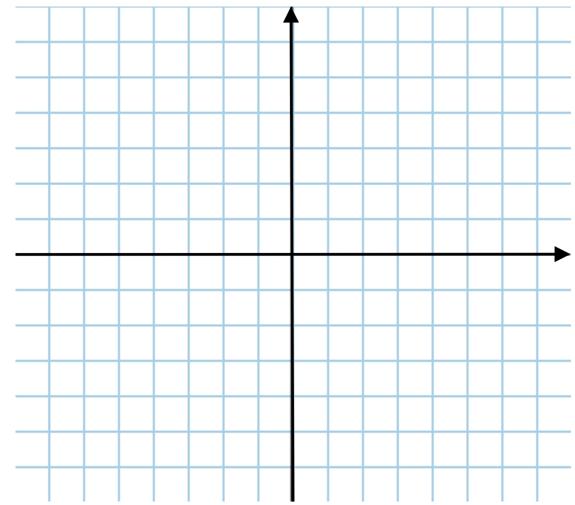
Calculus of Several Variables

Content

- Definition of functions of several variables
- Continuity of functions of several variables
- Partial derivatives

A **function of one variable** *f* (*x*) is a *rule* that transforms one number *x* into another number *f* (*x*).



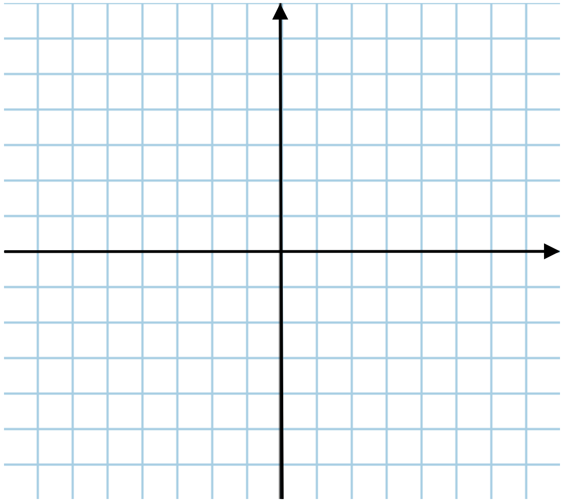
We often express this transformation by writing

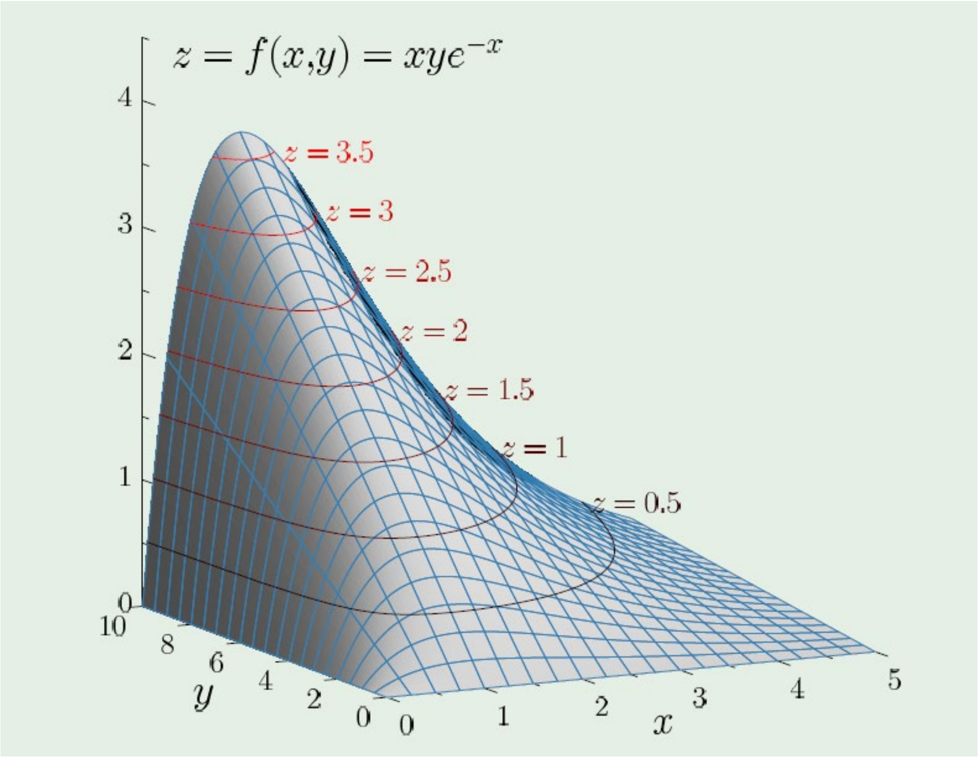
$$f : \mathrm{D} \to \mathrm{E}$$

E.g. Let f : R -> [-1,1] be the real-valued function f(x) = cos x.

A function of **two variables** $f(x,y)$ is a rule for transforming two numbers x and y into a unique number $f(x,y)$.

We often write that $z = f(x,y)$ to make it clear that for each pair (x,y) $f : \mathrm{R}^2 \to \mathrm{R}$ we get the output z.





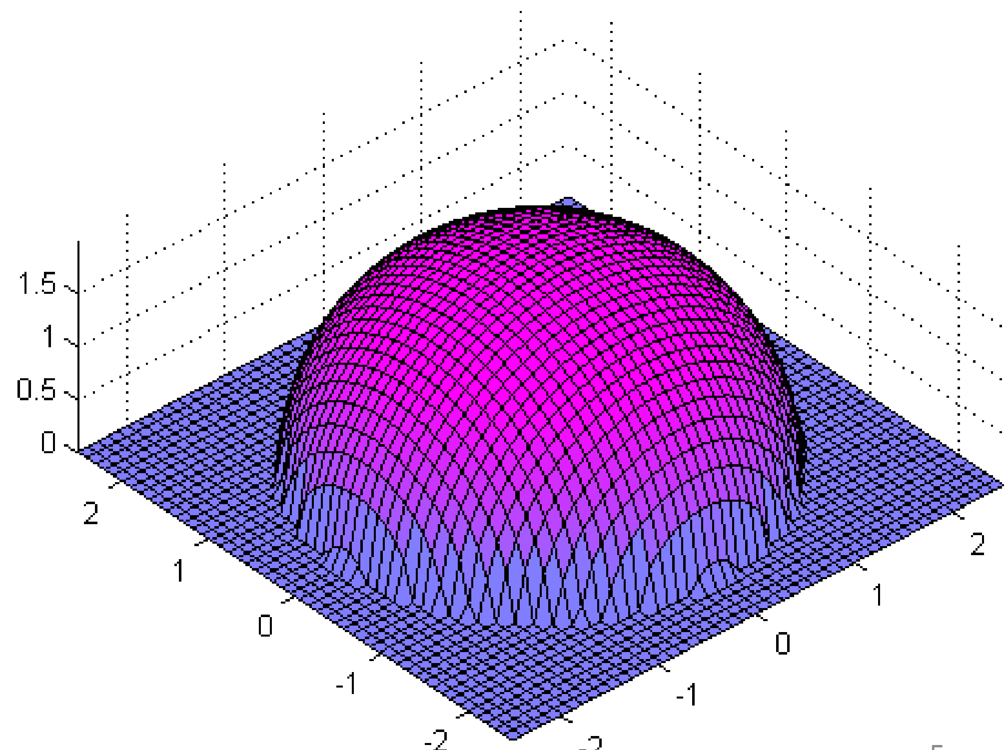
Because x and y are not connected, they are known as the *independent variables.*

The *domain D* of *f* is the set of all variables (*x,y*) such that *f* is defined and has a real value.

The *range R* of *f* is the set $\{f(x,y)|(x,y) \in D\}$

Example: $f(x,y) = \sqrt{4 - x^2 - y^2}$

Domain: $x^2 + y^2 \leq 4$



Range: $0 \leq z \leq 2$

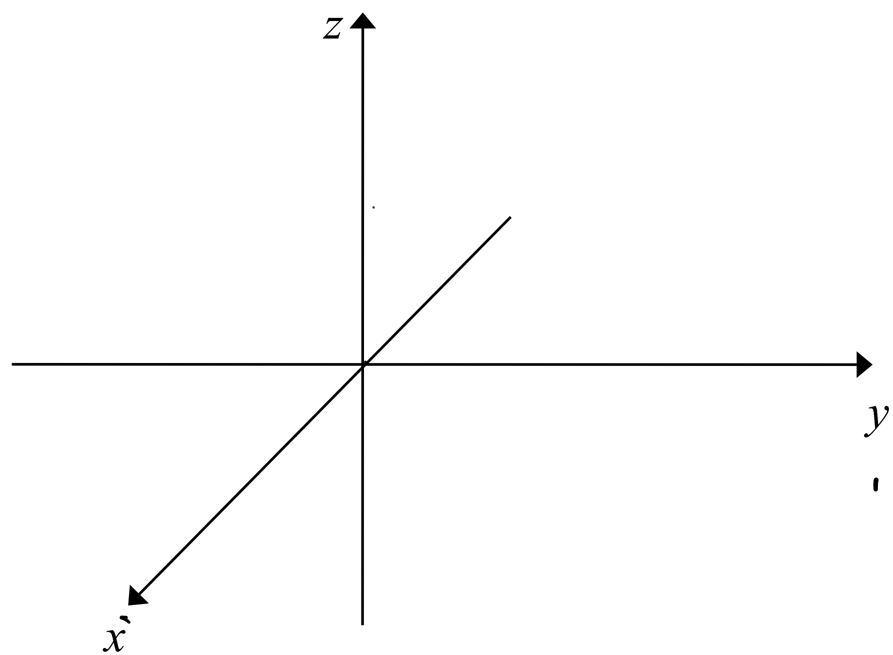
The **graph** of a function is the surface drawn in three dimensions with equation $z = f(x,y)$.

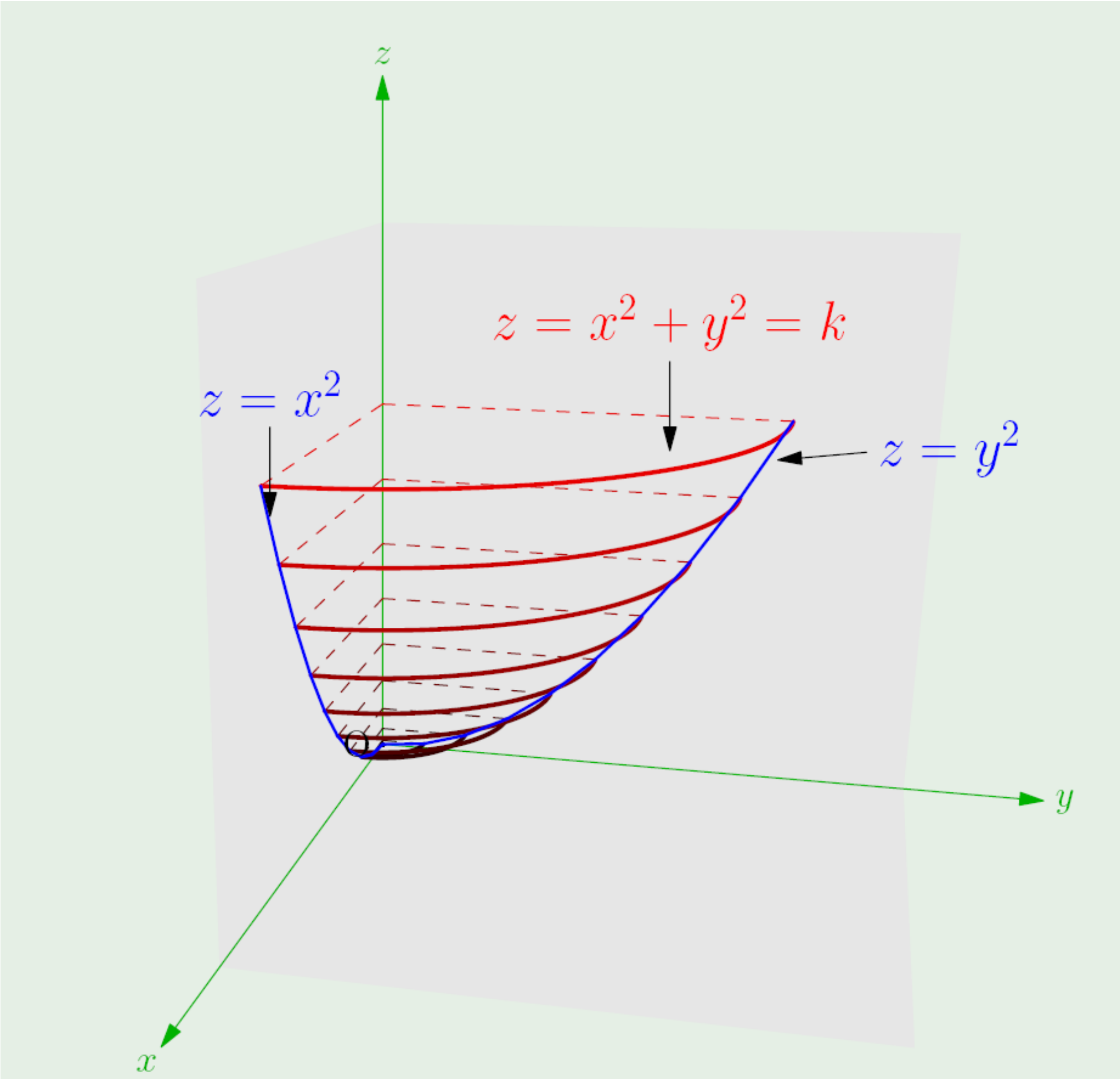
To draw a graph we use a technique called *slicing:*

1. Hold one variable constant (to some value k) while we plotting with respect to one of the other variables.

2. Change k, then repeat.

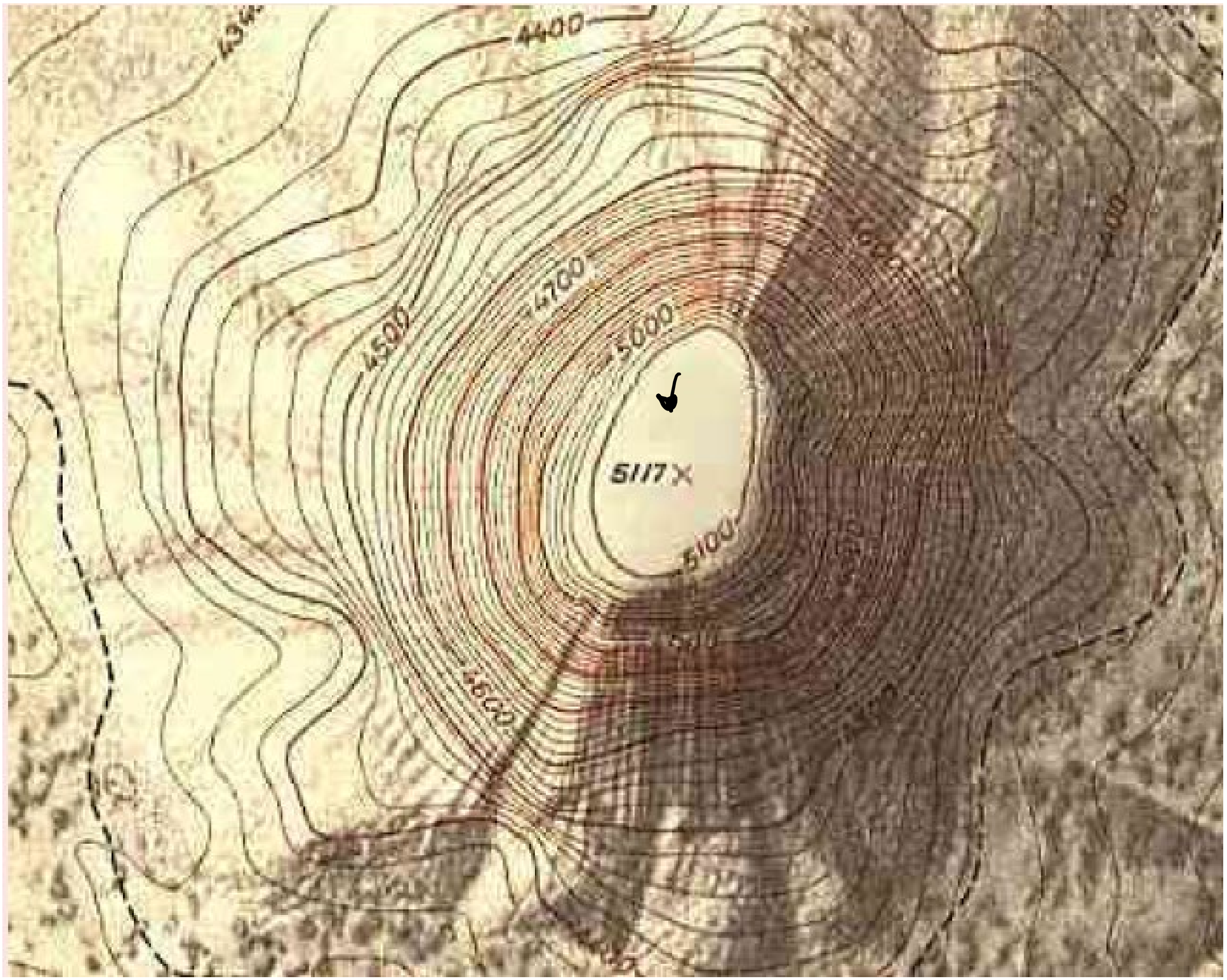
Example: Sketch a graph of $f(x,y) = x^2 + y^2$





The lines f(x,y) = const are known as **level curves.**

We can obtain a **contour plot** by plotting level curves f(x,y) = k at regular intervals of k, and then projecting the curves onto the x-y plane:



Recall: limit for a 1-variable function

Suppose that f(x) is a real valued function. The expression

$$\lim_{x \to a} f(x) = L$$

means that f(x) can be made arbitrarily close to L by taking x towards a.

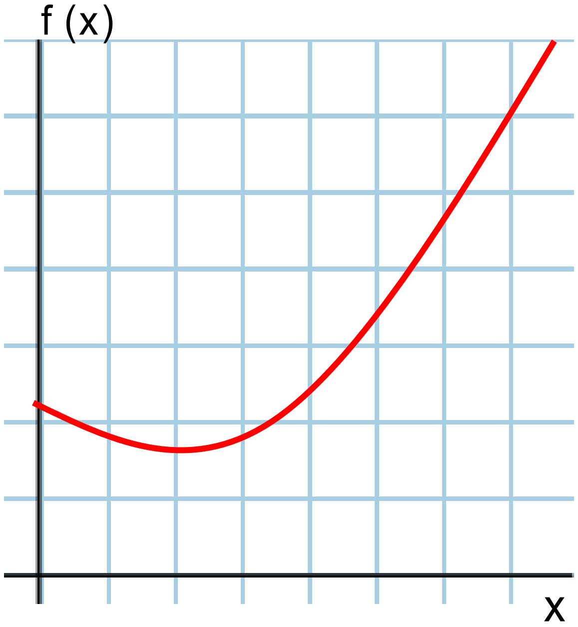
We can define a *left limit*

$$\lim_{x \to a^-} f(x) = L$$

and a *right limit*

$$\lim_{x \to a^+} f(x) = L$$

depending on the direction in which the limit is taken.



We only say that "a limit exists" if both these exist and are equal, i.e. if

$$L = \lim_{x \to a} f(x) = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

Limits in two dimensions

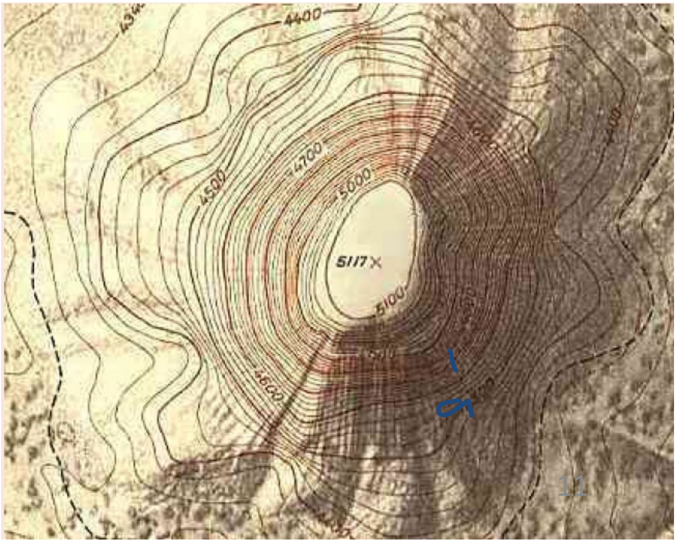
Definition: Given a function of two variables f(x,y), we write that

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

If the function *f* can be made arbitrarily close to *L* by taking the point (*x, y*) sufficiently close to (*a, b*).

Note that the limit *must be independent of the direction of approach*

If the limit process yields a different result for two directions, then the limit does not exist.



Example:

The function

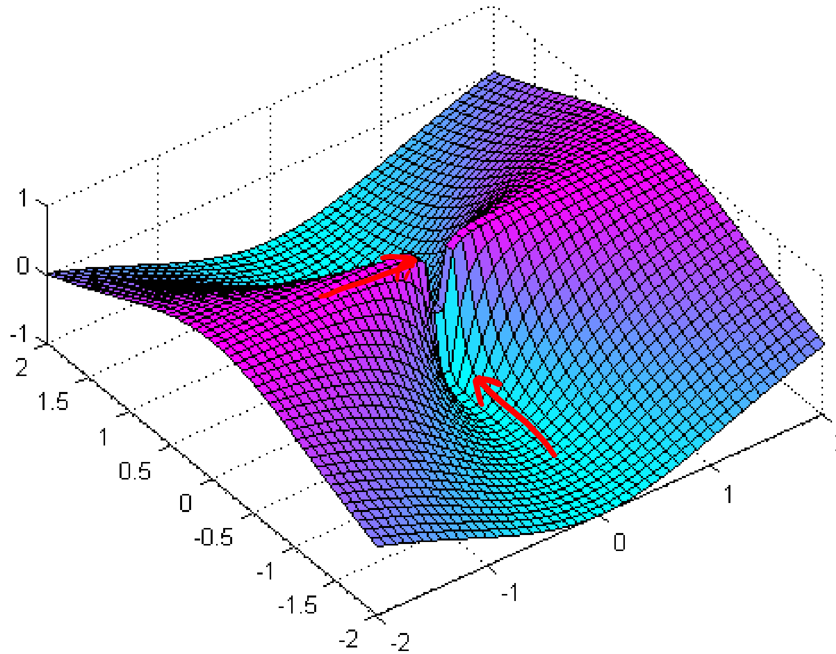
$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$$

has *no limit* as (x,y) -> (0,0).
Approaching to zero along x axis the limit is 1

$$\lim_{\substack{x \to 0 \\ y=0}} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{\substack{x \to 0 \\ y=0}} \frac{x^2 - 0}{x^2 + 0} = 1$$

Approaching zero along y axis the limit is -1.

$$\lim_{\substack{y \to 0 \\ x=0}} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{\substack{y \to 0 \\ x=0}} \frac{0 - y^2}{0 + y^2} = -1$$



Examples: Find the limits.

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{xy}{x^2 + y^2} =$$

$$\lim_{\substack{y \to 0 \\ x \to 0}} \frac{x^2 y}{x^4 + y^2} =$$

Examples: Find the limits.

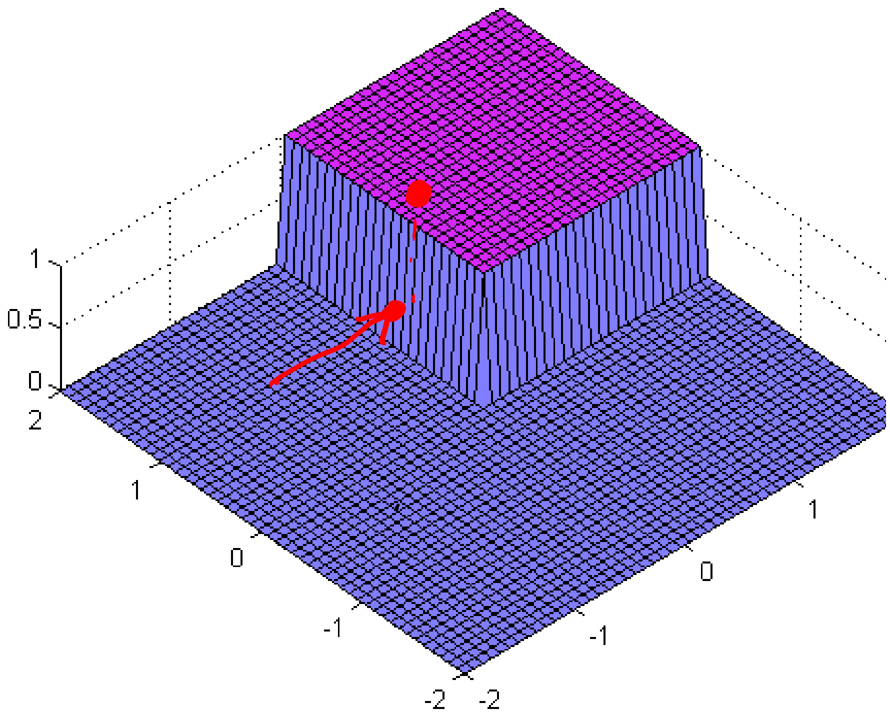
$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 y}{x^2 + y^2} =$$

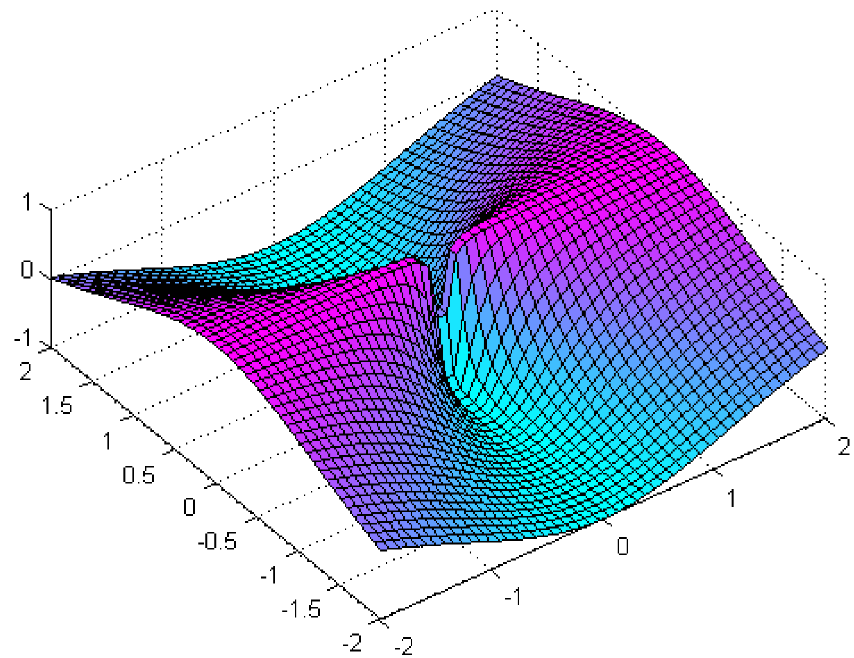
$$\lim_{\substack{y \to 0 \\ x \to 0}} \frac{xy}{\sqrt{x^2 + y^2}} =$$

Continuity in two dimensions

Definition A function f(x) is *continuous* at (*x*,*y*) = (*a*,*b*) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$





Rule of thumb: A discontinuity occurs if:

1. The function definition changes
2. There is a division by zero

Recall: The **derivative** of a real function is defined as

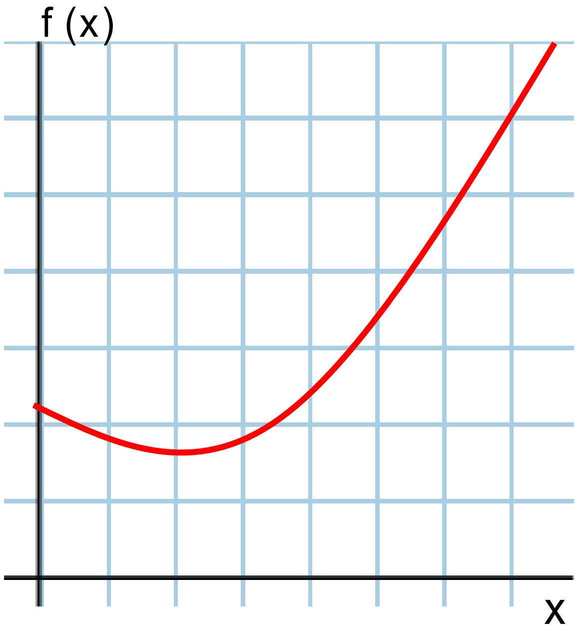
$$\frac{df}{dx} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

rise

run

$$x = x_0 + h$$

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$



A function f is *differentiable* at the point x_0 if the derivative *exists*.

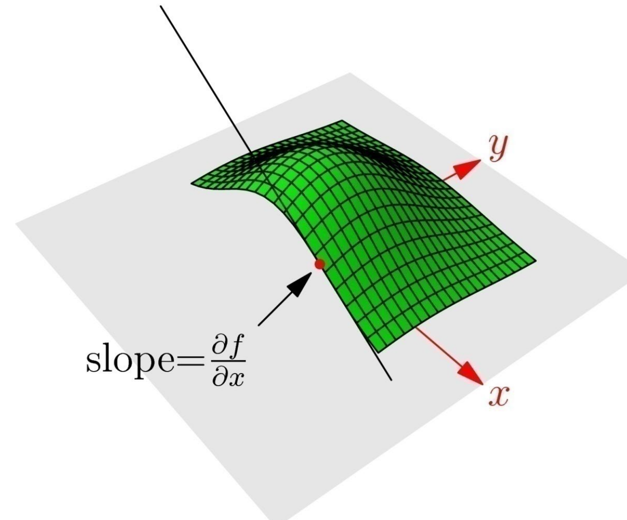
We often write the derivative as *f'(x)*:

$$f'(x) = \frac{df}{dx}$$

Definition of Partial Derivative

Let f : A ½ R^2 !   R be a real-valued function f(x,y). Then the partial Derivatives of f at the point (x_0,y_0) are defined as

$$\frac{\partial f}{\partial x} = \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}$$



$$\frac{\partial f}{\partial y} = \lim_{y \to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0}$$

We often write these as $f_x = \frac{\partial f}{\partial x}$ $f_y = \frac{\partial f}{\partial y}$

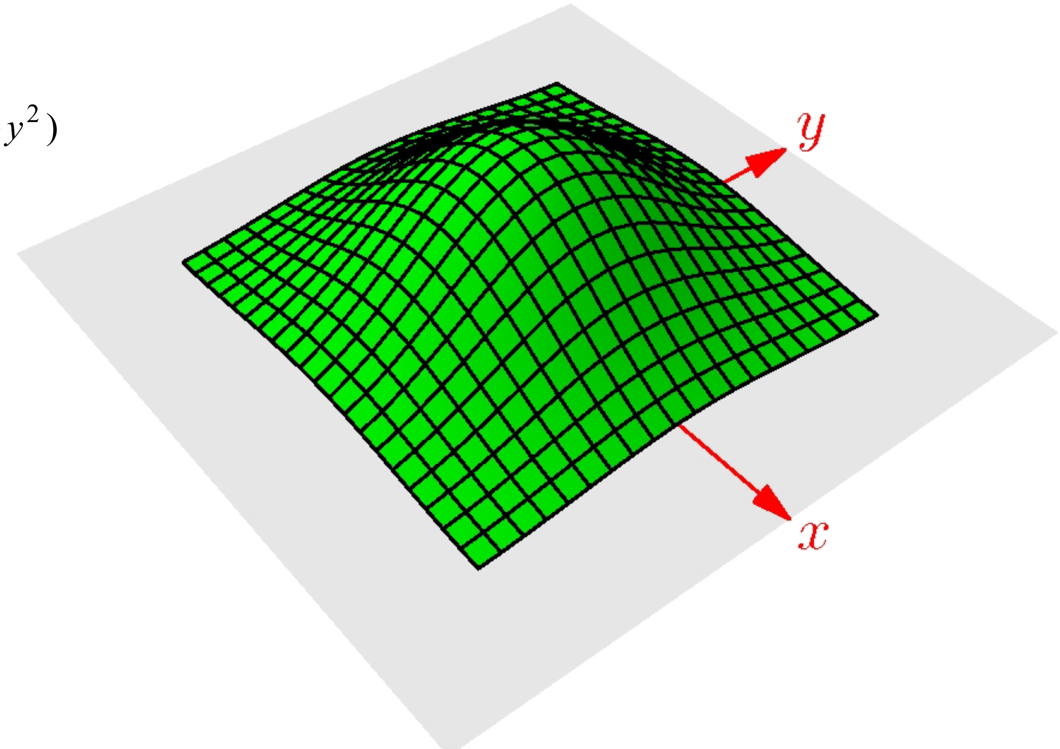
The partial derivative is the derivative *with all the other variables held constant.*

NB: it is very important to remember to write it with the partial "∂" symbol.

The partial derivative can be thought of as *the slope in a given direction*

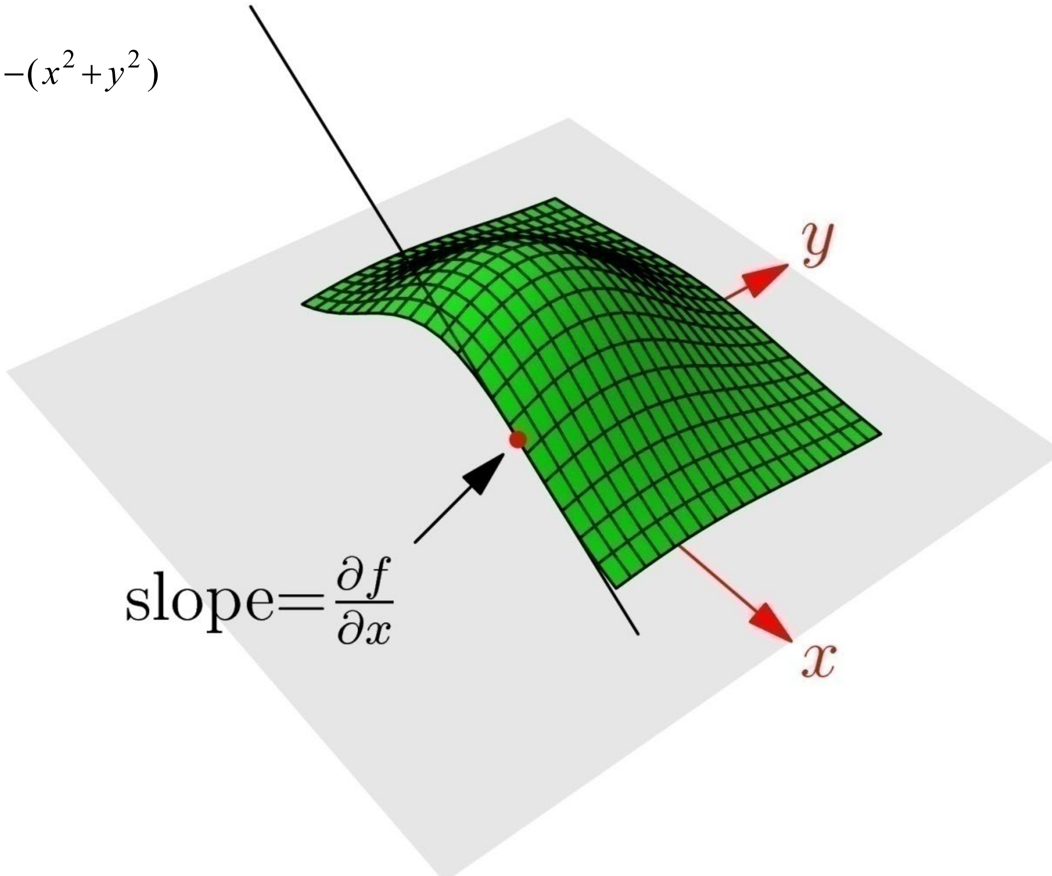
Example:

$$f(x,y) = e^{-(x^2+y^2)}$$



The partial derivative can be thought of as *the slope in a given direction*

Eg: $f(x, y) = e^{-(x^2 + y^2)}$



The partial derivative can be thought of as *the slope in a given direction*

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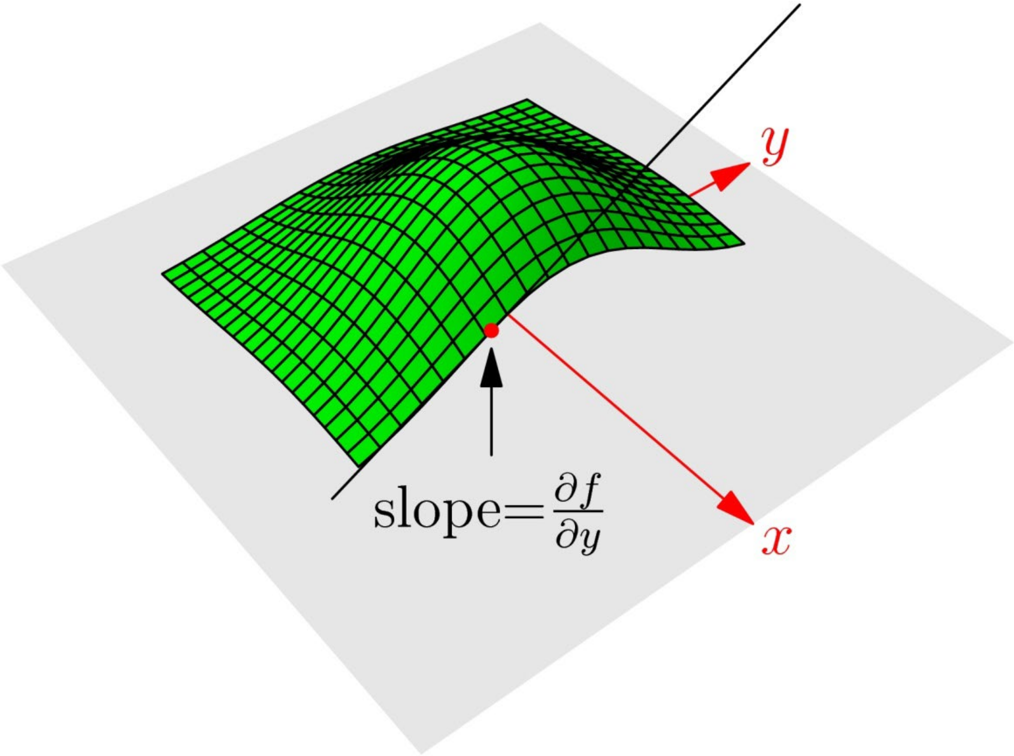
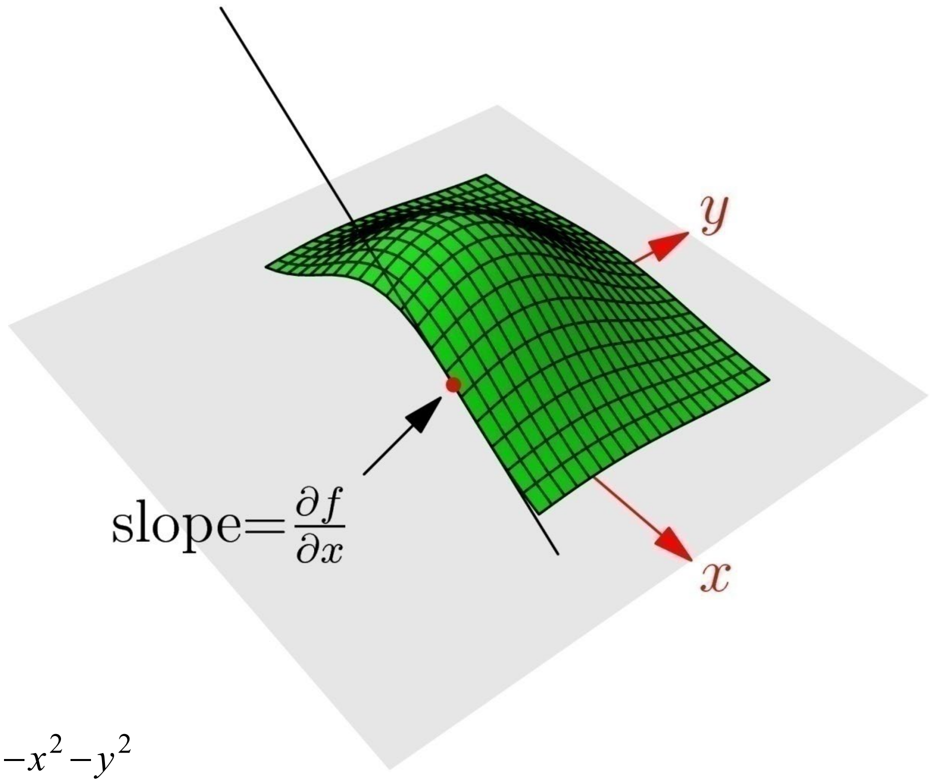


To compute the partial derivative we treat all other variables as constants:

Example:
Let

$$f(x, y) = e^{-x^2 - y^2}$$

then



$$f_x = \frac{\partial f}{\partial x} = e^{-x^2-y^2}(-x^2-y^2)'_x = -2xe^{-x^2-y^2}$$

$$f_y = \frac{\partial f}{\partial y} = e^{-x^2-y^2}(-x^2-y^2)'_y = -2ye^{-x^2-y^2}$$

Definition: A function *f* (*x*,*y*) with continuous first partial derivatives is called a differentiable function.

Example: Compute f_x and f_y for $f(x,y) = 2x^2 + 2xy + 2y^2$

$$f_x = \frac{\partial f}{\partial x} = (2x^2 + 2xy + 2y^2)'_x = 4x + 2y$$

$$f_y = \frac{\partial f}{\partial y} = (2x^2 + 2xy + 2y^2)'_y = 2x + 4y$$

Example:
Find f_y when

$$f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}$$

$$f_x = \frac{\partial f}{\partial x} = \frac{y(x^2+y^2)^{1/2} - xy\dfrac{x}{(x^2+y^2)^{1/2}}}{x^2+y^2} = \frac{y^3}{(x^2+y^2)^{3/2}}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{x(x^2+y^2)^{1/2} - xy\dfrac{y}{(x^2+y^2)^{1/2}}}{x^2+y^2} = \frac{x^3}{(x^2+y^2)^{3/2}}$$

Partial derivatives obey most of the usual rules of normal derivatives, such as the product rule, and the quotient rule.

However, there are differences too*:*

- It is *not* true that $\frac{\partial f}{\partial x} = 1/(\frac{\partial x}{\partial f})$
- The *chain rule* is more complicated

Example: Find f_x and f_y when:

$$f(x, y) = x \sin x \cos y$$

$$f_x = \frac{\partial f}{\partial x} = \cos y(\sin x + x \cos x)$$

$$f_y = \frac{\partial f}{\partial y} = -x \sin x \sin y$$

Example:
Find f_x and f_y of

$$f(x, y) = e^{-x^2}(xy^2 + 2)$$

$$f_x = \frac{\partial f}{\partial x} = e^{-x^2}(-2x)(xy^2 + 2) + e^{-x^2} y^2$$

$$f_y = \frac{\partial f}{\partial y} = e^{-x^2} 2xy$$

Higher order partial derivatives:

The partial derivatives f_x and f_y are known as the *first partial derivatives* or *the partial derivatives of first order.*

By differentiating once again, we obtain four partial derivatives of *second order:*

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$$

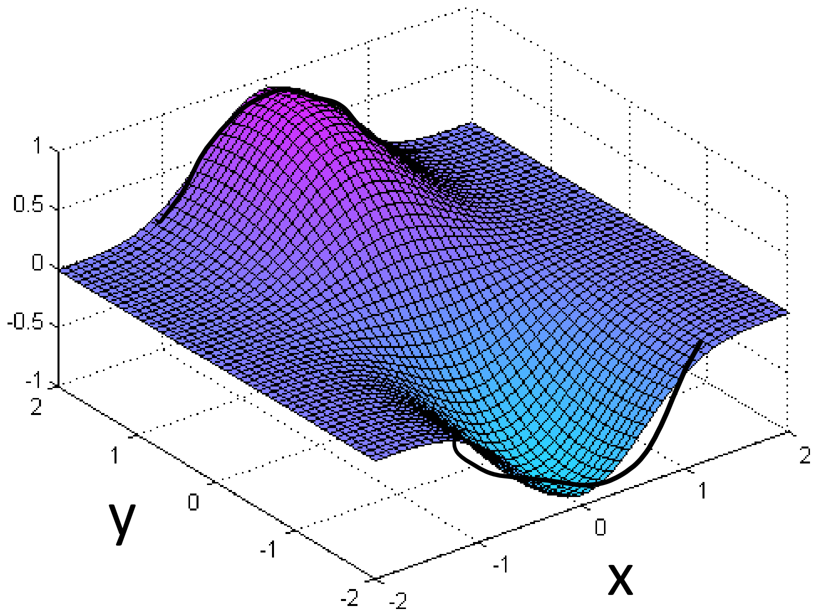
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

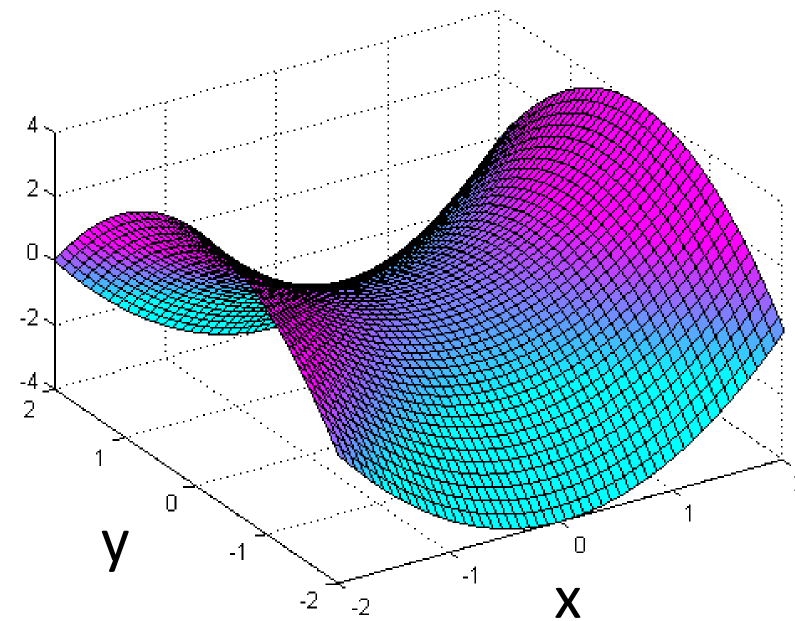
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{xy}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{yx}$$

We can also form *third* partial derivatives and so on.

We can think of the second-order partial derivatives as representing *degree of curvature* in each of the different directions.



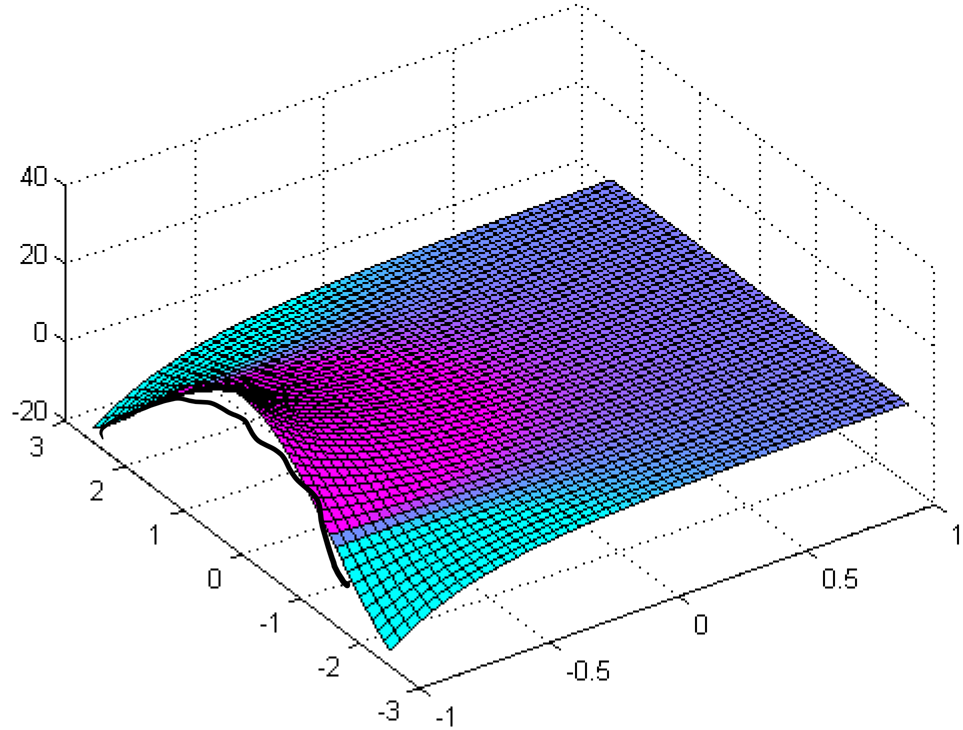


$f_{xx} > 0$ concave up $f_{yy} > 0$

$f_{xx} < 0$ concave down $f_{yy} < 0$

Example:

$$f(x,y) = \cos y e^{-3x}$$



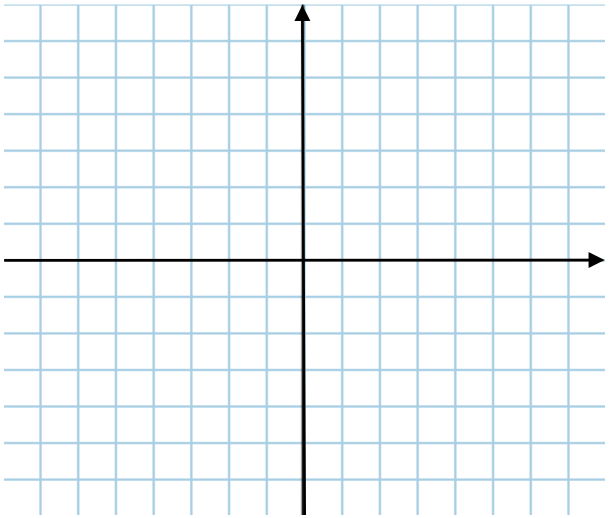
$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 9e^{-3x} \cos y$$

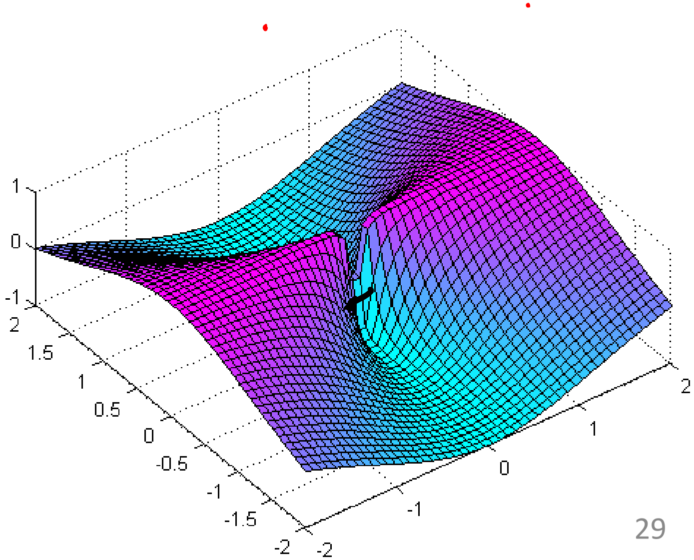
$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = -e^{-3x} \cos y$$

Clairaut's theorem: If both f_{xy} and f_{yx} are continuous on an open set surrounding (x_0,y_0), then

$$f_{xy} = f_{yx}$$

at the point (x_0,y_0).





Example: Compute f_{xy} and f_{yx} for

$$f(x,y) = x^2 y + x \sin y$$

$$f_x = 2xy + \sin y$$

$$f_{yx} = \frac{\partial^2 f}{\partial y \partial x} = 2x + \cos y$$

$$f_y = x^2 + x \cos y$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = 2x + \cos y$$

$$f_{yx} = f_{xy}$$

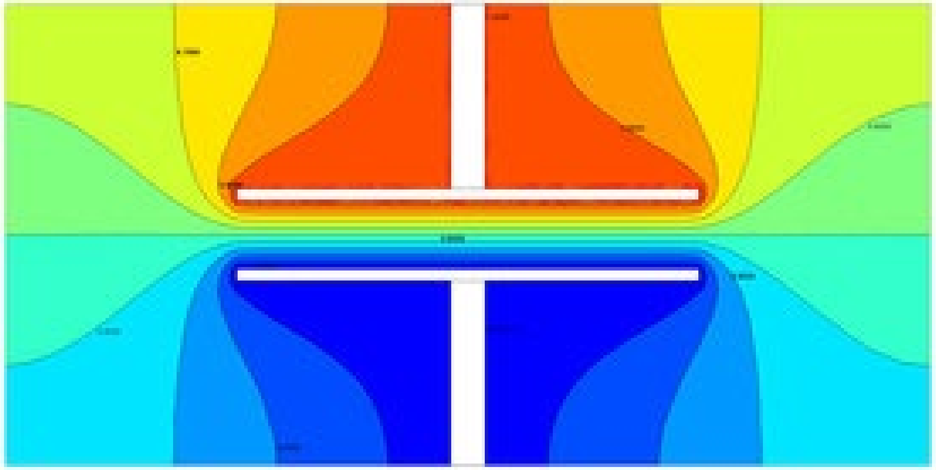
Ans: 2 x + cos y

Partial differential equations

Almost all mathematical models are expressed in terms of equations involving partial derivatives.

Laplace's equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$



The solution f can represent:

- Voltage across a capacitor
- Pressure in a fluid
- Velocity of steady-state fluid flow in a pipe

Solutions to Laplace's equation give the *functions with the smallest "overall curvature".*

The wave equation:

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$$

A solution to this equation is

$$u(x,t) = \cos(x - vt)$$

which is a wave moving in the positive x direction.