2$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{6}{5}x_2 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} -\frac{6}{5} \\ 1 \end{pmatrix} \qquad x_2 = 5$$

$$\mathbf{v} = \begin{pmatrix} -6 \\ 5 \end{pmatrix}; \quad \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} -6 \\ 5 \end{pmatrix} = \begin{pmatrix} 24 \\ -20 \end{pmatrix} = -4\begin{pmatrix} -6 \\ 5 \end{pmatrix};$$

Eigenvalues and Eigenvectors

Find eigenvalues and corresponding eigenvectors of matrix **A**

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\mathbf{Ax} = \boldsymbol{\lambda}\mathbf{x} \qquad \left(\begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\det \left| \begin{pmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{pmatrix} \right| = 0 \qquad \textit{Step} \quad 1$$

$$(2-\lambda)(-6-\lambda) - 9 = 0 \qquad -12 - 2\lambda + 6\lambda + \lambda^2 - 9 = 0$$

$$\lambda^2 + 4\lambda - 21 = 0 \quad \textit{characteristic} \quad \textit{equation}$$

$$\lambda_1 = -7, \quad \lambda_2 = 3$$

Find eigenvalues and corresponding eigenvectors of matrix **A**

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\mathbf{Ax} = \boldsymbol{\lambda}\mathbf{x}$$

$$\begin{pmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\lambda_1 = -7, \quad \lambda_2 = 3$$

$$\begin{pmatrix} 2-(-7) & 3 \\ 3 & -6-(-7) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \left(\begin{array}{cc|c} 9 & 3 & 0 \\ 3 & 1 & 0 \end{array}\right) \quad R_1 \to R_1/3 \quad \begin{pmatrix} 3 & 1 & 0 \\ 3 & 1 & 0 \end{pmatrix} R_2 \to R_2 - R_1$$

$$\begin{pmatrix} 3 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} x_2 \ \textit{free} \text{ var} \\ 3x_1 + x_2 = 0 \end{matrix} \quad x_1 = -\frac{x_2}{3} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{x_2}{3} \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix}$$

Find eigenvalues and corresponding eigenvectors of matrix **A**

$$x_2 = 3 \quad \mathbf{v}_1 = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad \lambda_1 = -7$$

$$\begin{pmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \lambda_2 = 3$$

$$\begin{pmatrix} 2-3 & 3 \\ 3 & -6-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 3 & 0 \\ 3 & -9 & 0 \end{pmatrix} \quad R_2 \to R_2/3 \quad \begin{pmatrix} -1 & 3 & 0 \\ 1 & -3 & 0 \end{pmatrix} R_2 \to R_2 + R_1$$

$$\begin{pmatrix} -1 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} x_2 \ \textit{free}\ \text{var} \\ -x_1 + 3x_2 = 0 \end{matrix} \quad x_1 = 3x_2 \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_2 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$x_2 = 1 \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \lambda_2 = 3$$

Eigenvalues and Eigenvectors

Find eigenvalues and corresponding eigenvectors of matrix **A**

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & -2 & 4 \end{pmatrix}; \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \mathbf{Ax} = \lambda\mathbf{x}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & -2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \lambda\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & -2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \lambda\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & -2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & -2 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Eigenvalues and Eigenvectors

$$\begin{pmatrix} 1-\lambda & 0 & 0 \\ -1 & 1-\lambda & 1 \\ -1 & -2 & 4-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\det\begin{vmatrix} 1-\lambda & 0 & 0 \\ -1 & 1-\lambda & 1 \\ -1 & -2 & 4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{vmatrix} = 0$$

Eigenvalues are 1, 2, 3.

$$(1-\lambda)((1-\lambda)(4-\lambda)+2) = 0$$

$$(1-\lambda) = 0 \Rightarrow \lambda_1 = 1$$

$$4-\lambda-4\lambda+\lambda^2+2 = 0, \quad \lambda^2-5\lambda+6 = 0$$

$$\lambda_2 = 2, \quad \lambda_3 = 3$$

Eigenvalues and Eigenvectors

$$\begin{pmatrix} 1-\lambda & 0 & 0 \\ -1 & 1-\lambda & 1 \\ -1 & -2 & 4-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \lambda_1 = 1$$

$$\begin{pmatrix} 1-1 & 0 & 0 \\ -1 & 1-1 & 1 \\ -1 & -2 & 4-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & -2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & -2 & 3 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & -2 & 3 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad R_2 \to R_2 - R_1$$

$$\begin{pmatrix} -1 & -2 & -1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad R_2 \to R_2/(-2) \begin{pmatrix} -1 & -2 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} R_1 \to R_1 + 2R_2$$

Eigenvalues and Eigenvectors

$$\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{matrix} x_3 \quad \textit{is free} \text{ var} \\ x_2 - x_3 = 0 \quad x_2 = x_3 \\ x_1 - x_3 = 0 \quad x_1 = x_3 \end{matrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \lambda_1 = 1$$

Eigenvalues and Eigenvectors

$$\begin{pmatrix} 1-2 & 0 & 0 \\ -1 & 1-2 & 1 \\ 1 & -2 & 4-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 0 & 0 \\ -1 & -1 & 1 \\ 1 & -2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 1 & -2 & 2 & 0 \end{pmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 + R_1 \end{matrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -2 & 2 & 0 \end{pmatrix} \quad R_3 \to R_3 - 2R_2$$

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{matrix} x_1 = 0 \\ -x_2 + x_3 = 0 \\ x_3 \textit{ is free} \quad \text{var} \end{matrix} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \lambda_2 = 2.$$

Eigenvalues and Eigenvectors

$$\begin{pmatrix} 1-3 & 0 & 0 \\ -1 & 1-3 & 1 \\ 1 & -2 & 4-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -2 & 0 & 0 \\ -1 & -2 & 1 \\ 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 0 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 1 & -2 & 1 & 0 \end{pmatrix} \quad \begin{matrix} R_1 \to R_1/(-2) \\ R_2 \to R_2 + R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 \end{pmatrix} \quad R_3 \to R_3 - R_2$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} x_1 = 0 \\ -2x_2 + x_3 = 0 \\ x_3 \ \textit{is free} \quad \text{var} \end{matrix} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ x_3/2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 0 \\ \frac{1}{2} \\ 1 \end{pmatrix}$$

$$\mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad \lambda_2 = 3, \quad \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & -2 & 4 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix} = 3\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

Eigenvalues and Eigenvectors

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & -2 & 4 \end{pmatrix}$$

$$\lambda_1 = 1, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \lambda_2 = 2, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \lambda_3 = 3, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

Eigenvalues and Eigenvectors

For the general square matrix **A** with sizes n by n we have

$$\mathbf{Ax} = \lambda\mathbf{x}$$

$$\mathbf{Ax} - \lambda\mathbf{x} = 0$$

$$\mathbf{Ax} - \lambda\mathbf{Ix} = 0$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$$

The eigenvalues λ of matrix **A** satisfy the condition

$$\det\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0$$

This equation is called characteristic equation

$$\lambda^n + c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} + \cdots + c_1\lambda + c_0 = 0$$

There are at most n eigenvalues (if we count the repeated roots)

Eigenvalues and Eigenvectors

$Let \quad \mathbf{A} \quad be \quad n \times n \quad matrix$

- Form the characteristic equation

$$Det\left|\mathbf{A} - \lambda \mathbf{I}\right| = 0$$

- It will be a polynomial equation of degree n

- Find all roots of the characteristic equation $\lambda_1, \lambda_2, \ldots \lambda_n$

- For each eigenvalue find its eigenvectors by solving the homogeneous linear system

$$\left(\mathbf{A} - \lambda_i \mathbf{I}\right)\mathbf{x} = 0$$

- This requires the row reduction of a square matrix. The resulting reduced row-echelon form must have at least one row of zeros.

Linear Transformations

- A matrix transformation from R^n to R^m is a rule that assigns each vector x in R^n a vector **Ax** in R^m : **x** to **Ax.**
- For any vector **x** in R^n, the vector T(**x**) of R^m is called the **image** of x.
- The set of all images T(**x**) is called the **range** of T.

Example 1 $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \quad R^2 \to R^2$

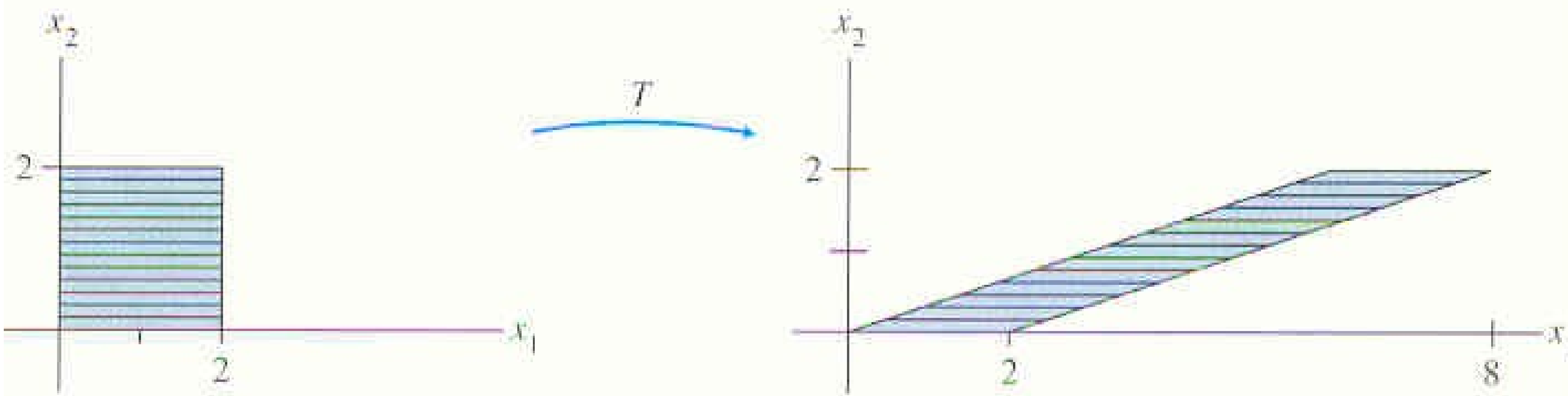
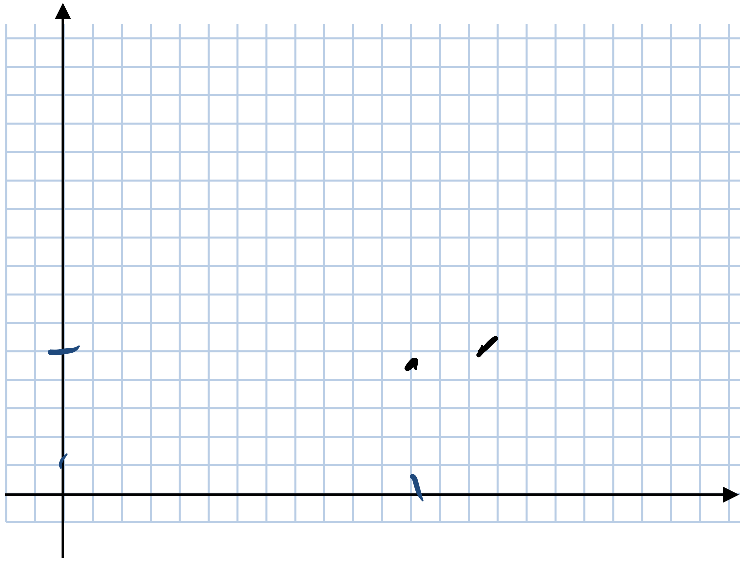


FIGURE 4 A shear transformation.

Matrix transformations can be thought of as combinations of *Dilations*, *Reflections* and *Rotations.*

Dilations stretch or shrink the components of a vector

E.g. $\mathbf{T} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$

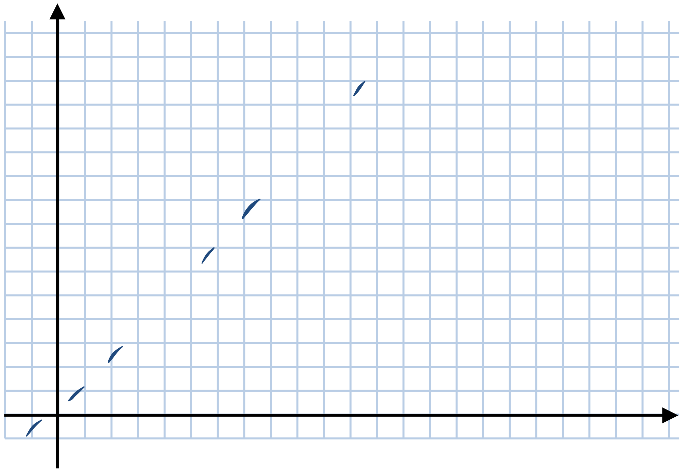


$$\begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} =$$

Reflections

E.g. $$\mathbf{T} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} =$$

Reflection Matricies in R^2

$$\mathbf{A}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \Rightarrow \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}y\\ x\end{pmatrix}$$

- Refelection through the line y=x

$$\mathbf{A}=\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix} \Rightarrow \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}-y\\ -x\end{pmatrix}$$

- Refelection through the line y=-x

$$\mathbf{A}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \Rightarrow \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}\underset{\sim}{x}\\ -y\end{pmatrix}$$

- Refelection through the x axis

$$\mathbf{A}=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} \Rightarrow \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}-x\\ y\end{pmatrix}$$

- Reflection through y axis

Linear Transformations

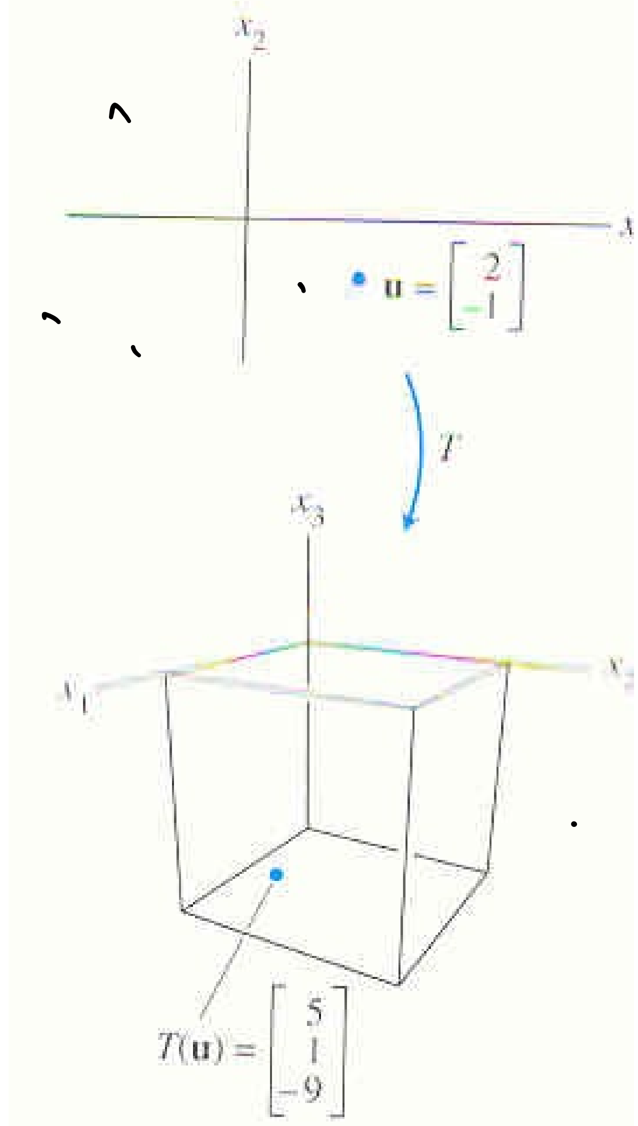
Example

$$\mathbf{A} = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix} \quad R^2 \to R^3$$

$$\begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -9 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 3 \\ 1 & 4 & -2 \end{pmatrix} \quad R^3 \to R^2$$

$$\begin{pmatrix} 1 & -2 & 3 \\ 1 & 4 & -2 \end{pmatrix}\begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \end{pmatrix}$$



LInear Transformations

Example 3 Projection onto xy plane

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad R^3 \to R^3$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$$

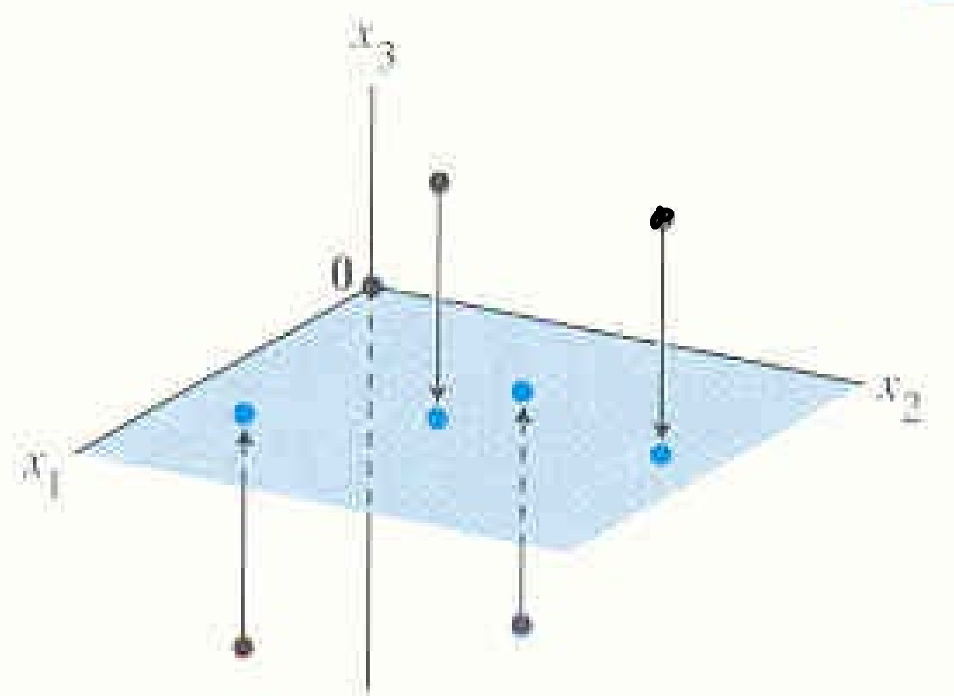


FIGURE 3

A projection transformation.

The transformation T is linear if $T(c_1\mathbf{v}+c_2\mathbf{u})=c_1 T(\mathbf{v}) + c_2 T(\mathbf{u})$. So **x** to **Ax** is linear transformation

Is this transformation $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix}$ linear?

Matrix of Linear Transformation

Let T be a linear transformation from R^n to R^m. Then there exist a unique matrix **A** such that T(**x**)=**Ax.**

The matrix **A** is the m by n matrix where jth column is the vector T(e_j).

$$\mathbf{x} = x_1\mathbf{e_1} + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$$

$$\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$T(\mathbf{x}) = T(x_1\mathbf{e_1} + x_2\mathbf{e}_2 + x_3\mathbf{e}_3) = T(x_1\mathbf{e_1}) + T(x_2\mathbf{e}_2) + T(x_3\mathbf{e}_3) =$$

$$= x_1T(\mathbf{e_1}) + x_2T(\mathbf{e}_2) + x_3T(\mathbf{e}_3) = \begin{pmatrix} T(\mathbf{e_1}) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{Ax}$$

$$\mathbf{A} = \begin{pmatrix} T(\mathbf{e_1}) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \end{pmatrix}$$

Standard Matrix of Rotations in R^2

Let T be the transformation that rotates each point in R^2 about the origin through the angle θ.

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$$

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \Rightarrow T(\mathbf{e}_1) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \Rightarrow T(\mathbf{e}_2) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\mathbf{A}^T\mathbf{A} = \mathbf{I}, \quad \mathbf{A}^{-1} = \mathbf{A}^T$$

Matricies with this property (**$\mathbf{A}^{-1}=\mathbf{A}^T$**) are called orthogonal matrices.
Rotation matrix is orthogonal matrix.

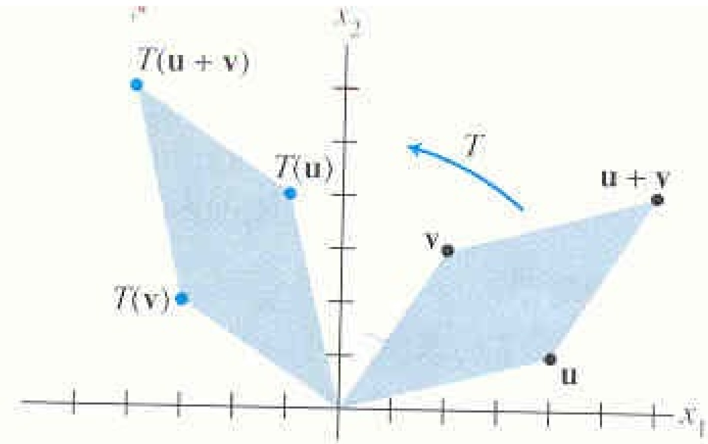


FIGURE 6 A rotation transformation.

Null Space of a Matrix

The null space of a matrix **A,** written as Nul **A,** is the set of all sulutions of the homogeneous equation **Ax=0**.

The vector **x** belongs to the Null space of matrix **A** if **Ax=0.**

A matrix **A** where det(**A**)=0 is called a *singular* matrix.
A system **Ax=0** will have an infinite number of solutions.

Find the null space of matrices:

$$\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$