Calculus of Several Variables

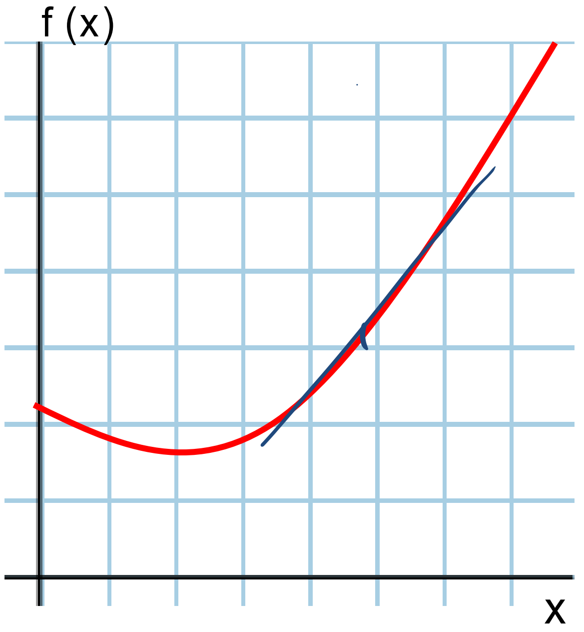
Content

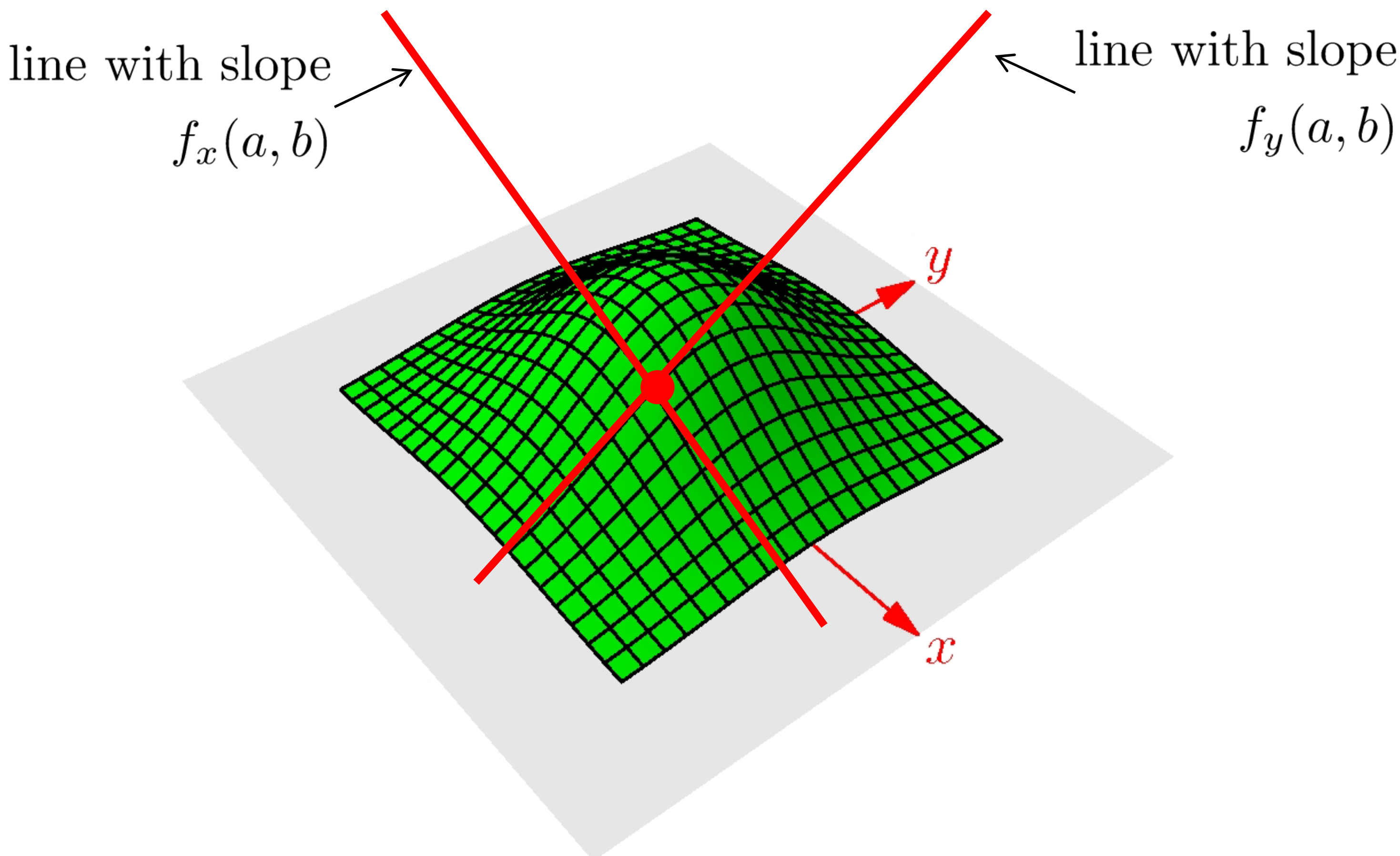
- Tangent plane
- Linear approximations
- Differentials
- Chain rule

Tangent lines and planes

Recall that for a function of a single variable y=f(x), we can find the the equation of a tangent line at a point x_0 from the derivative $f'(x_0)$:

$$y - f(x_0) = f'(x_0)(x - x_0)$$

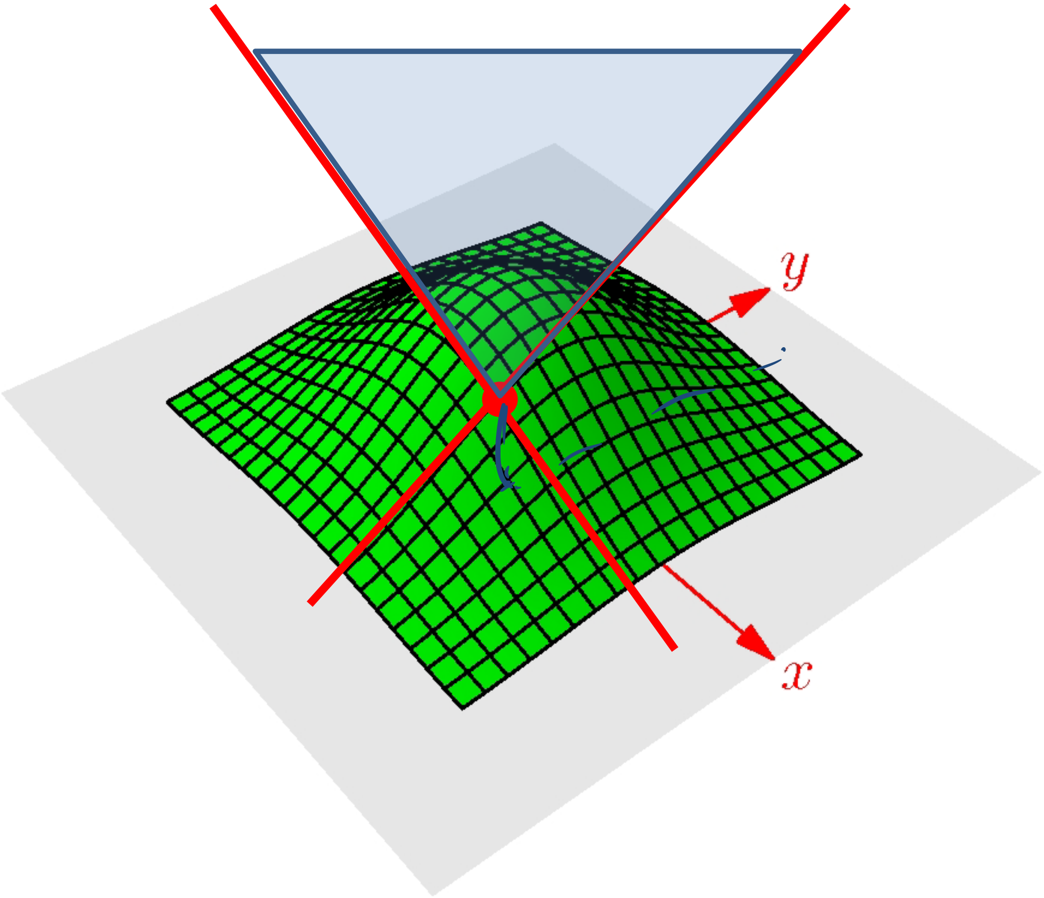




line with slope
$f_x(a,b)$
line with slope
$f_y(a,b)$
y
x

For a two-variable function f(x,y) the *tangent plane* top the function at a given point (a,b) is given by the equation

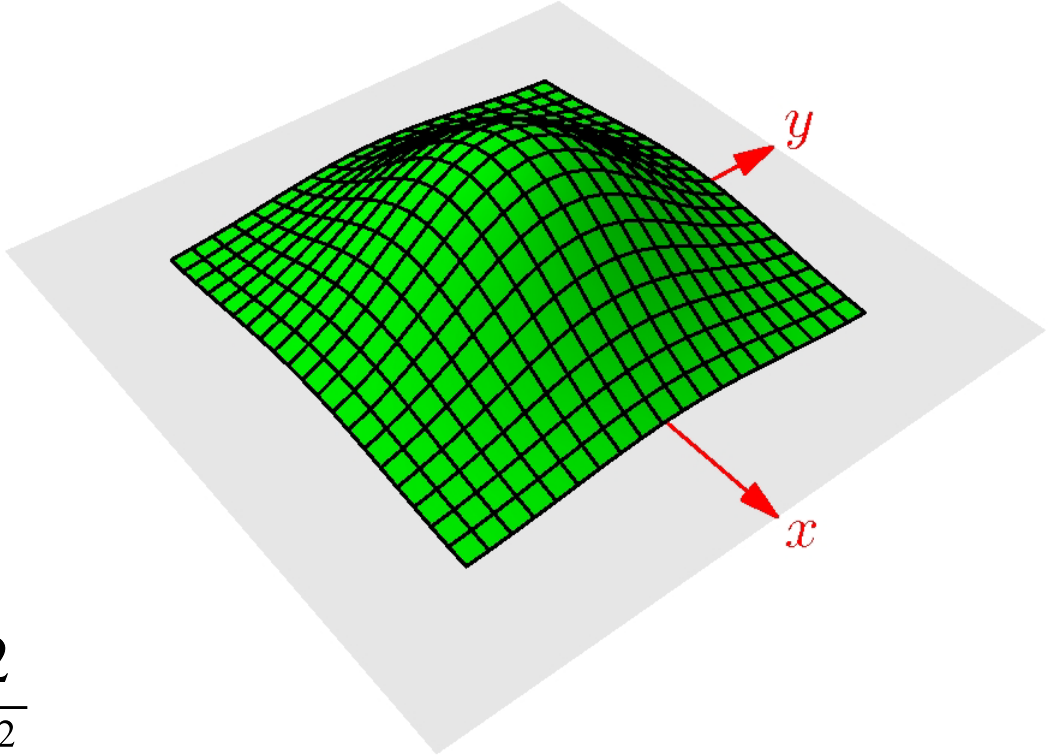
$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$



Example: Find the tangent plane to the surface

$$z = e^{-x^2-y^2}$$

at the point (1,1).



$$f(1,1) = e^{-1-1} = \frac{1}{e^2}$$

$$f_x = -2xe^{-x^2-y^2}, f_x(1,1) = -\frac{2}{e^2}$$

$$f_y = -2ye^{-x^2-y^2}, f_y(1,1) = -\frac{2}{e^2}$$

$$z - \frac{1}{e^2} = -\frac{2}{e^2}(x-1) - \frac{2}{e^2}(y-1)$$

$$e^2 z - 1 = -2x + 2 - 2y + 2$$

$$2x + 2y + e^2 z = 5$$

Example: Find the tangent plane to the surface

$$z = x^2 - y^2$$

at the point (0,1).

$$f(0,1) = -1$$

$$f_x = 2x, f_x(0,1) = 0$$

$$f_y = -2y, f_y(0,1) = -2$$

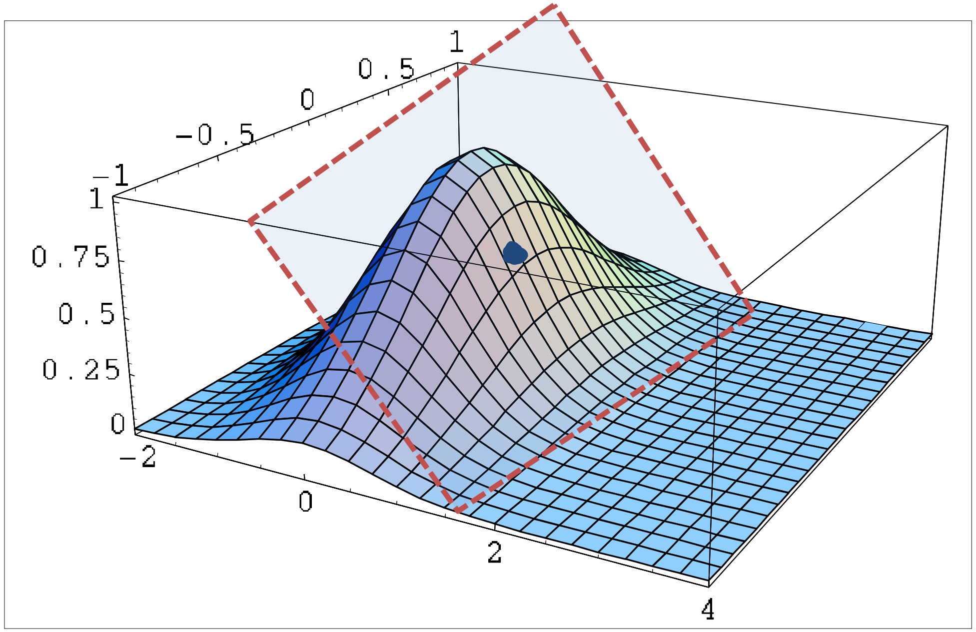
$$z + 1 = 0(x - 0) - 2(y - 1)$$

$$z + 1 = -2y + 2$$

$$2y + z = 1$$

Linear approximation

The tangent plane and the function f(x,y) have very similar values near the point of contact



This means that the equation of the tangent plane

$$z = L(x, y)$$

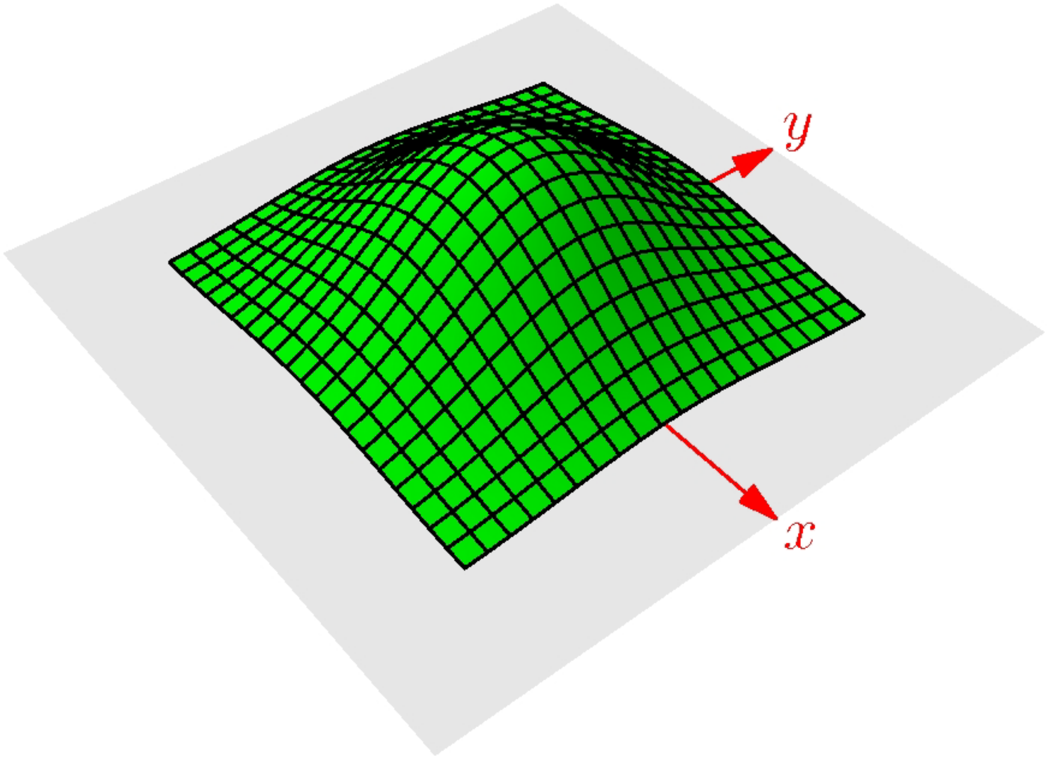
can be regarded as an approximation for f(x,y).
Rewriting the equation of the plane, we find:

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Example: Find a linear approximation for the function near the point (1,1).

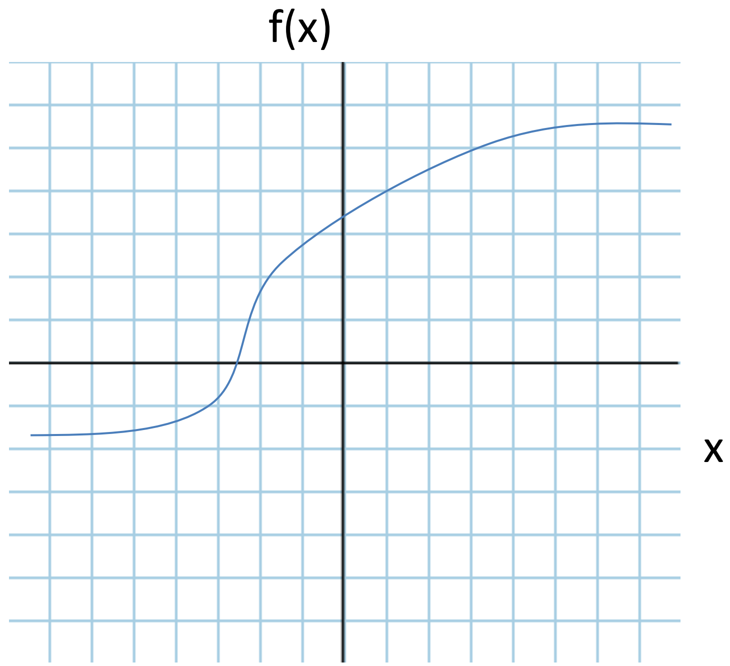
$$f(x,y) = e^{-x^2-y^2}$$

Construct the tangent plane at (1,1)



Differentials

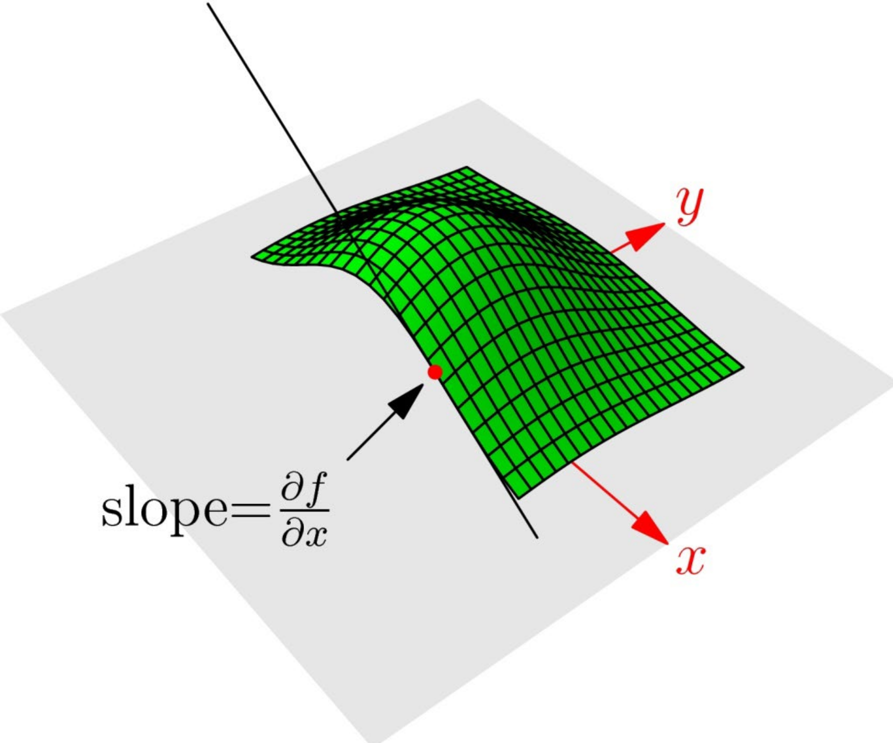
For a single variable function f(x), the *differential* df is the change in the function for a small change in x:



$$df = \frac{df}{dx}dx$$

- The differentials *df* and *dx* can be thought of as changes in *f* and *x* taken *in the limit that both become very small*

Can we do the same thing for functions of two variables?

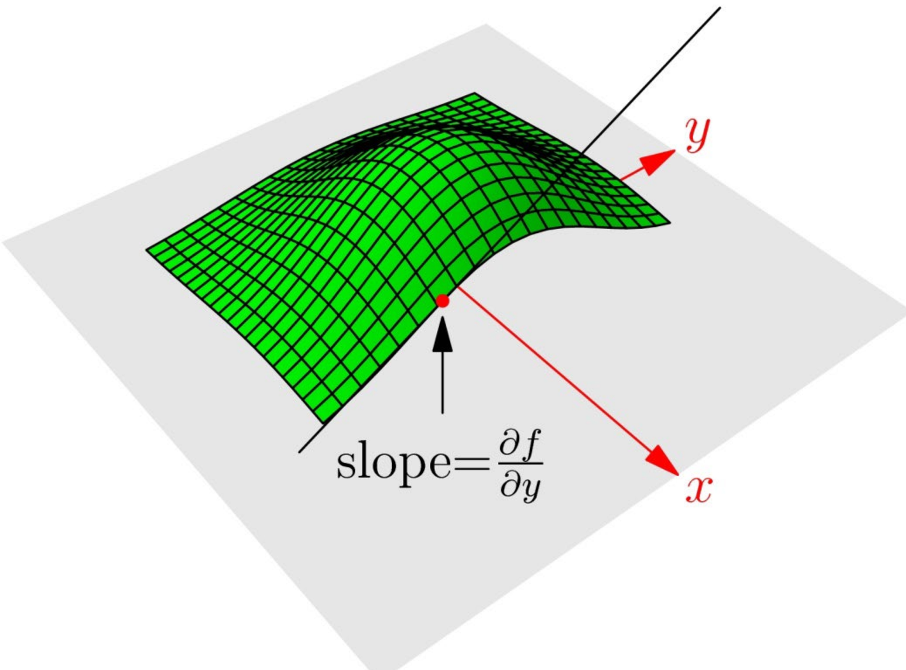


For a small step dx in the x direction the function changes by

$$df = \frac{\partial f}{\partial x} dx$$

For a small step dy in the y direction the function changes by

$$df = \frac{\partial f}{\partial y} dy$$



Therefore for a change in position in *both* dx *and* dy the function changes by

Definition:

For a differentiable function *f* (*x,y*), we define the *total differential df* as

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

where the quantities *dx* and *dy* represent infinitessimal changes in *x* and *y*.

Example:
Find the total differential of $f(x,y) = e^{-x} \sin(y+2x)$.

$$f_x = -e^{-x}\sin(y+2x) + 2e^{-x}\cos(y+2x)$$

$$f_y = e^{-x}\cos(y+2x)$$

$$df = f_x dx + f_y dy =$$

$$(-e^{-x}\sin(y+2x) + 2e^{-x}\cos(y+2x))dx + e^{-x}\cos(y+2x)dy$$

The chain rule with one independent variable

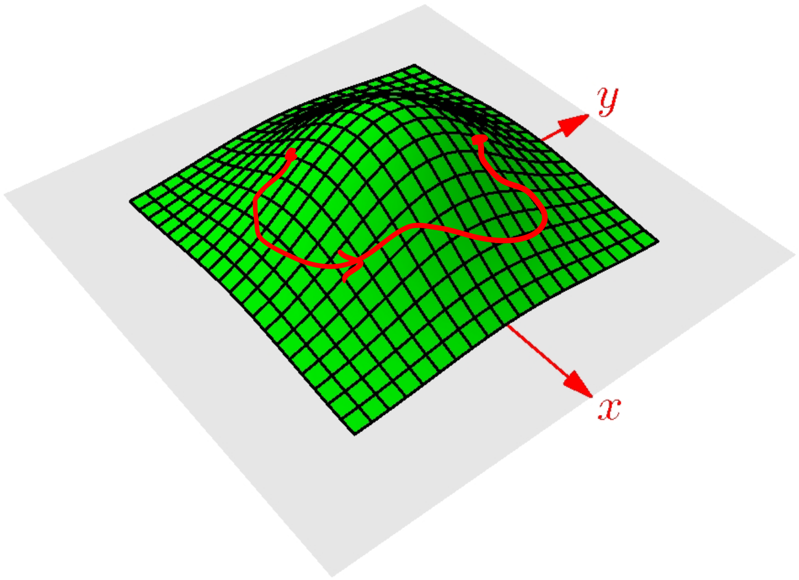
Suppose that $z = f(x,y)$ is a differentiable function of x and y, and that $x = x(t)$ and $y = y(t)$ are both differentiable functions of t.
$z = f(x(t), y(t))$

From the differential

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

we obtain the *chain rule*

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$



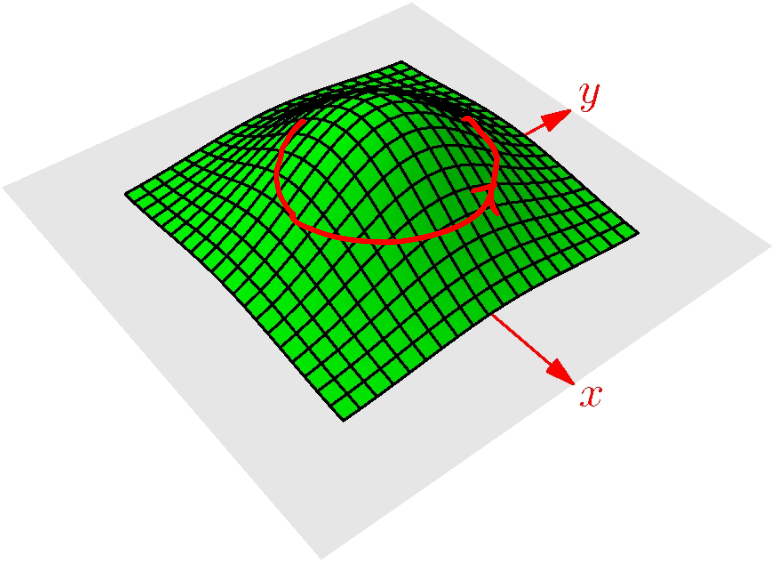
(x,y) are called *intermediate variables.*

t is the only *independent variable.*

Example: Let

$$z = f(x, y) = e^{-x^2 - y^2}$$

and let x(t) = cos t, y(t) = sin t. Compute dz/dt.

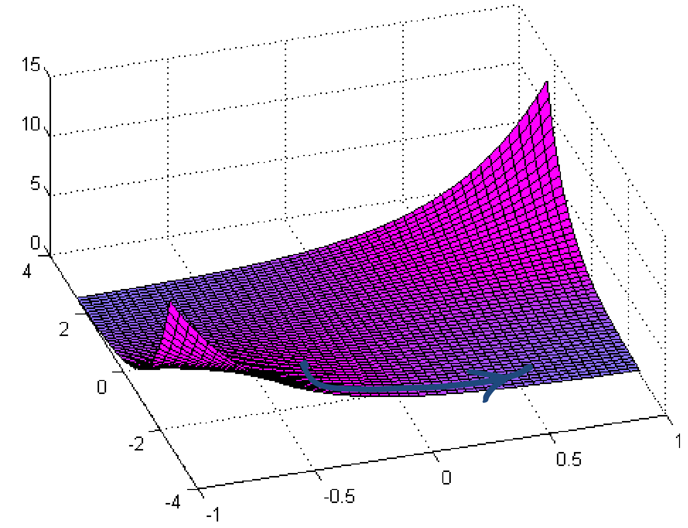


$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = -2xe^{-x^2-y^2}(-)\sin t - 2ye^{-x^2-y^2}\cos t =$$

$$= 2e^{-x^2-y^2}(x\sin t - y\cos t) = 2e^{-x^2-y^2}(\cos t\sin t - \sin t\cos t) = 0$$

Example: Let

$$z = f(x, y) = e^{xy}$$



and let x(t) = t^2, y(t) = t^3.
Compute dz/dt.

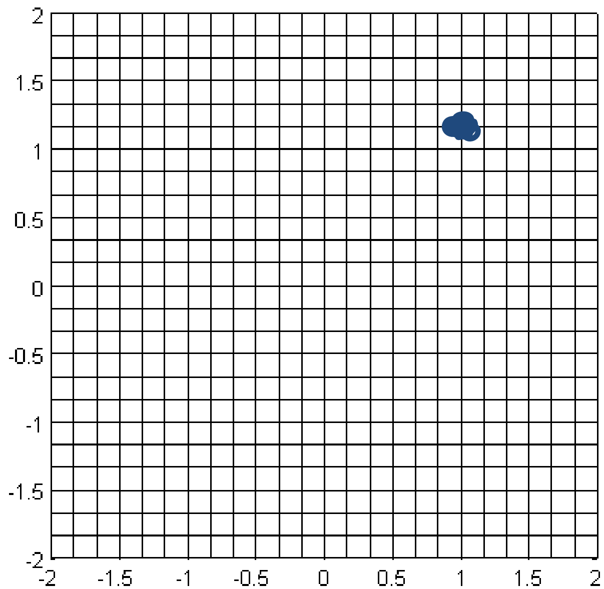
$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = ye^{xy}2t + xe^{xy}3t^2 = t^3e^{xy}2t + t^2e^{xy}3t^2 = 5t^4e^{t^5}$$

Changing coordinates

Suppose now that $x = x(s,t)$ and $y = y(s,t)$ are themselves functions of *new independent variables* s and t.

The new variables (s,t) also describe points in 2D space.

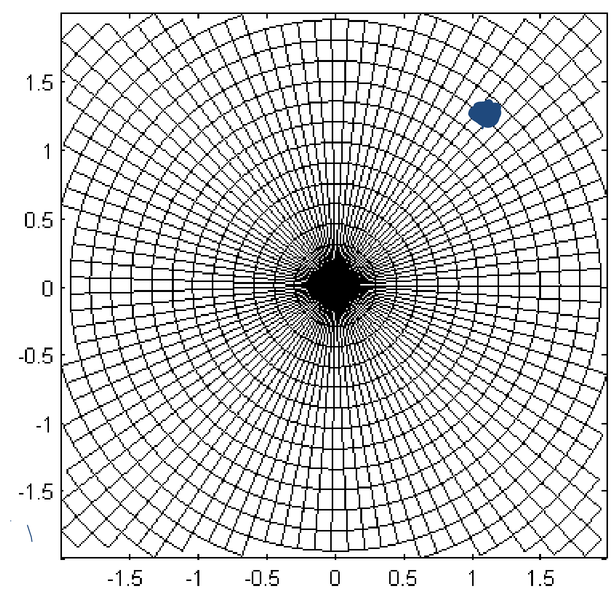
Old coordinate system (x,y)



$$s = s(x,y)$$
$$t = t(x,y)$$

$$x = x(s,t)$$
$$y = y(s,t)$$

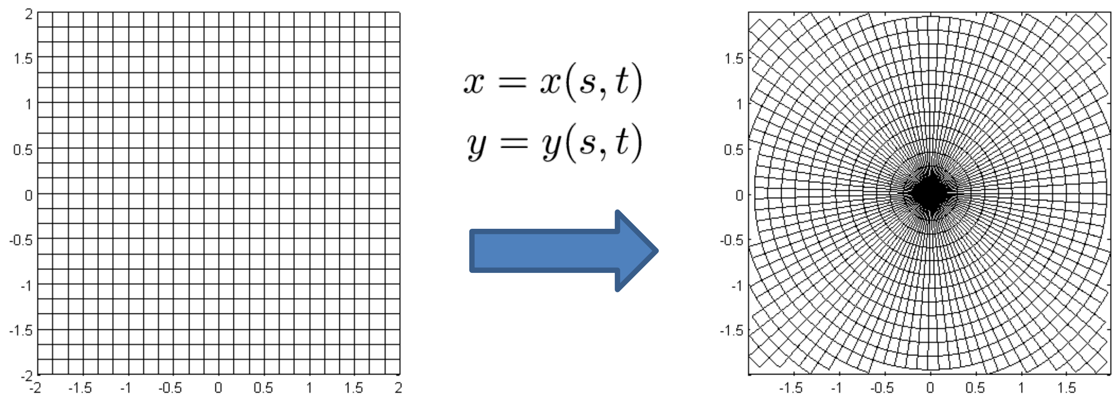
New coordinate system (s,t)



To move derivatives from one coordinate system to another we use the *chain rule for two independent variables:*

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$



Example:

For the function $f(x,y) = e^{-x^2-y^2}$
compute $\partial f / \partial r$, $\partial f / \partial \theta$ where $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial z}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = -2xe^{-x^2-y^2}\cos\theta - 2ye^{-x^2-y^2}\sin\theta =$$

$$= -2re^{-r^2}\cos^2\theta - 2re^{-r^2}\sin^2\theta = -2re^{-r^2}$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -2xe^{-x^2-y^2}(-r)\sin\theta - 2ye^{-x^2-y^2} r\sin\theta =$$

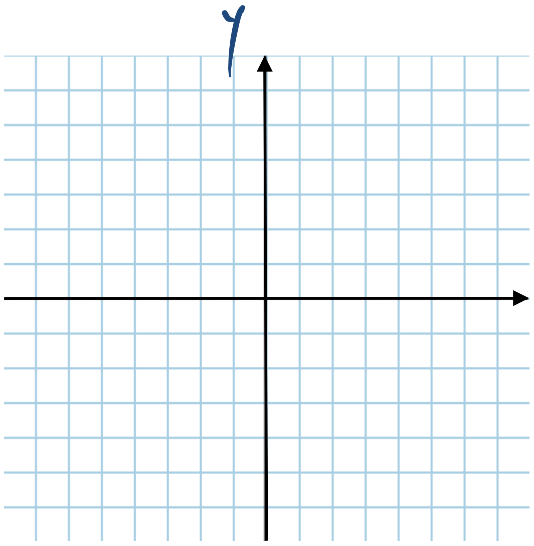
$$= 2re^{-r^2}\cos\theta\sin\theta - 2re^{-r^2}\sin\theta\cos\theta = 0$$

Implicit Differentiation

Recall that a function can be defined explicitly:

$$y = \pm\sqrt{4 - x^2}$$

$$y = f(x)$$



or implicitly:

$$F(x, y) = 0$$

$$x^2 + y^2 = 4$$

Now let x = t, and y = y(x) = y(x(t))).
Then using the chain rule on *F*(*x*,*y*), we can find *dy/dx*:

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Example: Find y' if $x^3 - 2xy + y^2 = 1$.

We can remember the chain rule using a tree diagram:

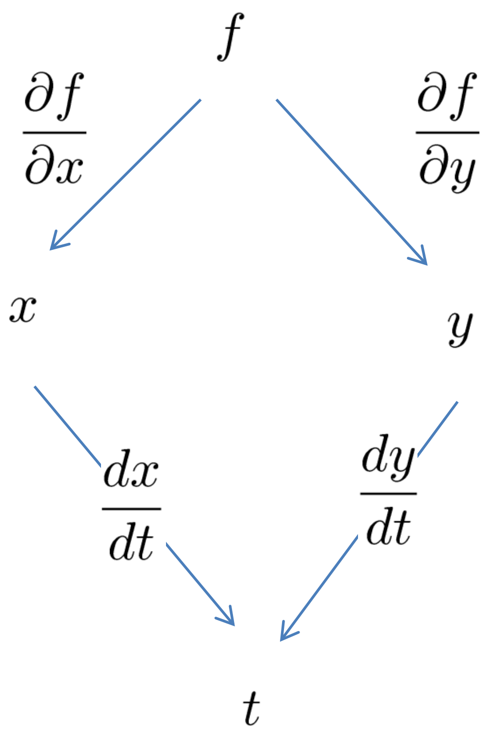
f

is a function of

x and y

which are functions of

t



How to remember the chain rule for two independent variables:

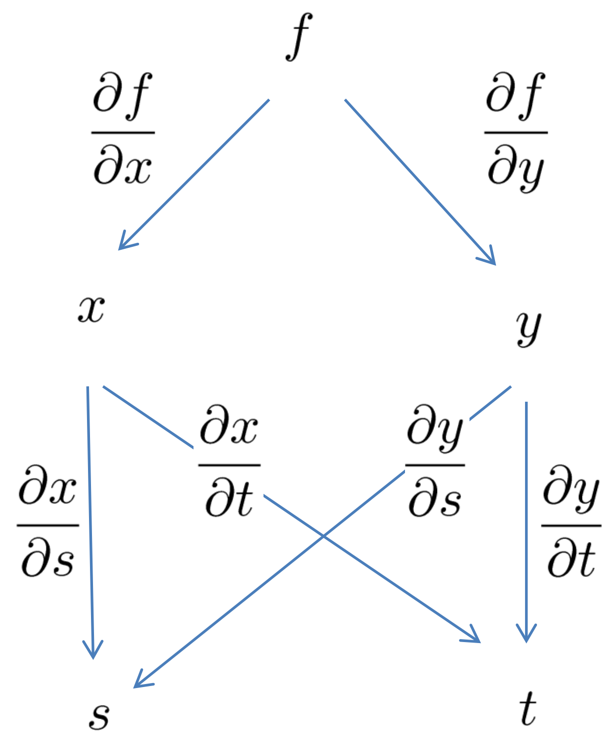
f

is a function of

x and y

which are functions of

s and t



General formulation of the chain rule:

Suppose that u($x_1, x_2, x_3, \ldots, x_n$) is a differentiable function of the n variables $x_1, x_2, x_3, \ldots, x_n$.

And each x_j ($t_1, t_2, \ldots t_m$) is a differentiable function of the m variables $t_1, t_2, \ldots t_m$.

Then the derivative of u with respect to each of the t_i variables is

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

Note that we can write this in matrix form, where **J** is transition (Jacobian) matrix

$$\begin{pmatrix} \frac{\partial u}{\partial t_1} \\ \frac{\partial u}{\partial t_2} \\ \vdots \\ \frac{\partial u}{\partial t_m} \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1}{\partial t_1} & \frac{\partial x_2}{\partial t_1} & \cdots & \frac{\partial x_n}{\partial t_1} \\ \frac{\partial x_1}{\partial t_2} & \frac{\partial x_2}{\partial t_2} & \cdots & \frac{\partial x_n}{\partial t_2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial x_1}{\partial t_m} & \frac{\partial x_2}{\partial t_m} & \cdots & \frac{\partial x_n}{\partial t_m} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x_1} \\ \frac{\partial u}{\partial x_2} \\ \vdots \\ \frac{\partial u}{\partial x_n} \end{pmatrix}, \mathbf{J} = \begin{pmatrix} \frac{\partial x_1}{\partial t_1} & \frac{\partial x_2}{\partial t_1} & \cdots & \frac{\partial x_n}{\partial t_1} \\ \frac{\partial x_1}{\partial t_2} & \frac{\partial x_2}{\partial t_2} & \cdots & \frac{\partial x_n}{\partial t_2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial x_1}{\partial t_m} & \frac{\partial x_2}{\partial t_m} & \cdots & \frac{\partial x_n}{\partial t_m} \end{pmatrix}$$

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=0,$$

Laplacian of function U(x,y)

Find Laplacian in cylindrical coordinate system

$$x=r\cos\theta,\quad y=r\sin\theta$$

$$U(x,y)=U(r\cos\theta,r\sin\theta)=\tilde{U}(r,\theta)=\tilde{U}(r(x,y),\theta(x,y))$$

$$\frac{\partial U}{\partial x}=\frac{\partial \tilde{U}}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial \tilde{U}}{\partial \theta}\frac{\partial \theta}{\partial x}=\tilde{U}_r\frac{\partial r}{\partial x}+\tilde{U}_\theta\frac{\partial \theta}{\partial x}$$

$$\frac{\partial^2 U}{\partial x^2}=\frac{\partial}{\partial x}\frac{\partial U}{\partial x}=\frac{\partial}{\partial x}\left(\tilde{U}_r\frac{\partial r}{\partial x}+\tilde{U}_\theta\frac{\partial \theta}{\partial x}\right)=\frac{\partial \tilde{U}_r}{\partial x}\frac{\partial r}{\partial x}+\tilde{U}_r\frac{\partial^2 r}{\partial x^2}+$$

$$+\frac{\partial \tilde{U}_\theta}{\partial x}\frac{\partial \theta}{\partial x}+\tilde{U}_\theta\frac{\partial^2 \theta}{\partial x^2}=\frac{\partial r}{\partial x}\left(\frac{\partial \tilde{U}_r}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial \tilde{U}_r}{\partial \theta}\frac{\partial \theta}{\partial x}\right)+\tilde{U}_r\frac{\partial^2 r}{\partial x^2}+$$

$$+\frac{\partial \theta}{\partial x}\left(\frac{\partial \tilde{U}_\theta}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial \tilde{U}_\theta}{\partial \theta}\frac{\partial \theta}{\partial x}\right)+\tilde{U}_\theta\frac{\partial^2 \theta}{\partial x^2}=\frac{\partial^2 \tilde{U}}{\partial r^2}\left(\frac{\partial r}{\partial x}\right)^2+\frac{\partial^2 \tilde{U}}{\partial \theta\partial r}\frac{\partial \theta}{\partial x}\frac{\partial r}{\partial x}+\frac{\partial \tilde{U}}{\partial r}\frac{\partial^2 r}{\partial x^2}$$

$$+\frac{\partial^2 \tilde{U}}{\partial \theta^2}\left(\frac{\partial \theta}{\partial x}\right)^2+\frac{\partial^2 \tilde{U}}{\partial r\partial \theta}\frac{\partial \theta}{\partial x}\frac{\partial r}{\partial x}+\frac{\partial \tilde{U}}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2}$$

$$\frac{\partial U}{\partial y}=\frac{\partial \tilde{U}}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial \tilde{U}}{\partial \theta}\frac{\partial \theta}{\partial y}=\tilde{U}_r\frac{\partial r}{\partial y}+\tilde{U}_\theta\frac{\partial \theta}{\partial y}$$

$$\frac{\partial^2 U}{\partial y^2}=\frac{\partial}{\partial y}\frac{\partial U}{\partial y}=\frac{\partial}{\partial y}\left(\tilde{U}_r\frac{\partial r}{\partial y}+\tilde{U}_\theta\frac{\partial \theta}{\partial y}\right)=\frac{\partial \tilde{U}_r}{\partial y}\frac{\partial r}{\partial y}+\tilde{U}_r\frac{\partial^2 r}{\partial y^2}+$$

$$+\frac{\partial \tilde{U}_\theta}{\partial y}\frac{\partial \theta}{\partial y}+\tilde{U}_\theta\frac{\partial^2 \theta}{\partial y^2}=\frac{\partial r}{\partial y}\left(\frac{\partial \tilde{U}_r}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial \tilde{U}_r}{\partial \theta}\frac{\partial \theta}{\partial y}\right)+\tilde{U}_r\frac{\partial^2 r}{\partial y^2}+$$

$$+\frac{\partial \theta}{\partial y}\left(\frac{\partial \tilde{U}_\theta}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial \tilde{U}_\theta}{\partial \theta}\frac{\partial \theta}{\partial y}\right)+\tilde{U}_\theta\frac{\partial^2 \theta}{\partial y^2}=\frac{\partial^2 \tilde{U}}{\partial r^2}\left(\frac{\partial r}{\partial y}\right)^2+\frac{\partial^2 \tilde{U}}{\partial \theta\partial r}\frac{\partial \theta}{\partial x}\frac{\partial r}{\partial y}+\frac{\partial \tilde{U}}{\partial r}\frac{\partial^2 r}{\partial y^2}$$

$$+\frac{\partial^2 \tilde{U}}{\partial \theta^2}\left(\frac{\partial \theta}{\partial y}\right)^2+\frac{\partial^2 \tilde{U}}{\partial r\partial \theta}\frac{\partial \theta}{\partial y}\frac{\partial r}{\partial y}+\frac{\partial \tilde{U}}{\partial \theta}\frac{\partial^2 \theta}{\partial y^2}=$$

$$\frac{\partial^2 \tilde{U}}{\partial r^2}\left(\frac{\partial r}{\partial y}\right)^2+2\frac{\partial^2 \tilde{U}}{\partial \theta\partial r}\frac{\partial \theta}{\partial x}\frac{\partial r}{\partial y}+\frac{\partial \tilde{U}}{\partial r}\frac{\partial^2 r}{\partial y^2}+\frac{\partial^2 \tilde{U}}{\partial \theta^2}\left(\frac{\partial \theta}{\partial y}\right)^2+\frac{\partial \tilde{U}}{\partial \theta}\frac{\partial^2 \theta}{\partial y^2}$$

$$\frac{\partial r}{\partial x} = \frac{\partial \left(x^2 + y^2\right)^{1/2}}{\partial x} = \frac{x}{\left(x^2 + y^2\right)^{1/2}} = \cos\theta$$

$$\frac{\partial \theta}{\partial x} = \frac{\partial \arctan \frac{y}{x}}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \frac{-y}{x^2} = -\frac{y}{x^2 + y^2} = -\frac{\sin\theta}{r}$$

$$\frac{\partial r}{\partial y} = \frac{\partial \left(x^2 + y^2\right)^{1/2}}{\partial y} = \frac{y}{\left(x^2 + y^2\right)^{1/2}} = \sin\theta$$

$$\frac{\partial \theta}{\partial y} = \frac{\partial \arctan \frac{y}{x}}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \frac{1}{x} = \frac{x}{x^2 + y^2} = \frac{\cos\theta}{r}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial r}{\partial x} = \frac{\partial \cos\theta}{\partial x} = -\sin\theta\frac{\partial\theta}{\partial x} = \frac{\sin^2\theta}{r}$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial \theta}{\partial x} = -\frac{\partial}{\partial x}\frac{\sin\theta}{r} = -\sin\theta\frac{\partial}{\partial x}\frac{1}{r} - \frac{1}{r}\frac{\partial \sin\theta}{\partial x} = \frac{2\sin\theta\cos\theta}{r^2}$$

$$\frac{\partial^2 r}{\partial y^2} = \frac{\cos^2\theta}{r}$$

$$\frac{\partial^2 \theta}{\partial y^2} = -\frac{2\sin\theta\cos\theta}{r^2}$$

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}$$