Calculus of Several Variables

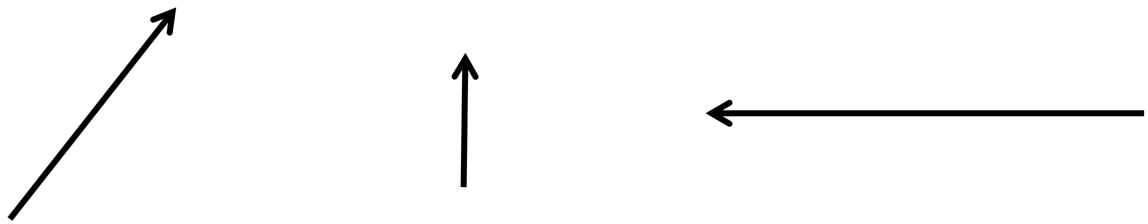
Content

- Directional derivative, gradient

- Maximum and Minimum of $f(x,y)$

Vectors: revision

A vector is something that has both *magnitude* and *direction*



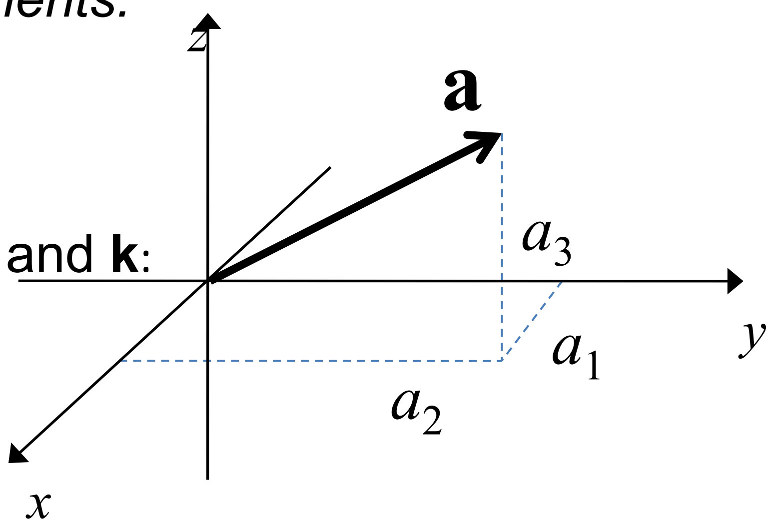
Force, acceleration, displacement, velocity, torque, momentum are all vectors.
(Mass, energy, time, speed, height, temperature are not)

We can write a vector in terms of its *components:*

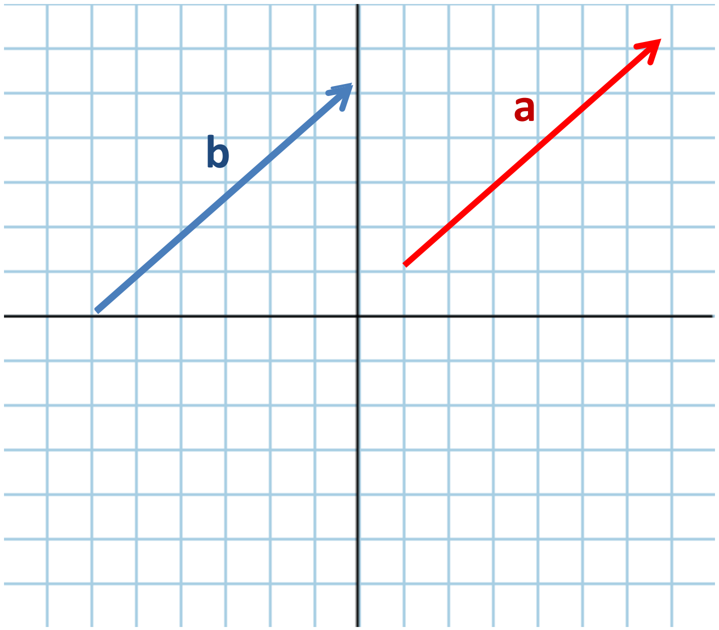
$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

or in terms of the coordinate unit vectors **i, j** and **k**:

$$\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$$



Vectors have only magnitude and direction, and so have no *position.*

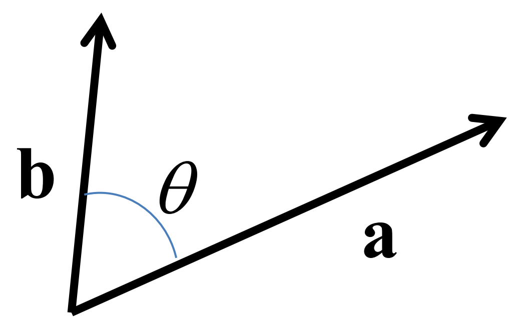


One of the most important vector operations is the *dot product.*
The dot product between two vectors **a** and **b** is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

This is equivalent to

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$



The dot product tells us how much two vectors are pointing in the same direction
The length of a vector

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

Unit vector

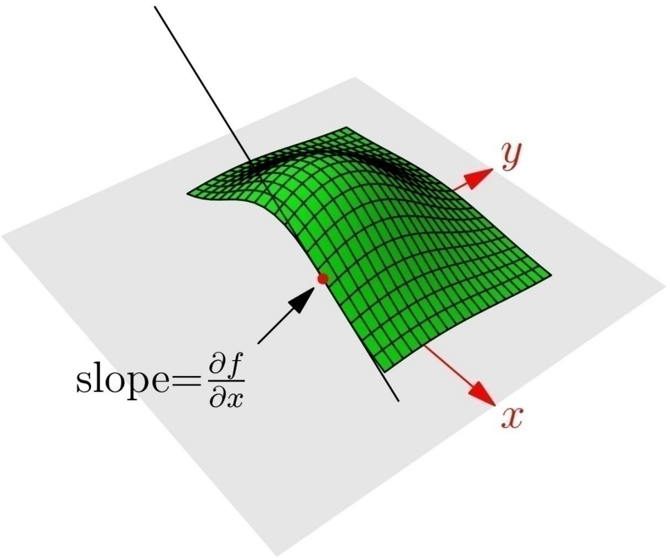
$$\widehat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

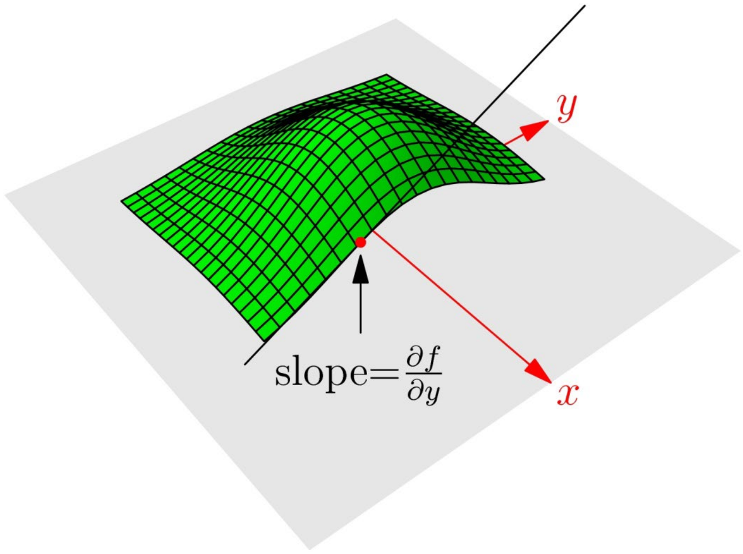
Directional derivative

Given a function of two variables f(x,y) *the partial derivative with respect to x* is the derivative when *y* is held constant:

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}, \quad \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

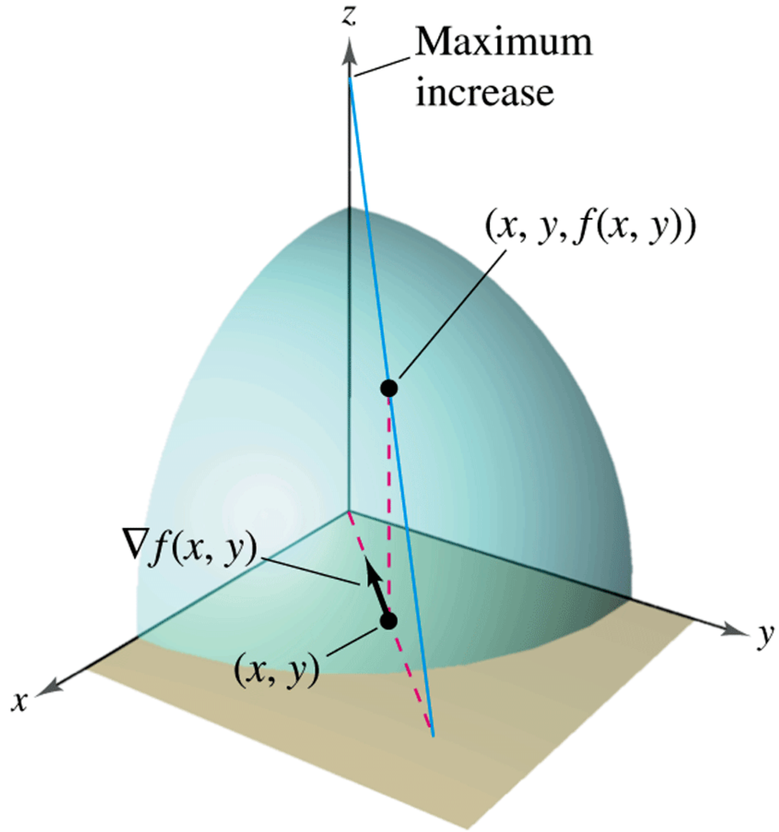
Partial derivative with *respect to y* is the derivative when *x* is held constant: We can think of this quantity as being *the slope in the direction of x and y.*





Can we define derivative in any direction?

Directional derivative

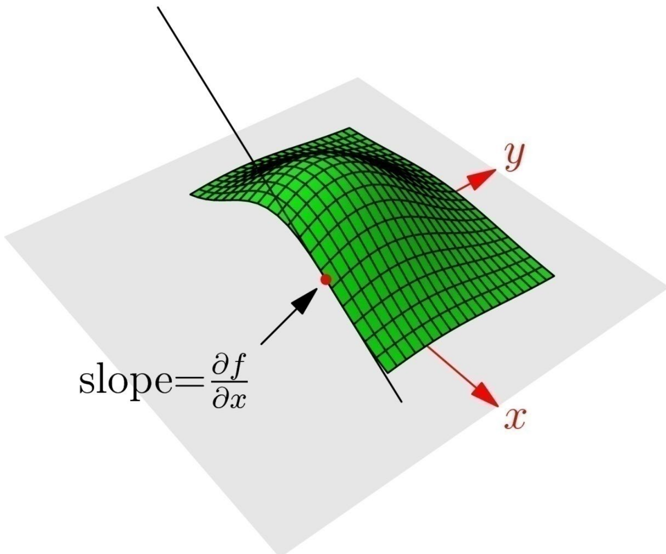


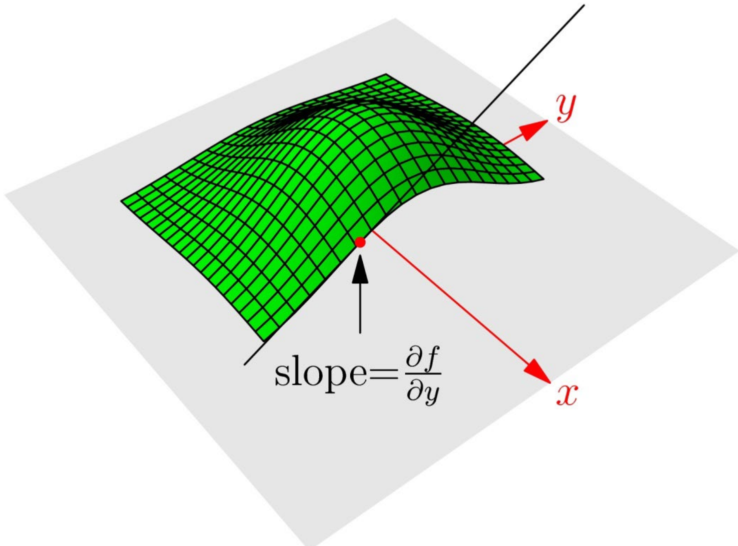
The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector **u**(a,b) is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha,\ y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

Directional derivative





If **u**(1,0)=**i** then $D_{\mathbf{i}}=f_x$ $D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha,\ y_0 + hb) - f(x_0, y_0)}{h}$

If **u**(0,1)=**j** then $D_{\mathbf{i}}=f_y$

.

Directional derivative

If *f(x,y)* is a differentiable function of *x* and *y*, then *f* has a directional derivative in the direction of any unit vector **u**(a, b) and

$$D_{\mathbf{u}} f(x,y) = \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial y} b$$

$$x = x_0 + ah, \quad y = y_0 + bh,$$

$$\phi(h) = f(x,y) = f(x_0 + ah, y_0 + bh)$$

$$D_{\mathbf{u}} f(x,y) = \frac{d\phi(h)}{dh} = \frac{\partial f}{\partial x}\frac{d(x_0 + ah)}{dh} + \frac{\partial f}{\partial y}\frac{d(y_0 + bh)}{dh} = a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y}$$

If the unit vector **u** makes an angle θ with the positive *x*-axis, then we can write **u**=(cos θ, sin θ)

and the formula for the directional derivative becomes

$$D_{\mathbf{u}} f(x, y) = f_x(x, y) \cos \vartheta + f_y(x, y) \sin \vartheta$$

Directional derivative

Directional derivative can be written in a vector form. Let

$$\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)=\nabla f,\quad \mathbf{u}=(a,b),\quad |\mathbf{u}|=1$$

$$D_{\mathbf{u}}f(x,y)=\frac{\partial f}{\partial x}a+\frac{\partial f}{\partial y}b=\nabla f\cdot\mathbf{u}$$

$$D_{\mathbf{u}}f(x,y)=f_x(x,y)\,a+f_y(x,y)\,b=\left\langle f_x(x,y),\ f_y(x,y)\right\rangle\cdot\langle a,b\rangle$$

$$=\left\langle f_x(x,y),\ f_y(x,y)\right\rangle\cdot\mathbf{u}=\nabla f\cdot\mathbf{u}$$

If f is a function of two variables *x* and *y*, then the **gradient** of f is the vector function defined by

$$\nabla f(x,y)=\left\langle f_x(x,y),\ f_y(x,y)\right\rangle=\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}$$

Another notation for the gradient is **grad** *f*.

Properties of directional derivative

Let

$$D_{\mathbf{u}} f(x,y) = \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial y} b = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}|\cos\theta = |\nabla f|\cos\theta$$

- Maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when **u** has the same direction as the gradient vector $\nabla f(x,y)$, $\theta = 0$, so f increases most rapidly along $\nabla f(\mathbf{x})$

$$D_{\mathbf{u}} f(x,y) = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}|\cos 0 = |\nabla f|$$

- Minimum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $-|\nabla f(\mathbf{x})|$ and it occurs when **u** has the opposite direction as the gradient vector $\nabla f(\mathbf{x})$. So f decreases most rapidly along $-\nabla f(\mathbf{x})$

- The direction **u** normal to $\nabla f(\mathbf{x})$ is the direction of zero change of f

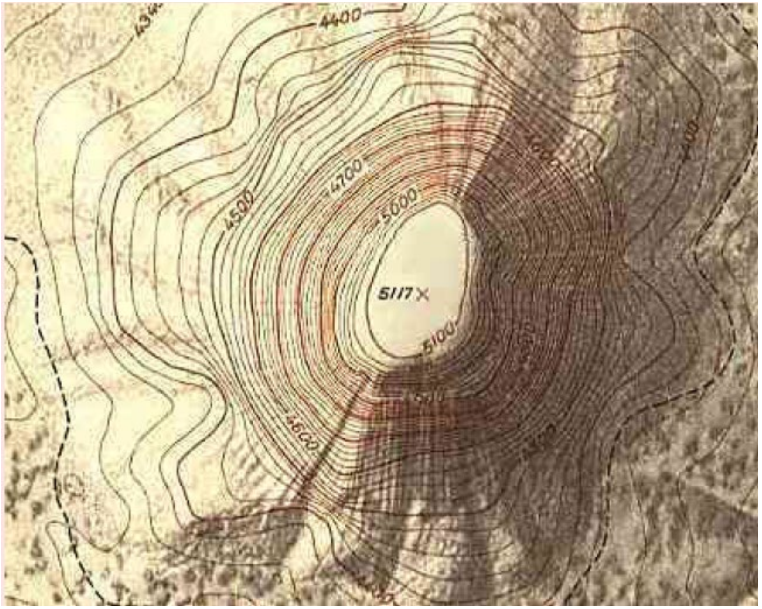
$$D_{\mathbf{u}} f(x,y) = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}|\cos\pi/2 = 0$$

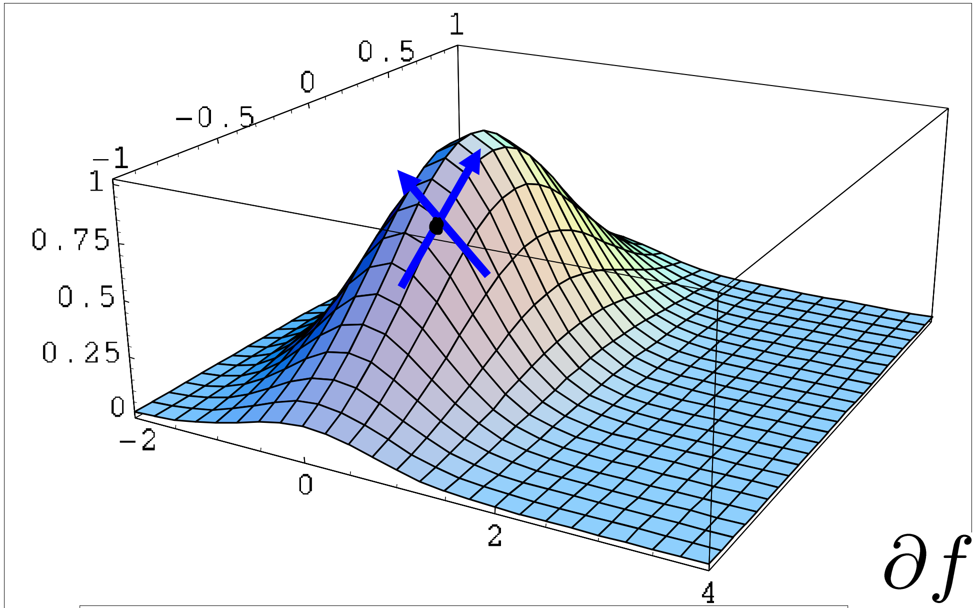
Properties of directional derivative

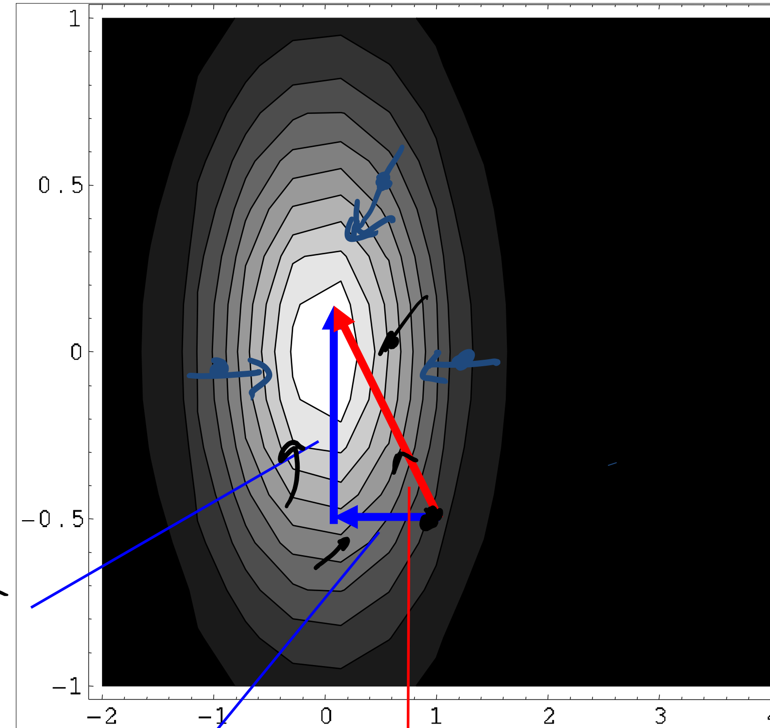
- At every point in the domain of f(x,y) the gradient of f $\nabla f(\mathbf{x})$ is normal to the level curve passing through the point

$$D_{\mathbf{u}} f(x,y) = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}| \cos \pi/2 = 0$$

$\nabla f(\mathbf{x})$



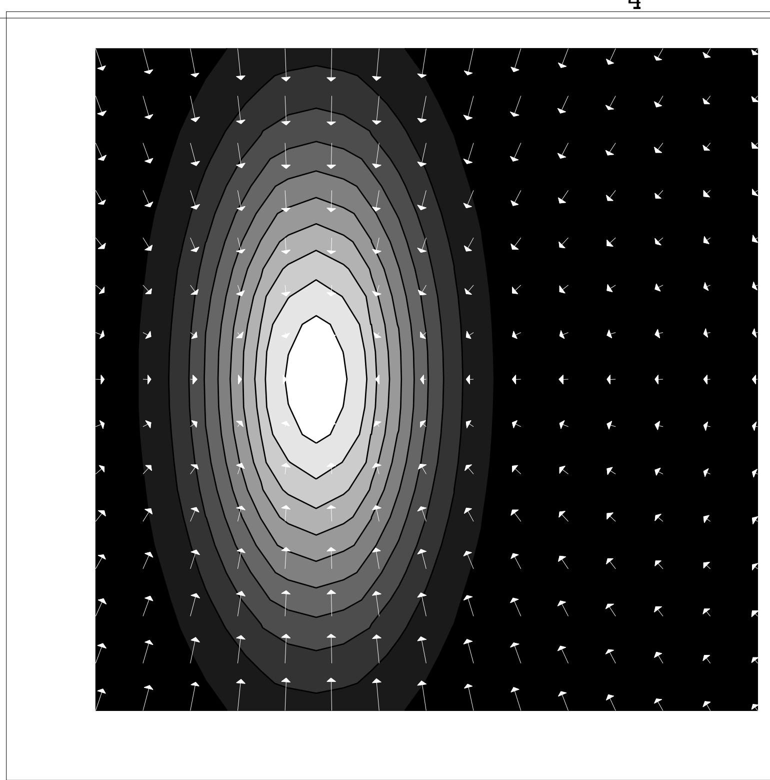




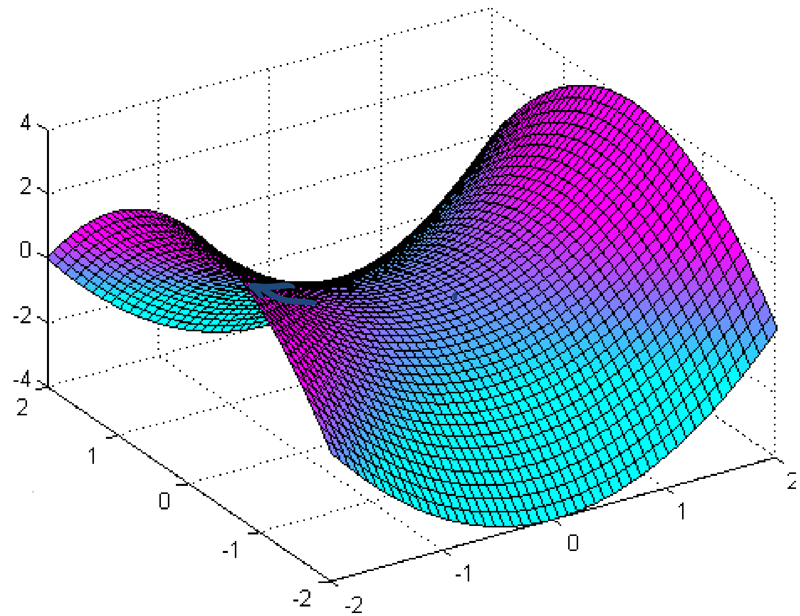
$$\frac{\partial f}{\partial y}\hat{\mathbf{j}}$$

$$\frac{\partial f}{\partial x}\hat{\mathbf{i}}$$

$$\frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}}$$



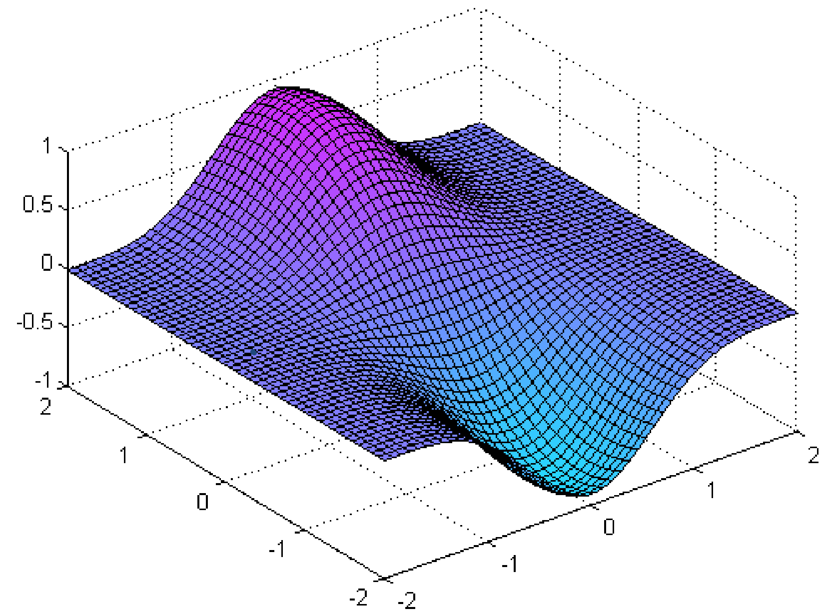
Example1 : Calculate the gradient of $f(x,y) = x^2 - y^2$



$$\nabla f = \mathbf{i}\frac{\partial(x^2 - y^2)}{\partial x} + \mathbf{j}\frac{\partial(x^2 - y^2)}{\partial y} = 2x\mathbf{i} - 2y\mathbf{j}$$

Example 2: Calculate the gradient of

$$f(x,y) = e^{-x^2}\sin y$$



$$\nabla f = \mathbf{i}\frac{\partial(e^{-x^2}\sin y)}{\partial x} + \mathbf{j}\frac{\partial(e^{-x^2}\sin y)}{\partial y} = -\mathbf{i}2xe^{-x^2}\sin y + \mathbf{j}e^{-x^2}\cos y$$

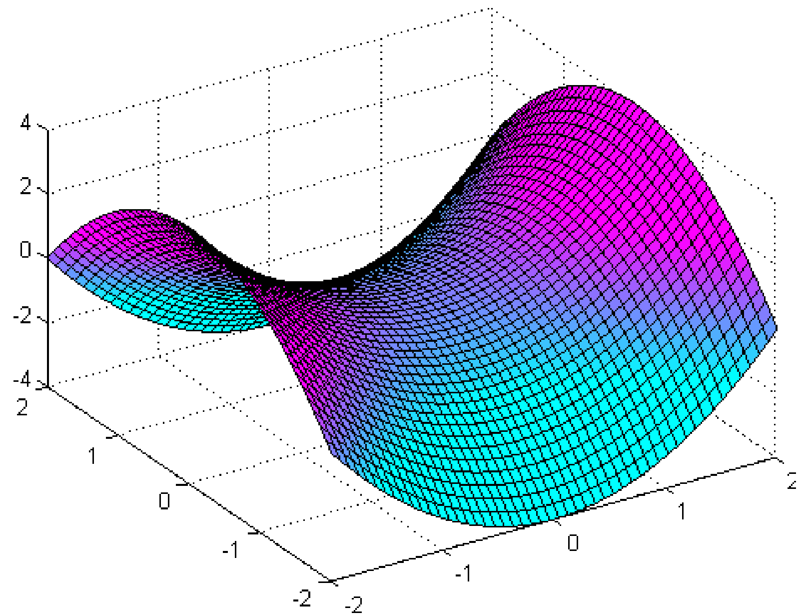
Example 3: Calculate the slope of the function

$$f(x, y) = x^2 - y^2$$

in the direction of the vector

$$v = 2\hat{\mathbf{i}} + \hat{\mathbf{j}}$$

at the point (1,1).



$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2\mathbf{i} + \mathbf{j}}{\sqrt{2^2 + 1}} = \frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}$$

$$D_{\mathbf{u}} f(x, y) = \nabla f \cdot \mathbf{u} = (\mathbf{i}2x - \mathbf{j}2y) \cdot (\frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}) =$$

$$= \frac{4x - 2y}{\sqrt{5}}; \quad D_{\mathbf{u}} f(1,1) = \frac{4-2}{\sqrt{5}} = \frac{2}{\sqrt{5}}$$

Properties of directional derivative

If f is a function of three variables, the **gradient vector** is $\nabla f(\mathbf{x})$

$$\nabla f = \left\langle f_x,\ f_y,\ f_z \right\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

The symbol r is a differential operator that *acts on* the scalar function f.

$$\nabla := \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

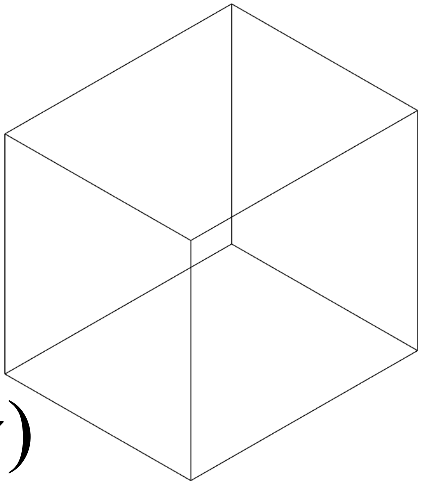
The directional derivative along a unit vector **u**(a,b,c) can be expressed in terms of the gradient as

$$D_{\mathbf{u}} f(x,y,x) = \nabla f(x,y,z)\cdot\mathbf{u} = a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} + c\frac{\partial f}{\partial z}$$

Example: The temperature in a rectangular box is given by

$$T(x,y,z) = xyz(1-x)(2-y)(4-z)$$

with $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 2$, $0 \leqslant z \leqslant 4$.
If a mosquito is at the point with coordinates (1/2,1,1), in which direction should it fly to cool off as fast as possible?



$$\nabla T(x,y,z) = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}$$

$$\frac{\partial T}{\partial x} = yz(1-x)(2-y)(4-z) - xyz(2-y)(4-z)$$

$$\frac{\partial T}{\partial y} = xz(1-x)(2-y)(4-z) - xyz(1-x)(4-z)$$

$$\frac{\partial T}{\partial z} = xy(1-x)(2-y)(4-z) - xyz(1-x)(2-y)$$

$$\nabla T(x,y,z) = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}$$

$$\left.\frac{\partial T}{\partial x}\right|_{(1/2,1,1)} = yz(1-x)(2-y)(4-z) - xyz(2-y)(4-z)\Big|_{(1/2,1,1)} = 0$$

$$\left.\frac{\partial T}{\partial y}\right|_{(1/2,1,1)} = xz(1-x)(2-y)(4-z) - xyz(1-x)(4-z)\Big|_{(1/2,1,1)} = 0$$

$$\left.\frac{\partial T}{\partial z}\right|_{(1/2,1,1)} = xy(1-x)(2-y)(4-z) - xyz(1-x)(2-y)\Big|_{(1/2,1,1)} = \frac{1}{2}$$

$$\mathbf{v} = -\nabla T\Big|_{(1/2,1,1)} = (0,0,-1/2)$$

The direction of the fastest decrease of T is in the opposite direction of z axis.

Minimum and maximum values

Definition

A function of two variables has a *local maximum at (a,b)* if

$$f(x,y) \leq f(a,b)$$

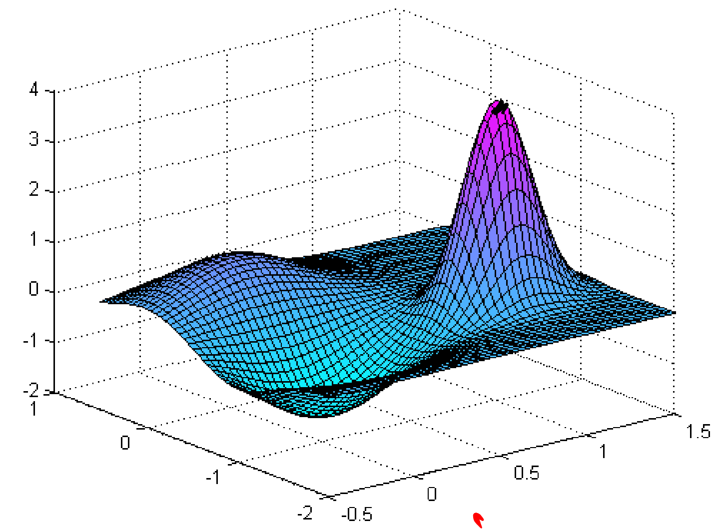
for all points *(x,y)* in some disk with centre (*a,b).*

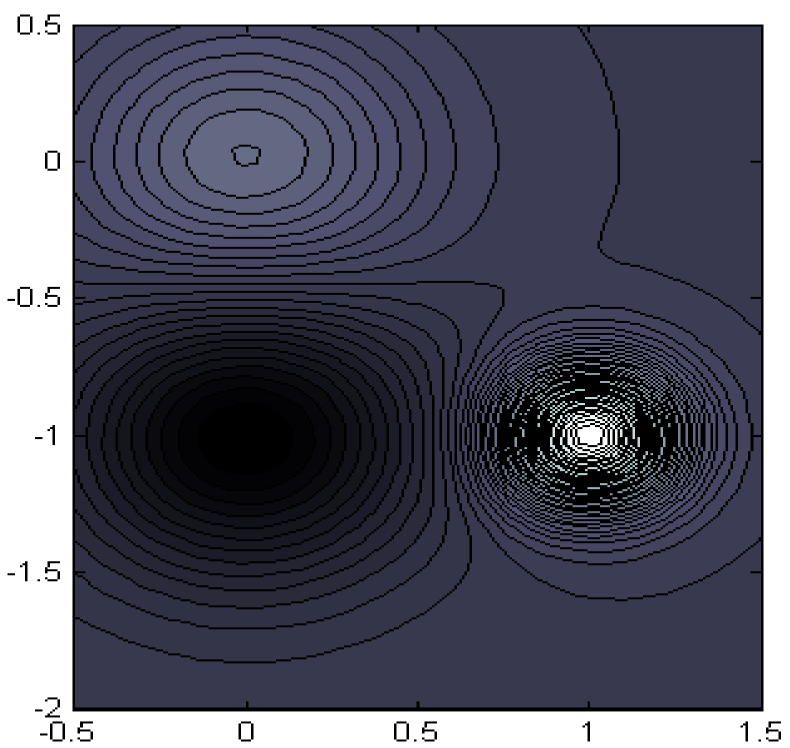
Similarly, if

$$f(x,y) \geq f(a,b)$$

then there is a *local minimum.*

If the inequality holds for *all* values of (x,y) then f has an *absolute maximum* (or *absolute minimum)* at *(a,b).*

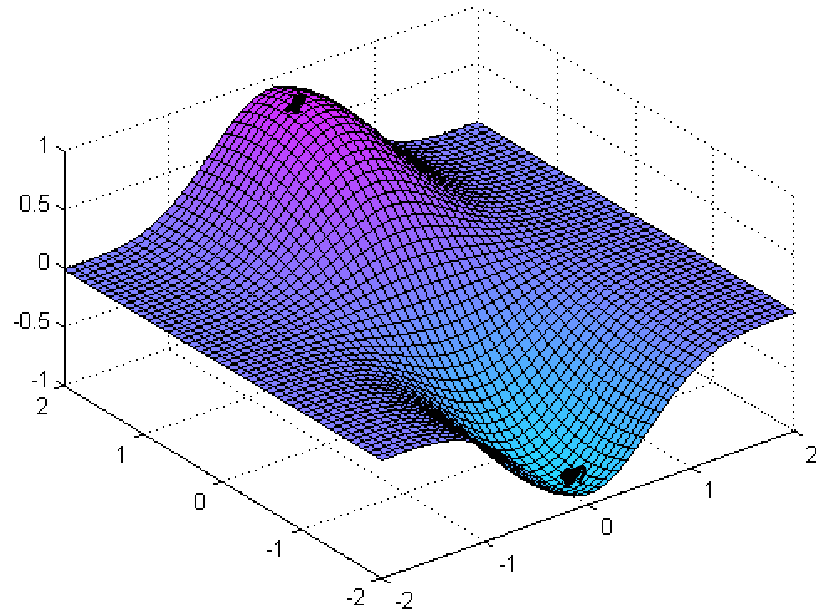




Fermat's theorem:

If f has a local extremum at (a,b) and the first partial derivatives exist, then the gradient is zero.

$$\nabla f|_{(a,b)} = 0$$



$$f(x, y) = e^{-x^2} \sin y$$

Note that

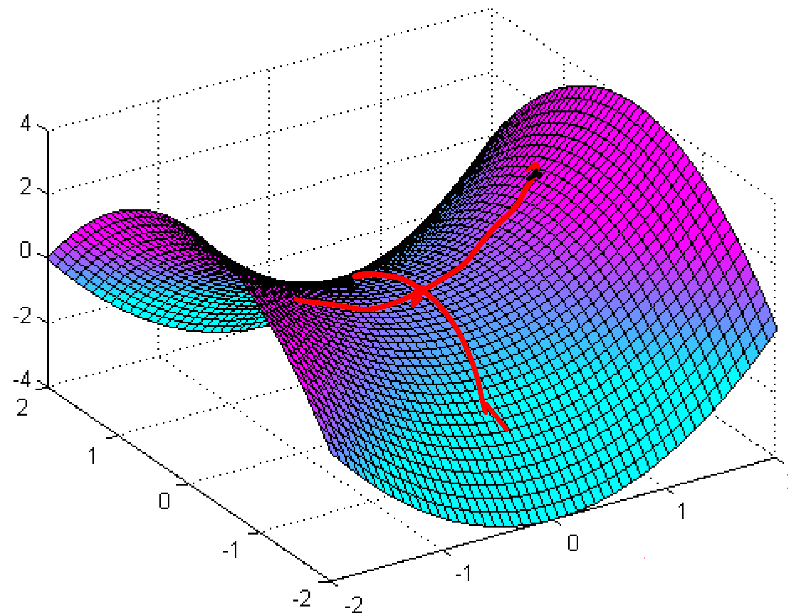
Local max or min $\xrightarrow{\text{implies}}$ $\nabla f|_{(a,b)} = 0$

But it is not necessarily true that

$\nabla f|_{(a,b)} = 0$ $\xrightarrow{\text{implies}}$ Local max or min

Counter-example:

$$f(x, y) = x^2 - y^2$$



saddle point

Maximum And Minimum values

- The extreme values of f(x,y) can occur only at:

- Boundary points of the domain f(x,y)

- **Critical points:** where $\nabla f|_{(a,b)} = 0$ is zero or does not exist

If the first and second order partial derivatives of f are continues around some region (a,b) and $\nabla f|_{(a,b)} = 0$ then the second derivative test can be applied to classify the extremum points

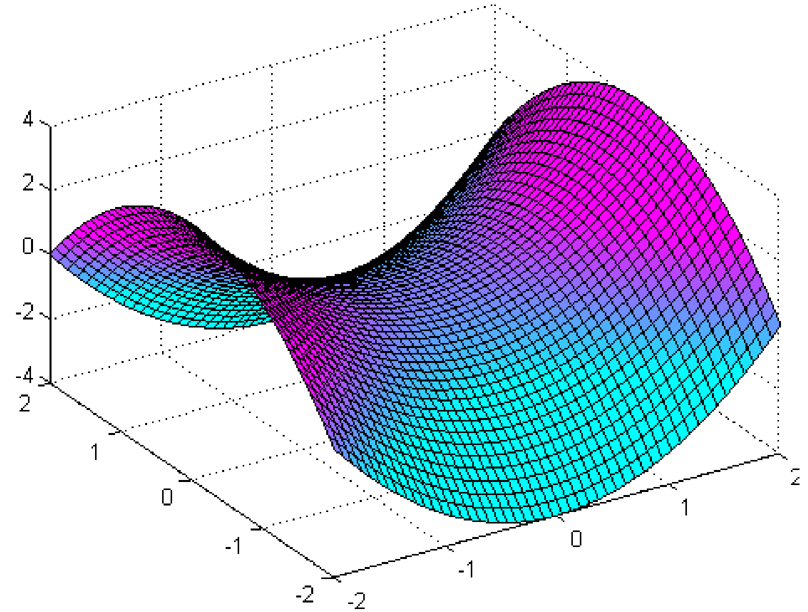
$$D = \left| \begin{array}{cc} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{array} \right|_{(a,b)}$$

Cases:

- If D>0 and f_{xx} (a,b) > 0, f(a,b) has a min
- If D>0 and f_{xx} (a,b) < 0, f(a,b) has a max
- If D<0 saddle point
- D=0 higher order derivatives are required

Example: Find all critical points of

$$f(x,y) = x^2 - y^2$$



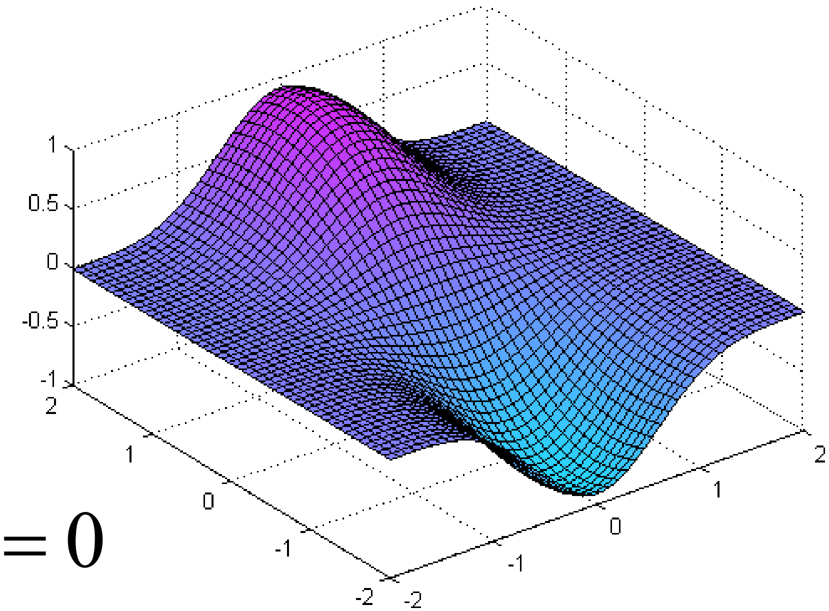
$$\frac{\partial f}{\partial x} = 2x = 0 \Rightarrow x = 0$$

$$\frac{\partial f}{\partial y} = -2y = 0 \Rightarrow y = 0$$

$$D = \left| \begin{array}{cc} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{array} \right|_{(a,b)}$$

Example: Find all critical points of

$$f(x,y) = e^{-x^2}\sin y$$



$$\frac{\partial f}{\partial x} = -2xe^{-x^2}\sin y = 0 \Rightarrow x = 0$$

$$\frac{\partial f}{\partial y} = e^{-x^2}\cos y = 0 \Rightarrow \cos y = 0 \Rightarrow y = \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$$

$$(0, \frac{\pi}{2} + \pi k)$$

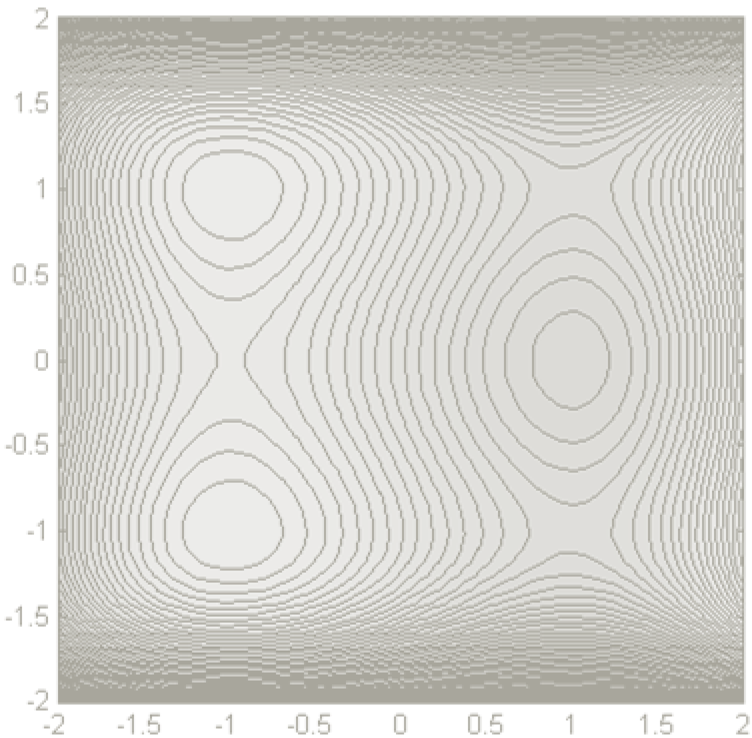
Example:

Find and classify the critical points of $f(x,y) = 3x - x^3 - 2y^2 + y^4$

$$\frac{\partial f}{\partial x} = 3 - 3x^2 = 0 \Rightarrow x = \pm 1$$

$$\frac{\partial f}{\partial y} = 4y^3 - 4y = 0 \Rightarrow y = 0, y = \pm 1$$

The critical points are: $(1,-1);(1,0);(1,1);(-1,-1);(-1,0);(-1,1)$



$$D = \begin{vmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial x \partial y} & \dfrac{\partial^2 f}{\partial y^2} \end{vmatrix} = \begin{vmatrix} -6x & 0 \\ 0 & 12y^2 - 4 \end{vmatrix} = -6x(12y^2 - 4)$$

$$D\Big|_{(1,-1)} = -6x(12y^2 - 4)\Big|_{(1,-1)} = -6(12-4) = -48 < 0$$

The critical point (1,-1) is a saddle point

$$D\Big|_{(-1,-1)} = -6x(12y^2 - 4)\Big|_{(-1,-1)} = 6(12-4) = 48 > 0$$

$$f_{xx} = 6 > 0$$

The critical point (-1,-1) is a minimum point.

The classification of the rest of the critical points are left as an exercise

$$D = \begin{vmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial x \partial y} & \dfrac{\partial^2 f}{\partial x^2} \end{vmatrix} = \begin{vmatrix} -6x & 0 \\ 0 & 12y^2 - 4 \end{vmatrix} = -6x(12y^2 - 4)$$

$$D\big|_{(1,0)} = -6\underline{x}(12y^2 - 4)\big|_{(1,0)} = -6(0-4) = 24 > 0$$

$$f_{xx} = -6 \times 1 = -6 < 0$$

The critical point (1,0) is a maximum point

$$D\big|_{(1,1)} = -6x(12y^2 - 4)\big|_{(+1,+1)} = -6(12-4) = -48 < 0$$

The critical point (1,-1) is a saddle point.

The classification of the rest of the critical points are left as an exercise

Finding the absolute minimum or maximum

Theorem:
The absolute maximum or minimum of a continuous function f defined on a domain D must either occur at the critical points of f, or at extreme values of f on the boundary of D.

To find the absolute maximum and minimum in a domain:

1. Find all critical values of *f* within D

2. Find all extreme values on the boundary of D

3. Compare these, and take the maximum/minimum.

Example: Find the absolute maximum of the function

$$f(x,y) = x^2 - y^2$$

in the domain $D = \{(x,y) | x^2 + y^2 \leq 2\}$