Calculus of Several Variables

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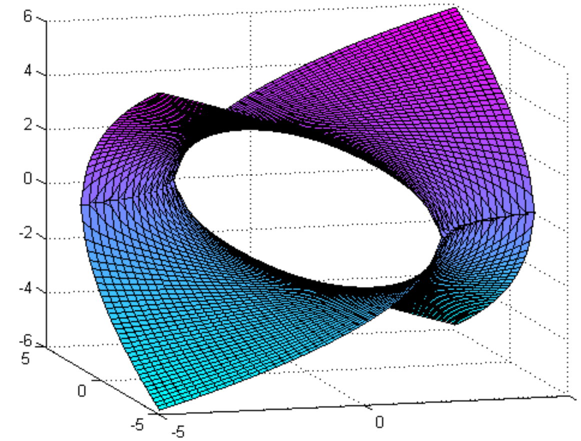
Example:

Find the points on the surface

$$y^2 = 9 + xz \quad \bullet$$

that are closest to the origin.



$$d = \sqrt{x^2 + y^2 + z^2}$$

$$d^2 = x^2 + y^2 + z^2 = x^2 + z^2 + xz + 9 = f(x, z)$$

$$\nabla f = 0 \Rightarrow \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{k}\frac{\partial f}{\partial z} = 0$$

$$f_x = 2x + z = 0$$

$$f_z = 2z + x = 0$$

$$x = 0,\ z = 0$$

$$D = \begin{vmatrix} f_{xx} & f_{xz} \\ f_{zx} & f_{zz} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0 \quad f_{xx} = 2 > 0$$

$$f(0,0) = 9$$

$$y^2 = xz + 9 = 9$$

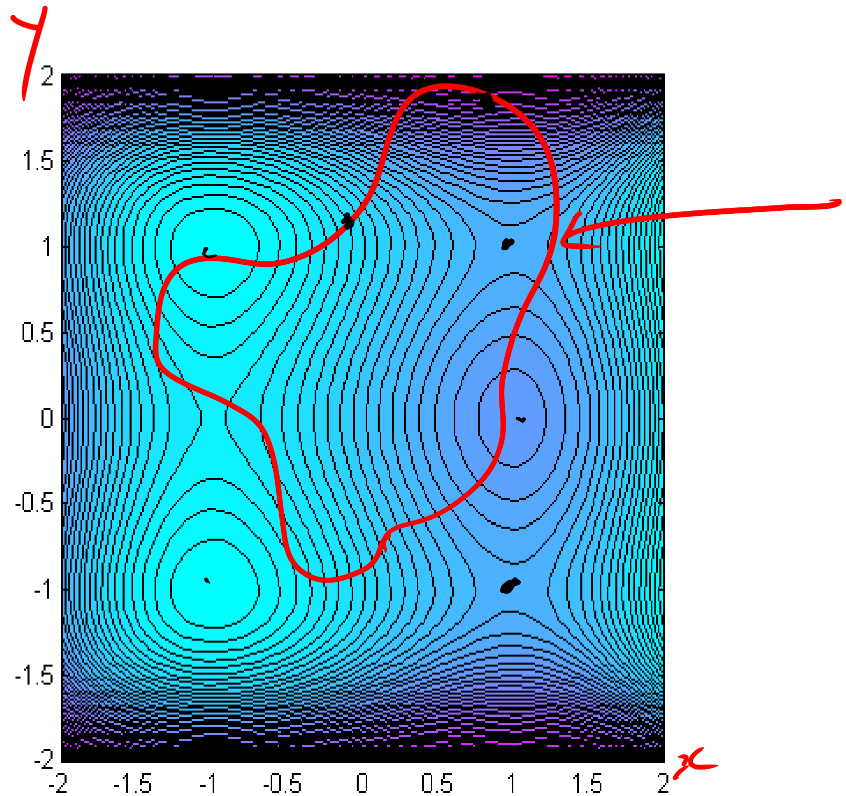
$$y = \pm 3$$

$$(0, 3, 0), \quad (0, -3, 0)$$

Constrained optimization and Lagrange multipliers

We often have to find the maximum of a function subject to one (or more) *constraints.*

A *constraint* is a function that connects the *independent variables* and so is represented by a curve or line. The problem of *constrained optimization* is to find the maximum along this curve.



A constraint in two dimensions can be represented by a function

g(x; y) = 0

Examples:

$$g(x,y) = x^2 + y^2 - 1 = 0$$

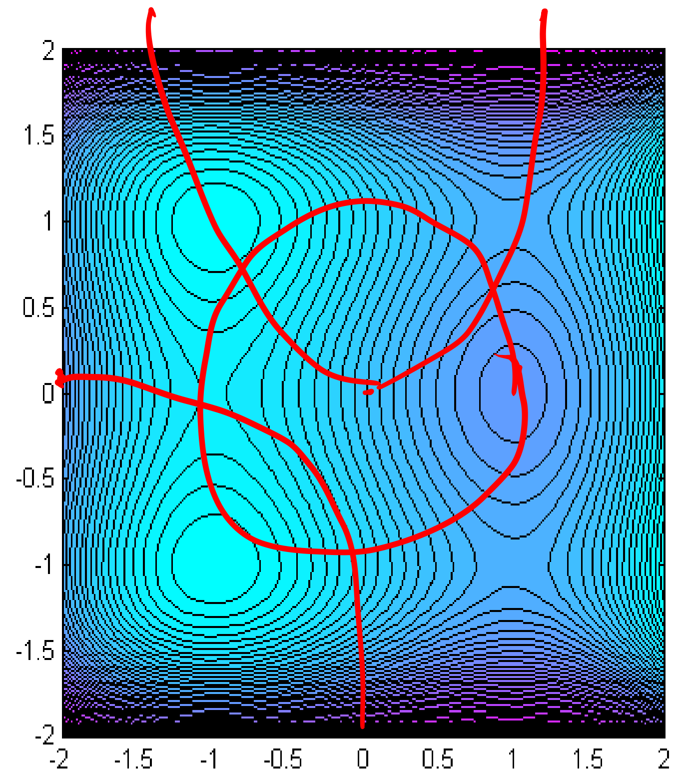
$$x^2 + y^2 = 1$$

$$g(x,y) = y - x^2 = 0$$

$$y = x^2$$

$$g(x,y) = (x+2)^2 + (y+2)^2 - 4 = 0$$

$$(x+2)^2 + (y+2)^2 = 4$$



A vector perpendicular to the constraint curve g(x,y) = 0 is

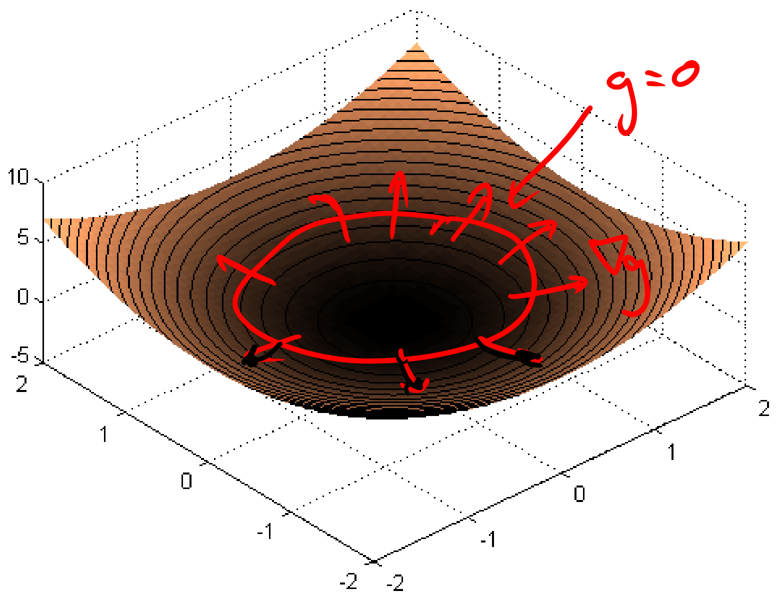
$$\nabla g(x,y)$$

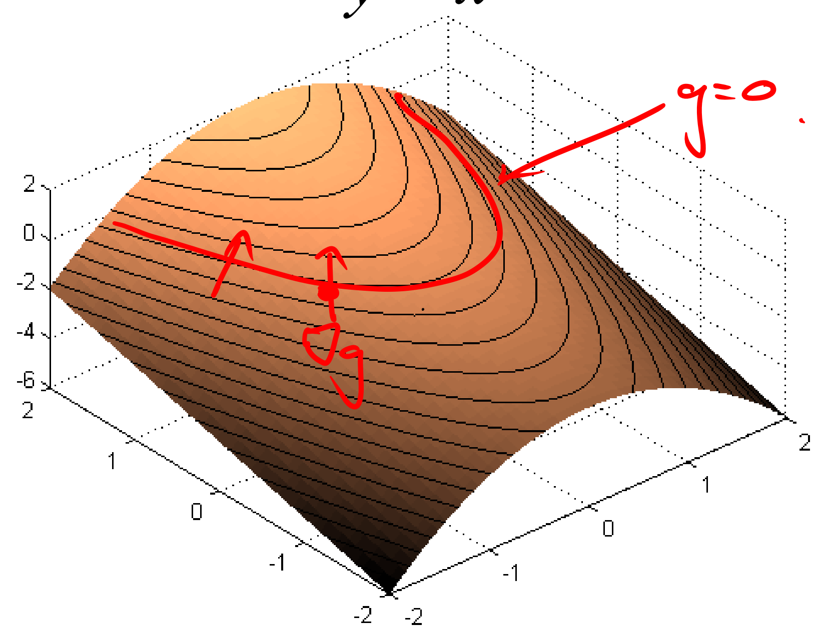
To see this, we can plot the function g(x,y) itself:

$$g(x,y) = x^2 + y^2 - 1 = 0$$

$$g(x,y) = y - x^2 = 0$$

$$y = x^2$$

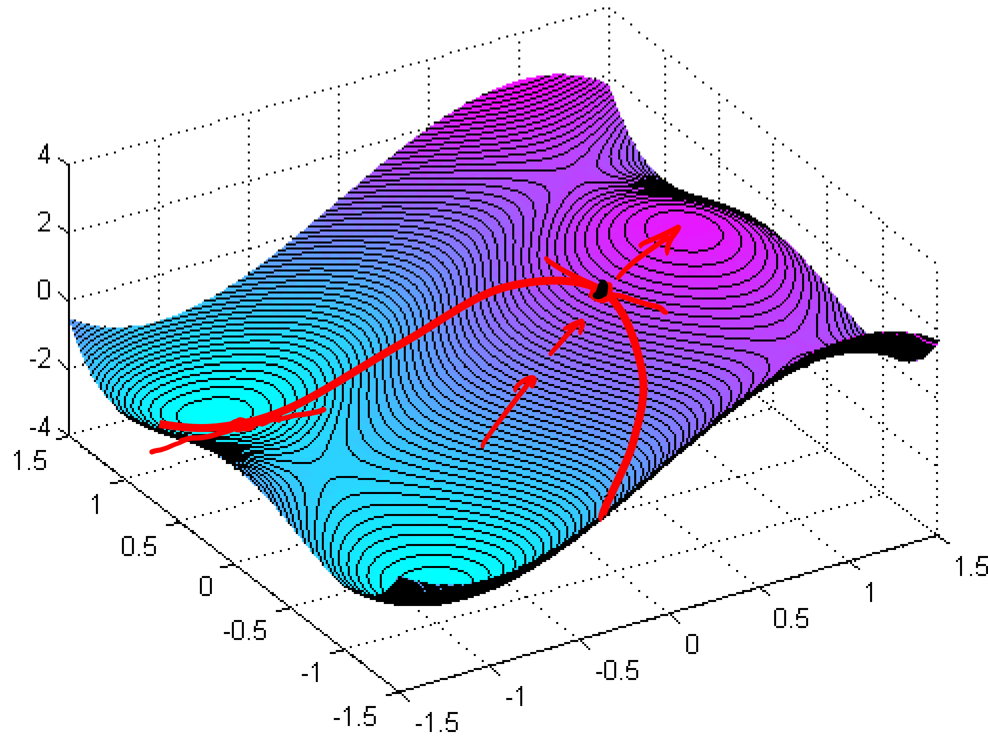




The constraint curve is a particular level curve of this function, so grad g is always perpendicular to g(x,y)=0.

The problem of constrained optimization can be solved using the method of **Lagrange multiplie**

The central idea:



The maximum along the line occurs when the constraint curve is parallel to the level curves of the function f(x,y).

This occurs when the (tangent to) *constraint curve is perpendicular to the gradient of f(x,y).*

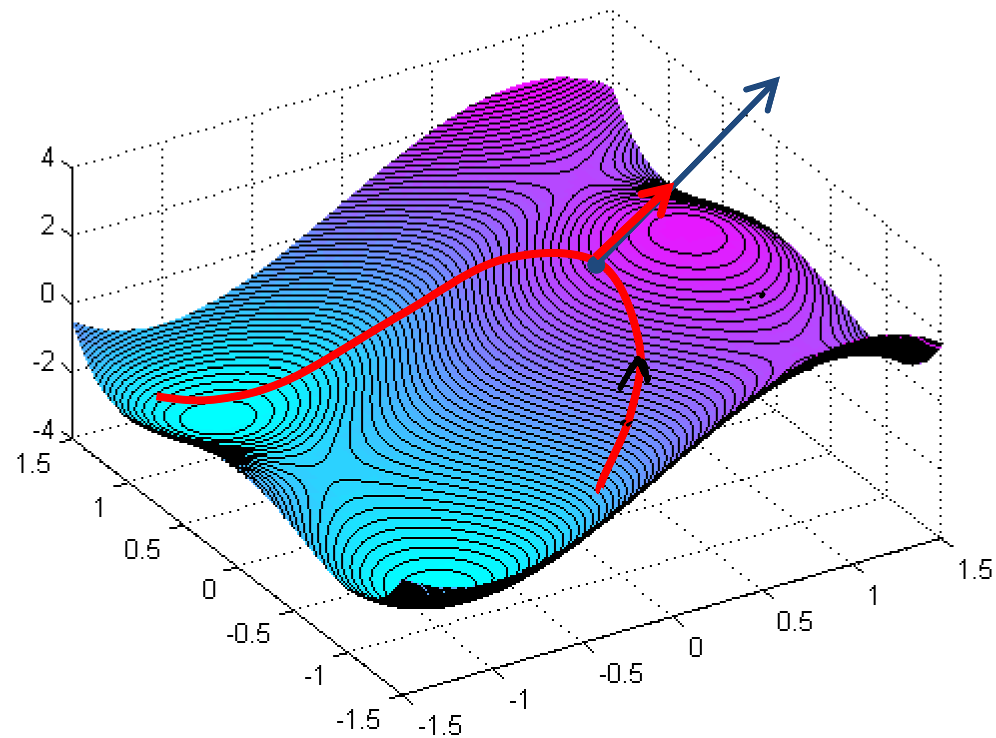
Why? Because parallel to the curves the function does not increase or decrease.

The method of Lagrange multipliers

To find the maximum and minimum of a function f(x,y,z) subject to a constraint g(x,y,z) = 0, we find all values of x, y, z and λ such that

$$\nabla f = \lambda \nabla g$$

We then evaluate f at these points; the largest is the maximum, the smallest is the minimum.



Example 1: Find the maximum of

$$f(x; y) = e^{xy}$$

subject to the constraint

$$x^3 + y^3 = 16$$

$$g(x, y) = x^3 + y^3 - 1$$

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

$$\mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} = \lambda(\mathbf{i}\frac{\partial g}{\partial x} + \mathbf{j}\frac{\partial g}{\partial y})$$

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \Rightarrow ye^{xy} = \lambda 3x^2$$

$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \Rightarrow xe^{xy} = \lambda 3y^2, \qquad \frac{ye^{xy}}{xe^{xy}} = \frac{\lambda 3x^2}{\lambda 3y^2} \Rightarrow x^3 = y^3$$

Using the constraint condition $x^3 + y^3 = 16$ and $x^3 = y^3$

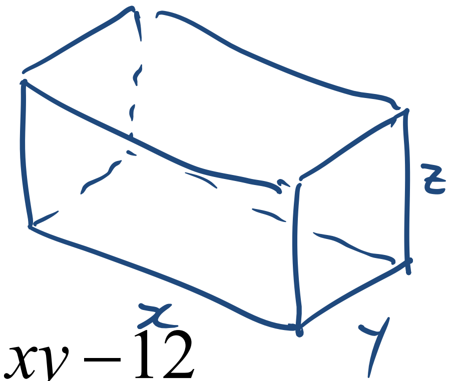
$$2x^3 = 16 \Rightarrow x^3 = 8 \Rightarrow x = 2, y = 2$$

$f(x;y) = e^{xy}$

$$f(2,2) = e^4$$

Example 2:

A rectangular box without a lid is to be constructed from $12m^2$ of sheet metal. Find the maximum volume of such a box.



$$V = xyz = f(x, y, z)$$

$$S = 2xz + 2yz + xy = 12 \Rightarrow g(x, y, z) = 2xz + 2yz + xy - 12$$

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \Rightarrow yz = \lambda(2z + y)$$

$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \Rightarrow xz = \lambda(2z + x)$$

$$\frac{\partial f}{\partial z} = \lambda \frac{\partial g}{\partial z} \Rightarrow xy = \lambda(2x + 2y)$$

$$f(x, y, z) = xyz$$

$$2xz + 2yz + xy = 12$$

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \Rightarrow yz = \lambda(2z + y) \quad xyz = 2\lambda xz + \lambda xy \quad Eq(1)$$

$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \Rightarrow xz = \lambda(2z + x) \quad xyz = 2\lambda yz + \lambda xy \quad Eq(2)$$

$$\frac{\partial f}{\partial z} = \lambda \frac{\partial g}{\partial z} \Rightarrow xy = \lambda(2x + 2y) \quad xyz = 2\lambda xz + 2\lambda yz \quad Eq(3)$$

$$Eq(1) - Eq(2) \Rightarrow 2\lambda z(x - y) = 0 \Rightarrow x = y$$

$$Eq(3) - Eq(2) \Rightarrow \lambda x(2z - x) = 0 \Rightarrow x = 2z = y$$

$$2xz + 2yz + xy = 12$$

$$4z^2 + 4z^2 + 4z^2 = 12$$

$$z = 1, x = y = 2z = 2$$

Example 3:

Use the method of Lagrange multipliers to find the maximum and minimum values of the function

$$f(x, y, z) = xyz$$

subject to the constraint

$$x^2 + 2y^2 + 3z^2 = 6.$$

$$f(x, y, z) = xyz$$

$$x^2 + 2y^2 + 3z^2 = 6$$

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \Rightarrow yz = 2\lambda x$$

$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \Rightarrow xz = 4\lambda y$$

$$\frac{\partial f}{\partial z} = \lambda \frac{\partial g}{\partial z} \Rightarrow xy = 6\lambda z$$

$$yz = 2\lambda x \quad xyz = 2\lambda x^2 \quad x^2 = \frac{xyz}{2\lambda}$$

$$xz = 4\lambda y \quad xyz = 4\lambda y^2 \quad 2y^2 = \frac{xyz}{2\lambda}$$

$$xy = 6\lambda z \quad xyz = 6\lambda z^2 \quad 3y^2 = \frac{xyz}{2\lambda}$$

$$(xyz)^3 = 48\lambda^3 (xyz)^2 \Rightarrow xyz = 48\lambda^3$$

$$\frac{xyz}{2\lambda} + \frac{xyz}{2\lambda} + \frac{xyz}{2\lambda} = 6 \Rightarrow \frac{48\lambda^3}{2\lambda} + \frac{48\lambda^3}{2\lambda} + \frac{48\lambda^3}{2\lambda} = 6$$

$$\lambda^2 = \frac{1}{12} \Rightarrow \lambda_1 = \frac{1}{2\sqrt{3}}, \lambda_2 = -\frac{1}{2\sqrt{3}}$$

$$f(x, y, z) = xyz = 48\lambda^3 = \frac{2}{\sqrt{3}} \text{ max}$$

$$f(x, y, z) = xyz = -48\lambda^3 = -2 / \sqrt{3} \text{ min}$$