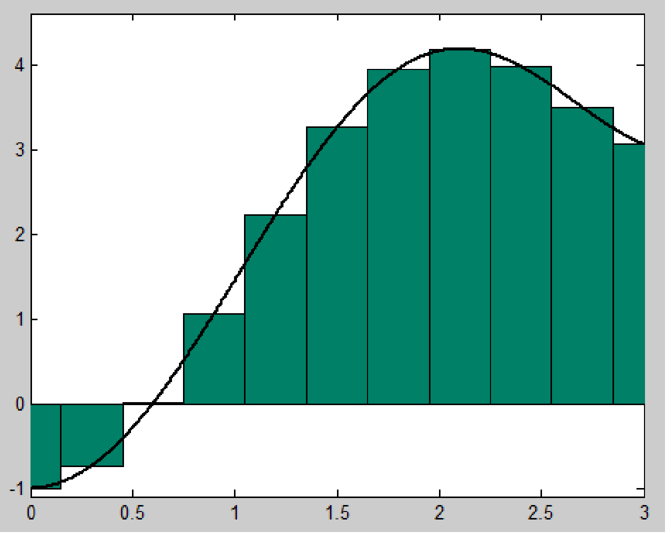
Integration in Two Dimensions

1D definite integrals (review)

We think of a one-dimensional definite integral
as the sum of areas of infinite number of rectangles:



$$A = \int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

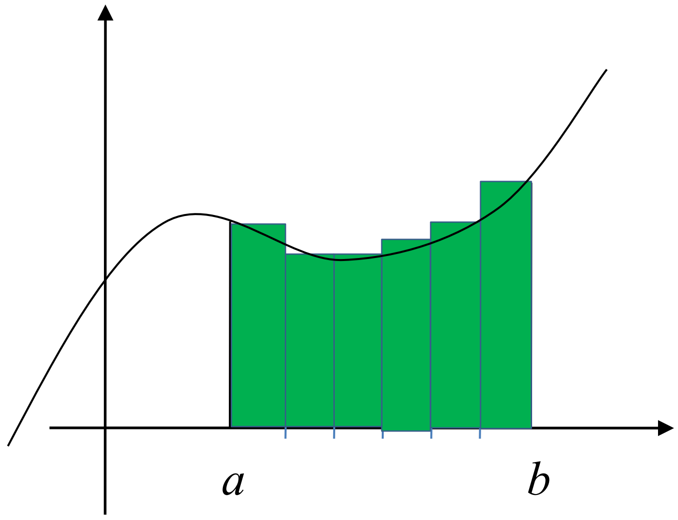
where

$$\Delta x = \frac{b-a}{n} \qquad x_i = a + i\Delta x$$

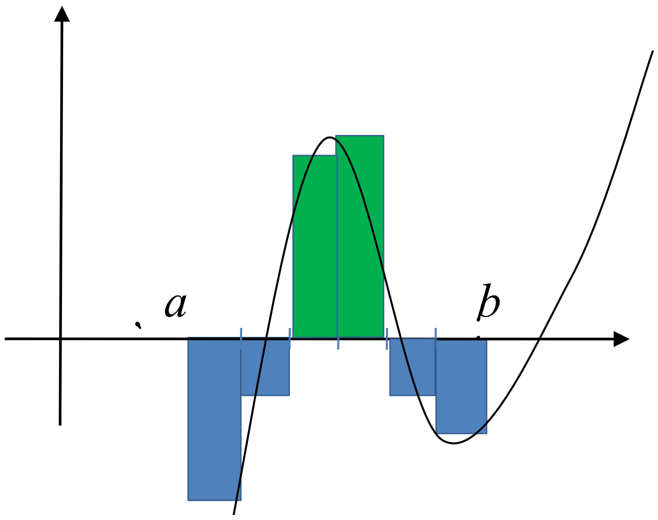
This is known as the *Riemann sum* of the integral.

As the number of rectangles increases a better and better approximation for the area under the curve is obtained.

<u>NB:</u> The integral is often thought of as the *area* under a graph.



However, integrals can also be *negative* or *zero* (unlike areas).



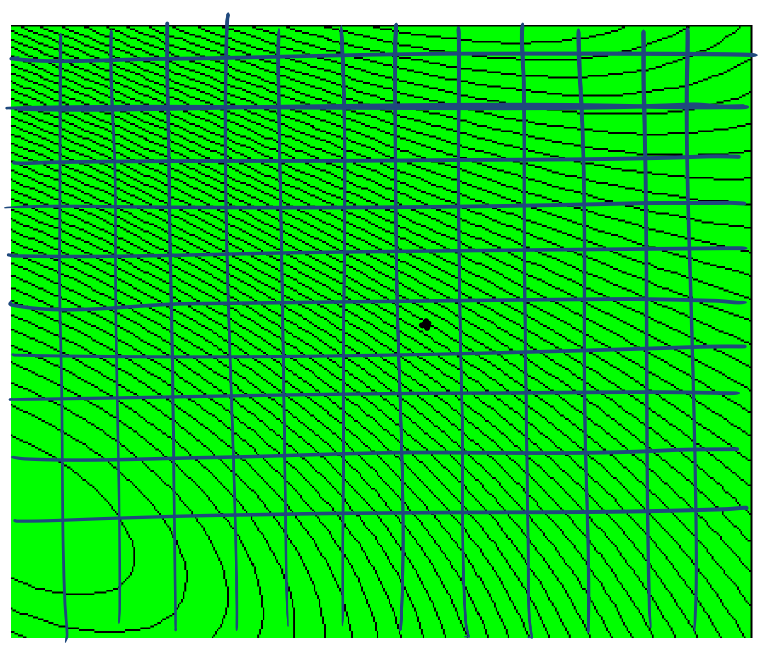
We can extend this definition to integrals of 2D functions over *rectangular domains* .

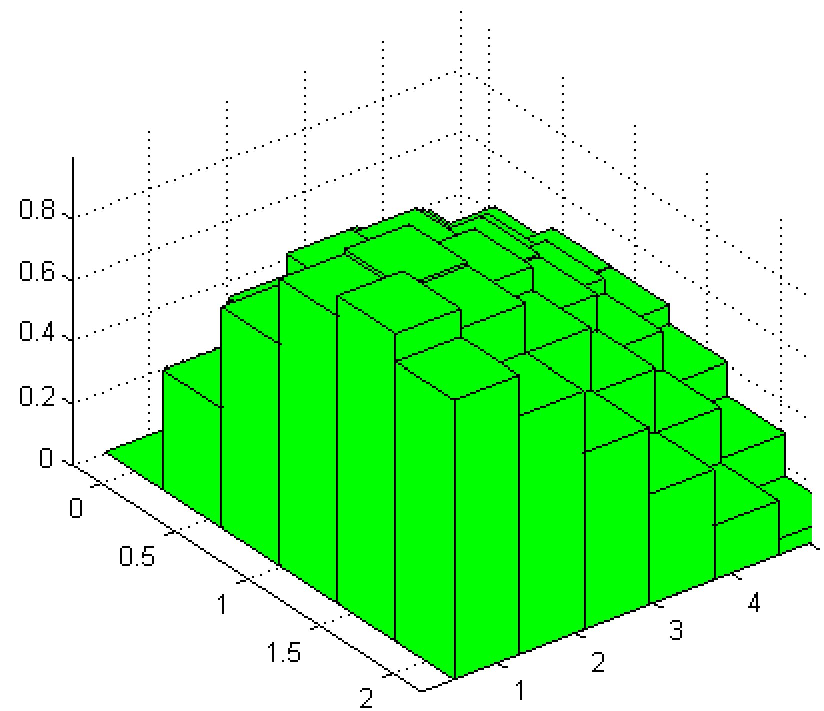
$$A = \int_c^d \int_a^b f(x,y)\,dx\,dy = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_i)\Delta x \Delta y, \qquad \Delta y = \frac{d-c}{m}, \quad \Delta x = \frac{b-a}{n}$$

$$\Delta A = \Delta x \Delta y$$

This time, the integral represents a *signed volume* under the 2D surface.

$$x_i = a + i\Delta x$$
$$y_i = c + j\Delta y$$





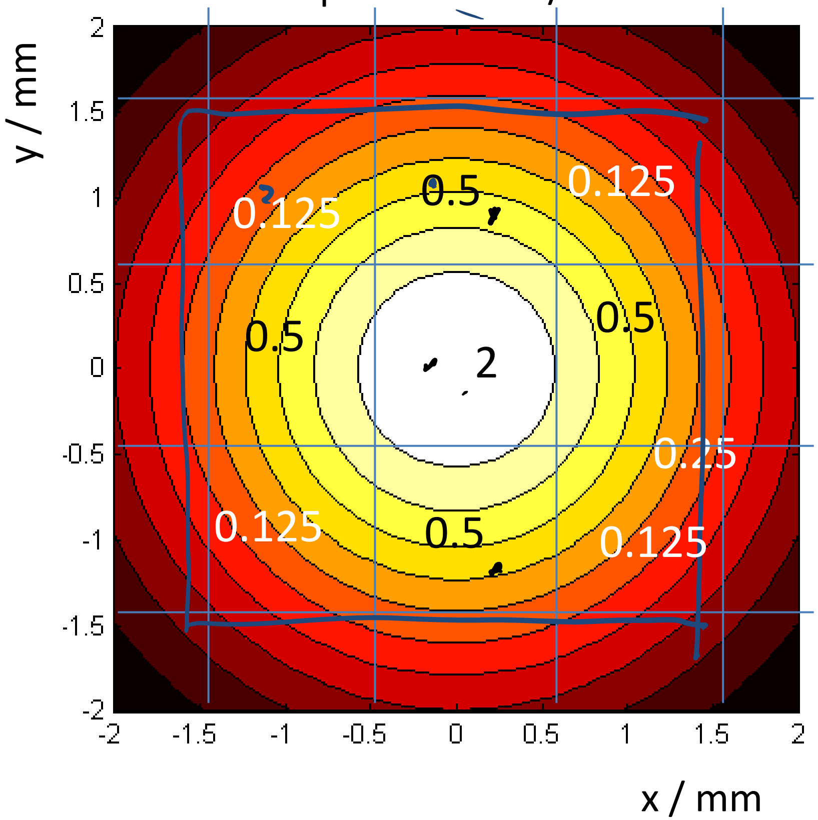
Given a function of two variables, we can *approximate* the integral by summing the rectangles.

$$A = \int_c^d \int_a^b f(x,y)dxdy = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\Delta x \Delta y,$$

Example: Estimate the total power in a laser beam

$$\Delta A = \Delta x \Delta y = 1mm^2$$

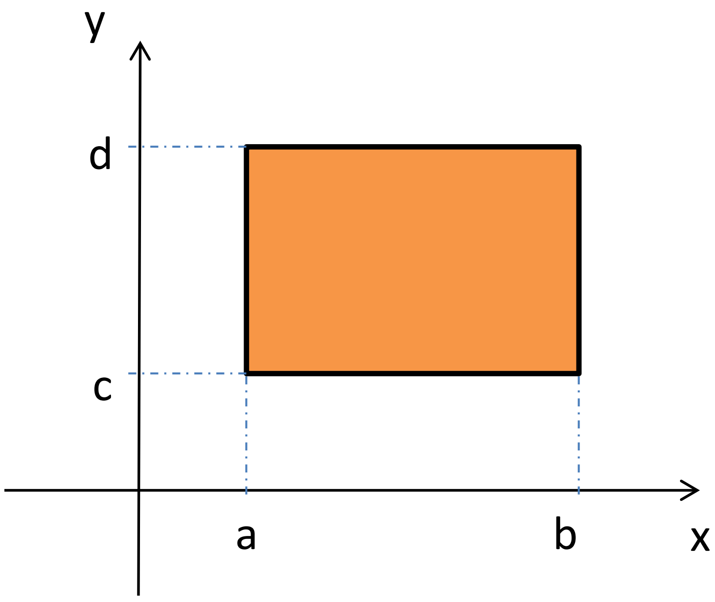
Beam profile in GW/m^2



0.125	**0.5**	**0.125**
0.5	2	0.5
0.125	0.5	0.125

To perform an integral in 2D, we use nested (or *iterated)* integration:

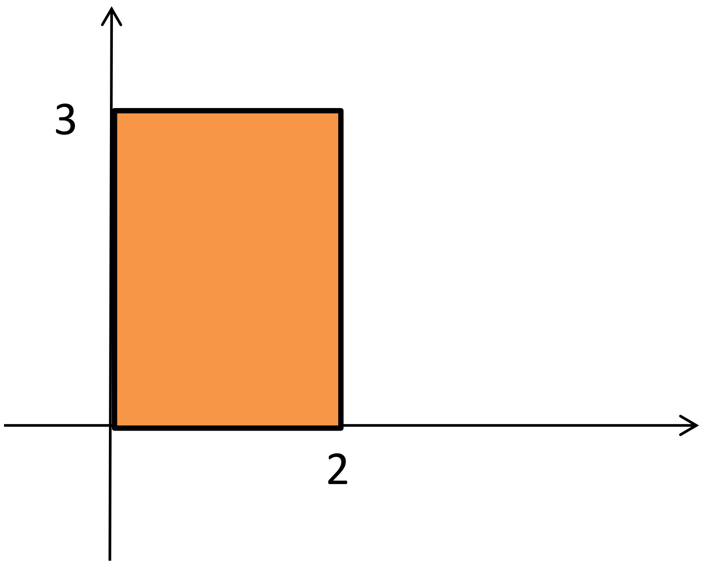
For a rectangular domain, this means that we pick one variable to integrate over first and evaluate this while keeping the other variable constant.



$$A = \iint_D f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_a^b \left[\int_c^d f(x,y)dy\right] dx$$

Example: evaluate

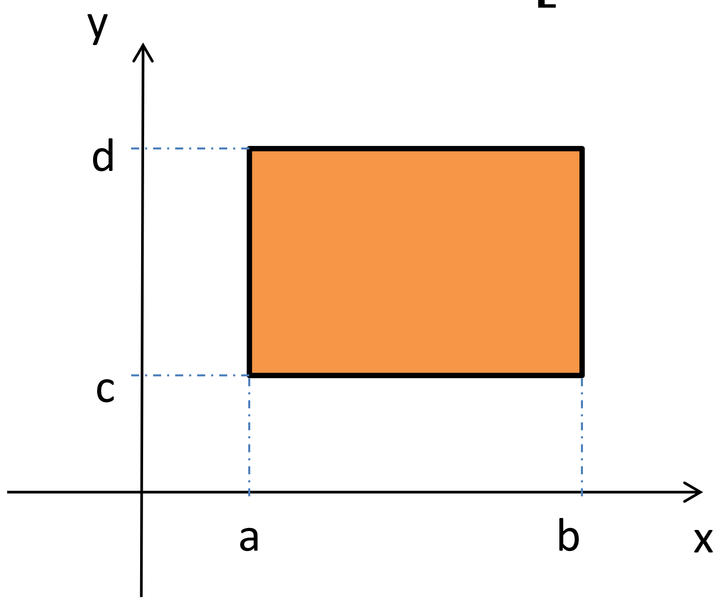
$$A = \int_0^3 \int_0^2 x^2 y dx dy$$



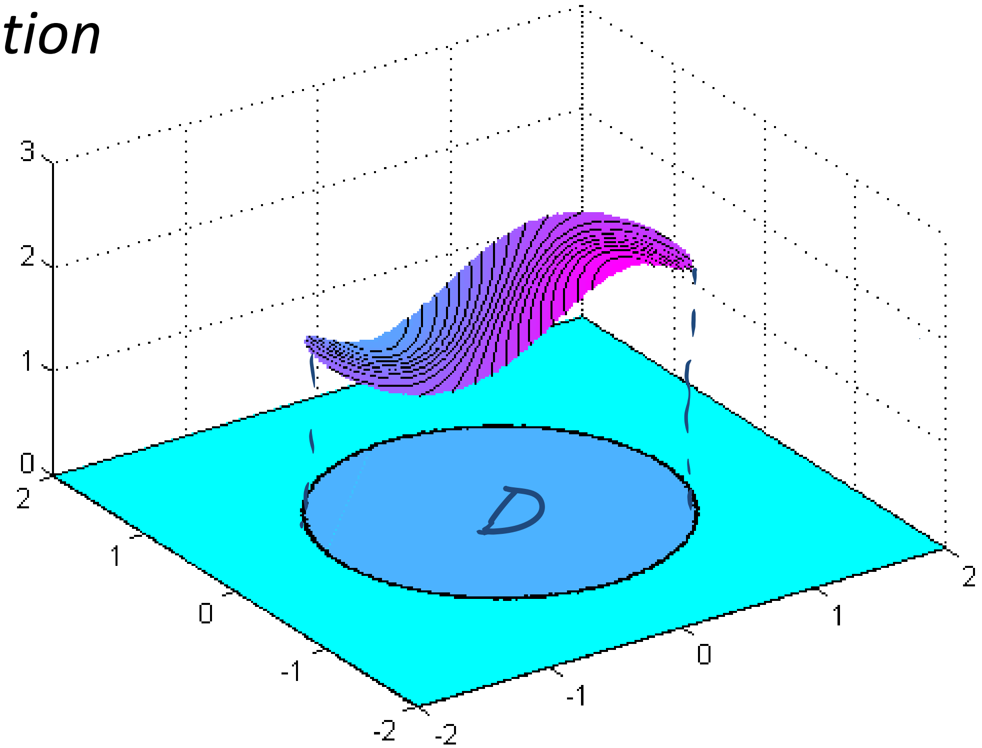
$$A = \int_0^3 \int_0^2 x^2 y dx dy = \int_0^3 \left. \frac{x^3 y}{3} \right|_{x=0}^{x=2} dy = \frac{8}{3} \left. \frac{y^2}{2} \right|_{y=0}^{y=3} = \frac{8}{3} \frac{9}{2} = 12$$

Fubini's theorem states that for integrals of continuous functions over rectangular domains, the *order of integration is not important.*

$$A = \iint\limits_{D} f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$$



Unlike in 1D, the *domain of integration* in 2D can be complicated.



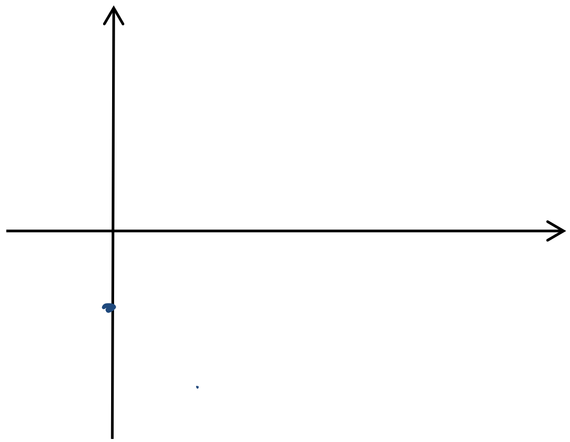
To integrate in 2D, we first have to describe the domain of integration.

The general form is:

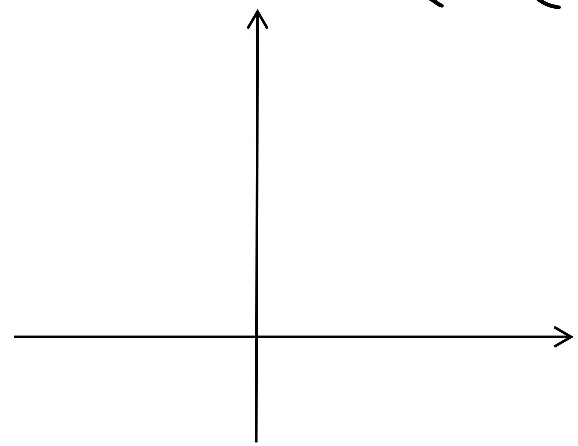
$$D = \{(x, y) \mid some\ inequalities\ involving\ x, y\}$$

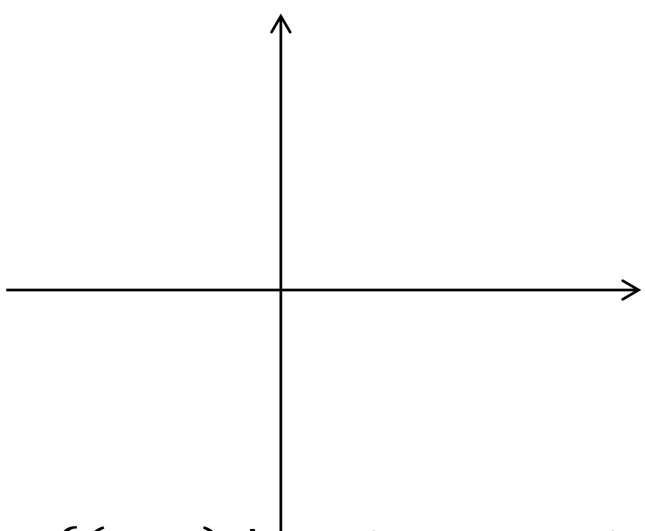
Examples: $D = \{(x, y) \mid \textit{some inequalities involving } x, y\}$

$D = \{(x, y) \mid x > 1, y > 1\}$

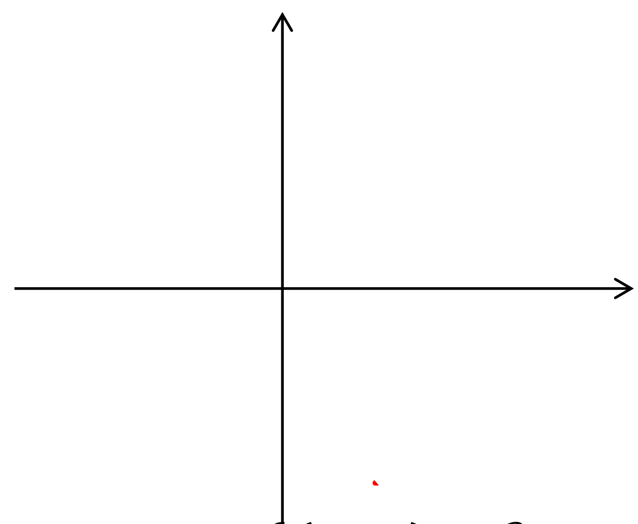


$D = \{(x, y) \mid -1 \leq x \leq 2, y \leq x^2\}$





$D = \{(x, y) \mid -1 \leq y \leq 1, x > y^3\}$

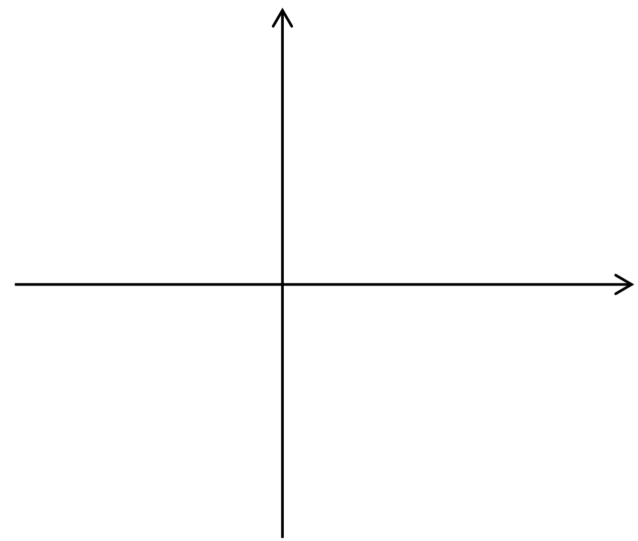


$D = \{(x, y) \mid x^2 + y^2 \leq 4\}$

Integrating over more complicated domains:

First, write down and draw the domain in 2D, e.g.

$$D = \{(x,y) \mid -1 \le x \le 2, \qquad 0 < y < x+1\}$$



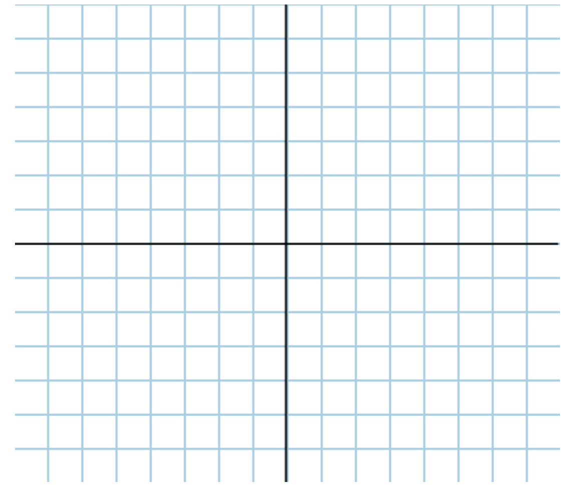
Pick the inequalities and use them as the limits for you integral:

$$A = \int_{-1}^{2} \int_{0}^{x+1} f(x,y)\,dy\,dx$$

(Important: make sure that the outer limits do not depend on x or y. If this happens, swap the order of integration).

Then integrate, starting with the inner integrals.

Example: Find $\iint_D (6x+6y)\,dx\,dy$



over the domain

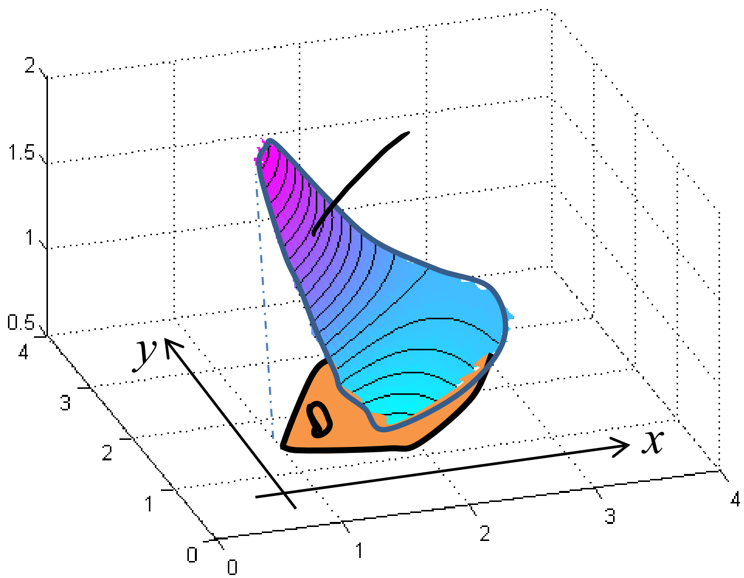
$$D = \{(x,y) \mid 0 \le x \le 1, \quad 0 \le y \le x\}$$

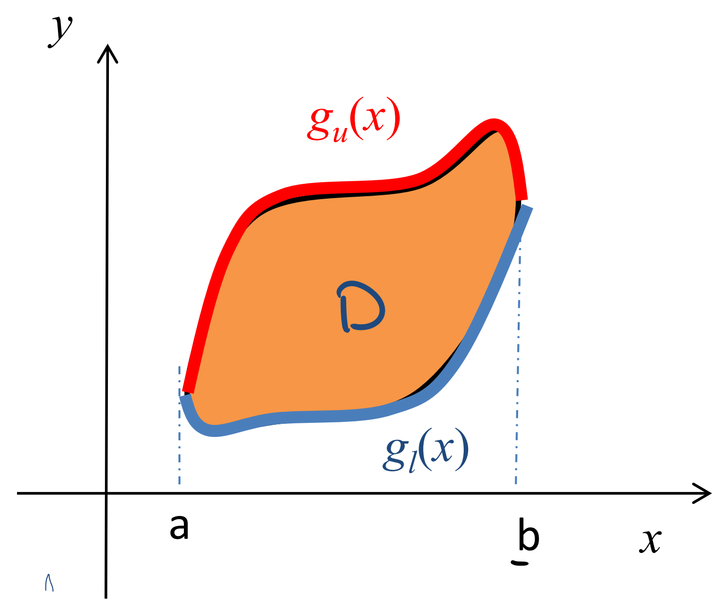
$$A = \int_0^1 \int_0^x (6x+6y)\,dy\,dx$$

$$A = \int_0^1 \int_0^x (6x+6y)\,dy\,dx = \int_0^1 (6xy+3y^2)\Big|_{y=0}^{y=x} dx =$$

$$= \int_0^1 (6x^2+3x^2)\,dx = \frac{9}{3}x^3\Big|_{x=0}^{x=1} = 3$$

Given a function f(x,y) on a domain D,





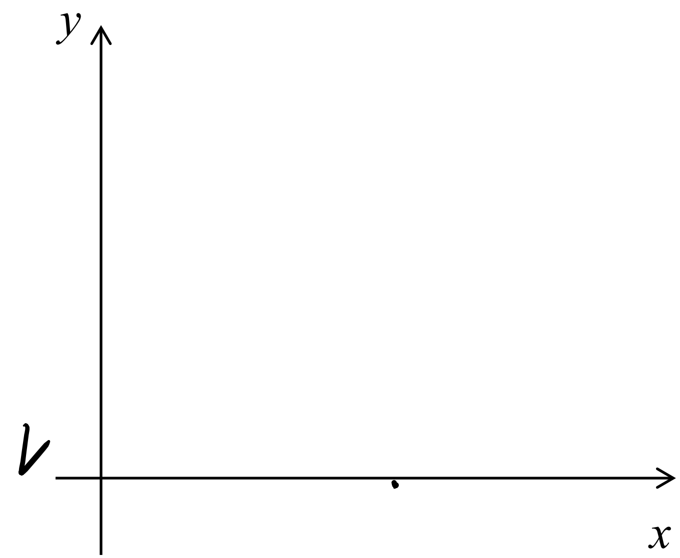
the integral of a function f(x,y) over this domain D is

$$\int_D f\,dA = \iint_D f(x,y)\,dy\,dx = \int_a^b \int_{g_l(x)}^{g_u(x)} f(x,y)\,dy\,dx$$

Example: Integrate

$$\int_D x\, cos\, y\, dA$$

where D is the region bound by y = 0, y = x^2, and $x = \sqrt{\pi}$.

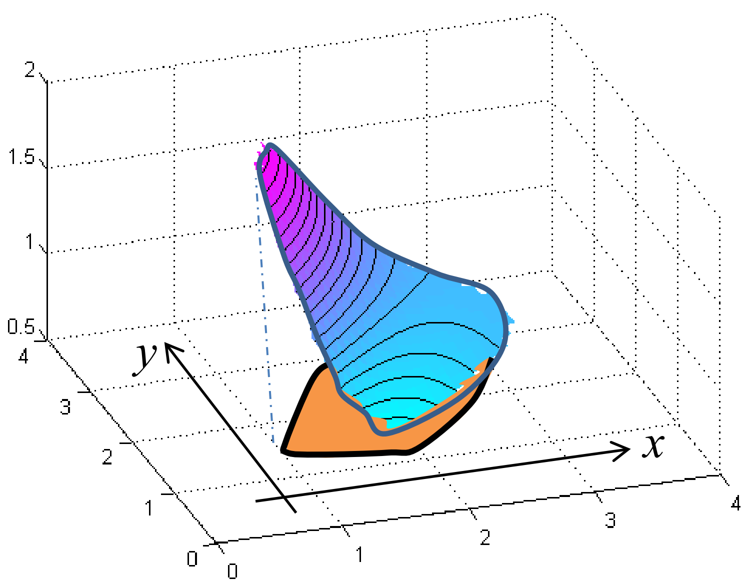


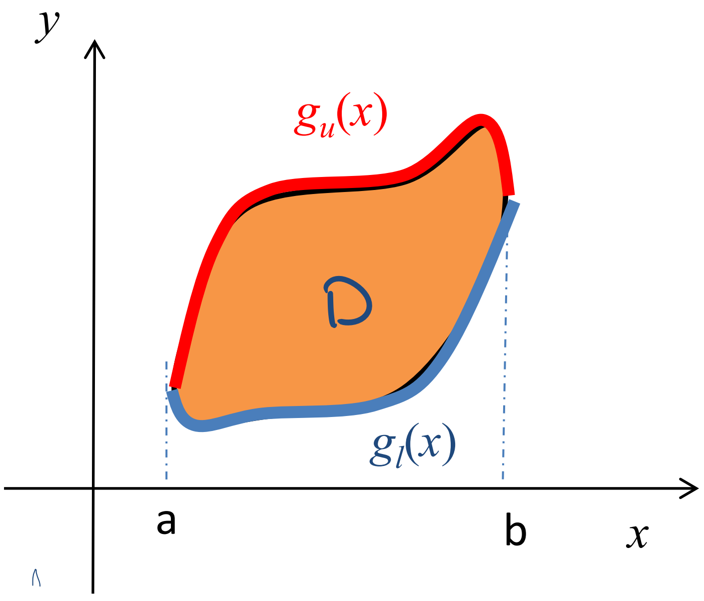
$$A = \int_0^{\sqrt{\pi}} \int_0^{x^2} x \cos y dy dx = \int_0^{\sqrt{\pi}} x \sin y \Big|_{y=0}^{y=x^2} dx =$$

$$= \int_0^{\sqrt{\pi}} x \sin x^2 dx = \frac{1}{2}\int_0^{\pi} \sin u du = -\frac{1}{2}(\cos\pi - \cos 0) = 1$$

$$u = x^2 \Rightarrow du = 2xdx$$

Given a function f(x,y) on a domain D,





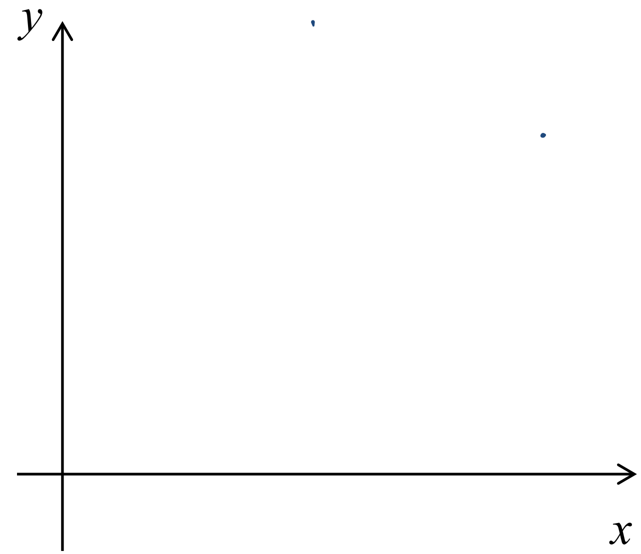
the integral of a function f(x,y) over this domain D is

$$\int_D f\,dA = \iint_D f(x,y)\,dy\,dx = \int_a^b \int_{g_l(x)}^{g_u(x)} f(x,y)\,dy\,dx$$

Example: Integrate

$$\int_D x \cos y \, dA$$

where D is the region bound by $y = 0$, $y = x^2$, and $x = \sqrt{\pi}$.

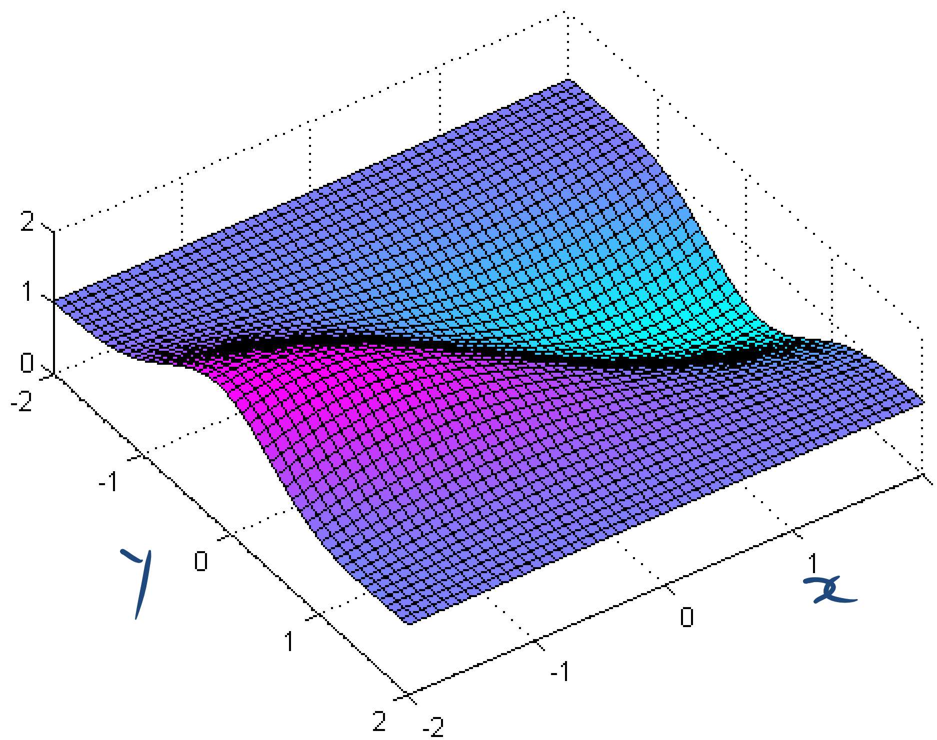


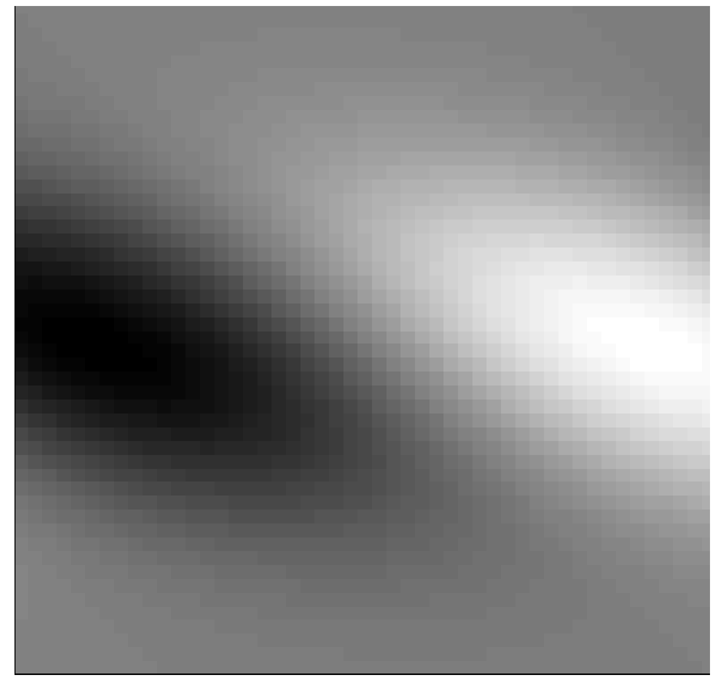
$$A = \int_0^{\sqrt{\pi}} \int_0^{x^2} x \cos y \, dy \, dx = \int_0^{\sqrt{\pi}} x \sin y \Big|_{y=0}^{y=x^2} dx =$$

$$= \int_0^{\sqrt{\pi}} x \sin x^2 \, dx = \frac{1}{2} \int_0^{\pi} \sin u \, du = -\frac{1}{2}(\cos \pi - \cos 0) = 1$$

$$u = x^2 \Rightarrow du = 2x \, dx$$

The integral can be thought of as a *weighted sum* over the domain, where the function f(x,y) gives the weighting.



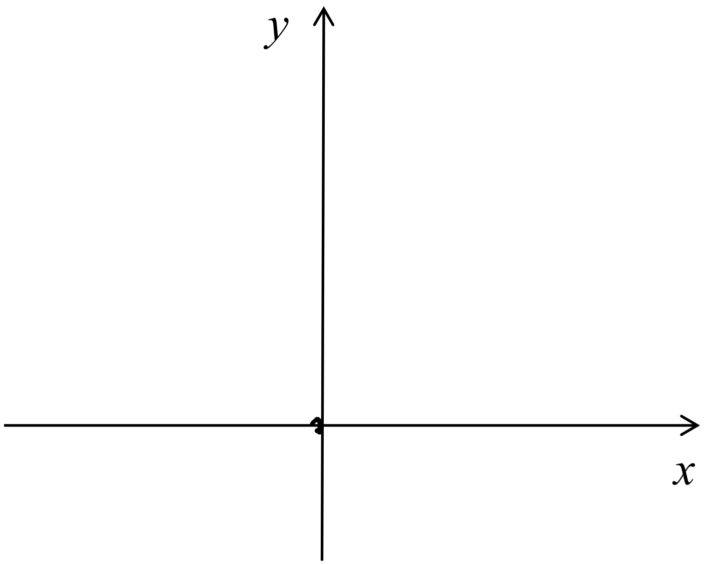


Integrating the function f(x,y) = 1 over the domain D gives the area of the domain.

$$A = \iint_D 1\, dxdy$$

Example:

Find the area of the domain between the curves y = x^2 and y = x^3.



$$A = \iint_D dxdy = \int_0^1 \int_{x^3}^{x^2} dydx = \int_0^1 (x^2 - x^3)dx = \frac{x^3}{3} - \frac{x^4}{4}\Big|_{x=0}^{x=1} = \frac{1}{12}$$

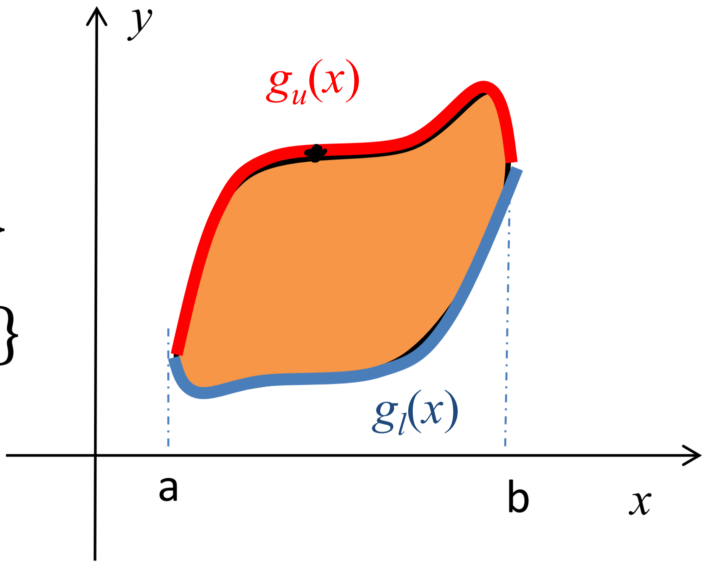
Swapping the order of integration

Some domains of integration can be described in two ways:

$$D = \{(x, y) \mid a < x < b, gl(x) < y < gh(x)\}$$

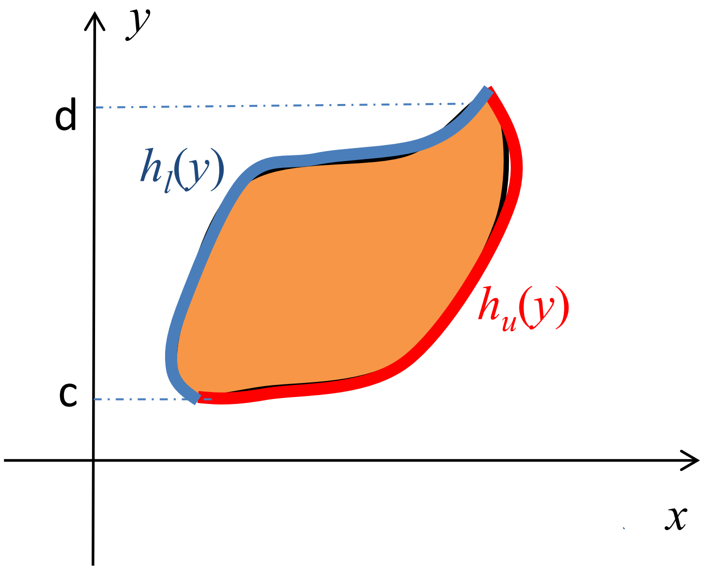
$$D = \{(x, y) \mid c < y < d, hl(y) < x < hu(y)\}$$

When this happens we can swap the order (via a generalisation of Fubini's theorem):



$$\iint_D f(x,y)dydx = \int_a^b \int_{g_l(x)}^{g_u(x)} f(x,y)dydx =$$

$$\int_c^d \int_{h_l(y)}^{h_u(y)} f(x,y)dxdy$$

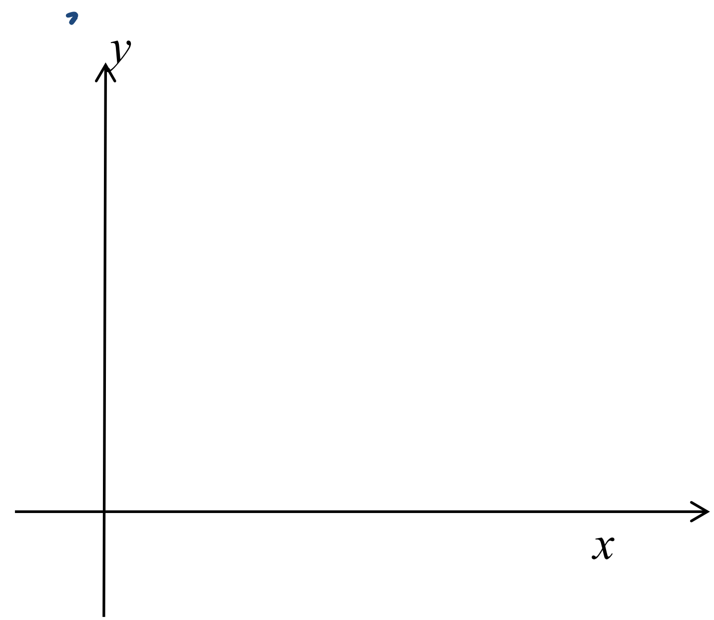


Some integrals are easier performed one way than another.
To swap the order:

1. Draw a graph

2. From the graph write down the limits of the domain

3. Put these limits into the integral



E.g.

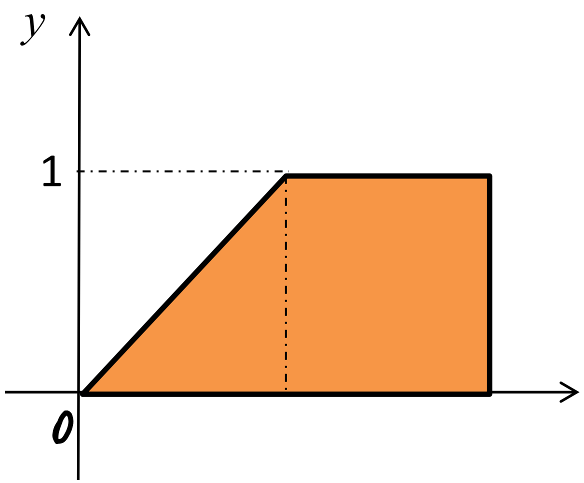
$$\int_0^{\pi/2}\int_y^{\pi/2}\frac{\sin x}{x}\,dxdy$$

$$A=\int_0^{\pi/2}\int_0^{x}\frac{\sin x}{x}\,dydx=\int_0^{\pi/2}\frac{\sin x}{x}\,y\Big|_{y=0}^{y=x}\,dx=$$

$$=\int_0^{\pi/2}\sin x\,dx=-(\cos\frac{\pi}{2}-\cos 0)=1$$

We can also change the order to avoid “splitting” an integral.

E.g. Evaluate $\int_0^1 \int_y^2 xy \, dx dy$

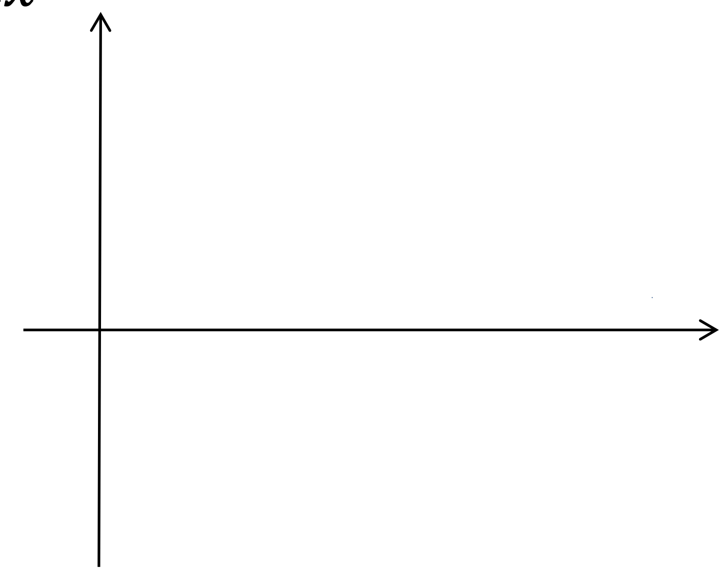


Over the region shown:

$D = \{(x, y) \mid y \leq x \leq 2, \qquad 0 \leq y \leq 1\}$

Double integrals can be separated: if the limits do not depend on x and y

$$\int_c^d \int_a^b f(x)g(y)\,dy = \int_c^d g(y)dy \int_a^b f(x)dx$$

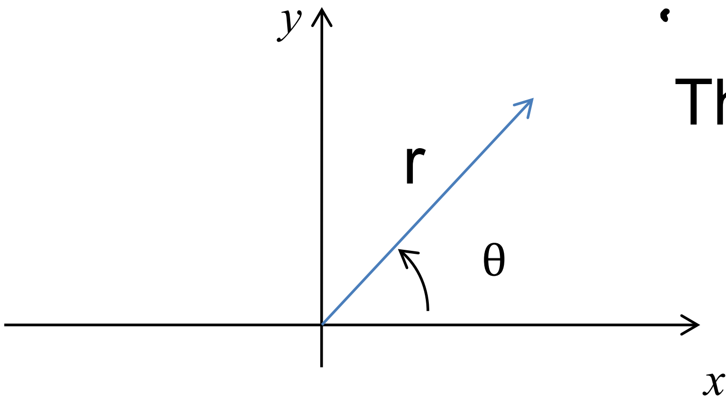


Example: evaluate

$$\int_0^1 \int_{-1}^1 e^{x+y}\,dxdy = \int_0^1 e^y dy \int_{-1}^1 e^x\,dx$$

Polar coordinates

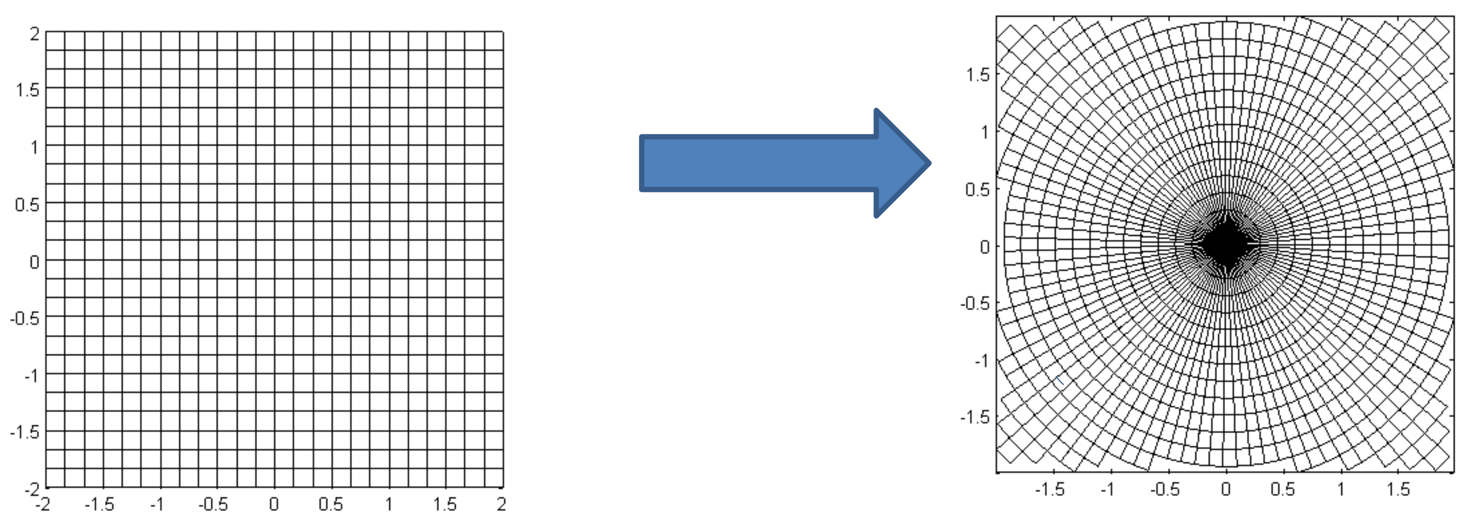
We often want to integrate circular domains, or regions with round elements. For this we need polar coordinates.



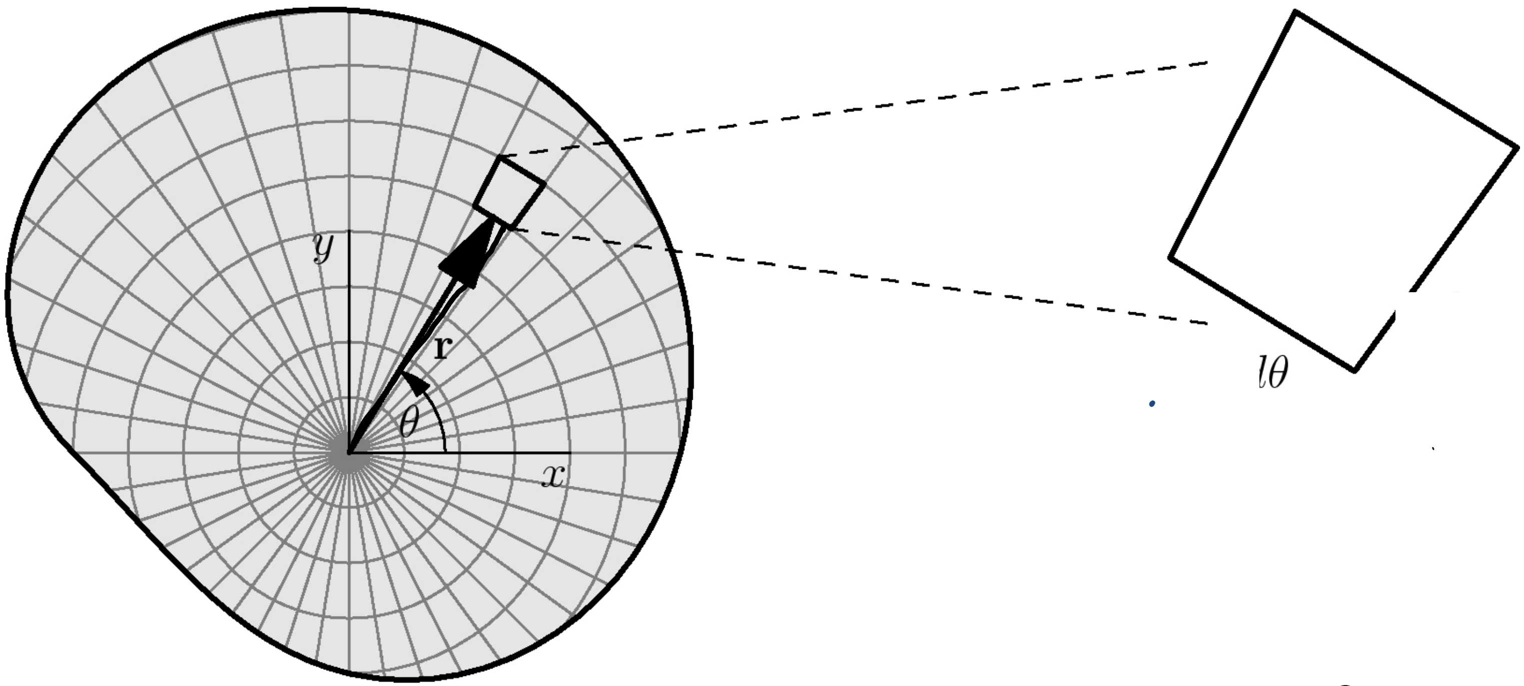
The transformation from (x,y) to (r,θ) is

$x = r\cos\theta$
$y = r\sin\theta$

To integrate, we also need to change from dx dy to dr dμ



To integrate, we divide the domain into a large number of small sections, each with area dA.



x=rcosθ

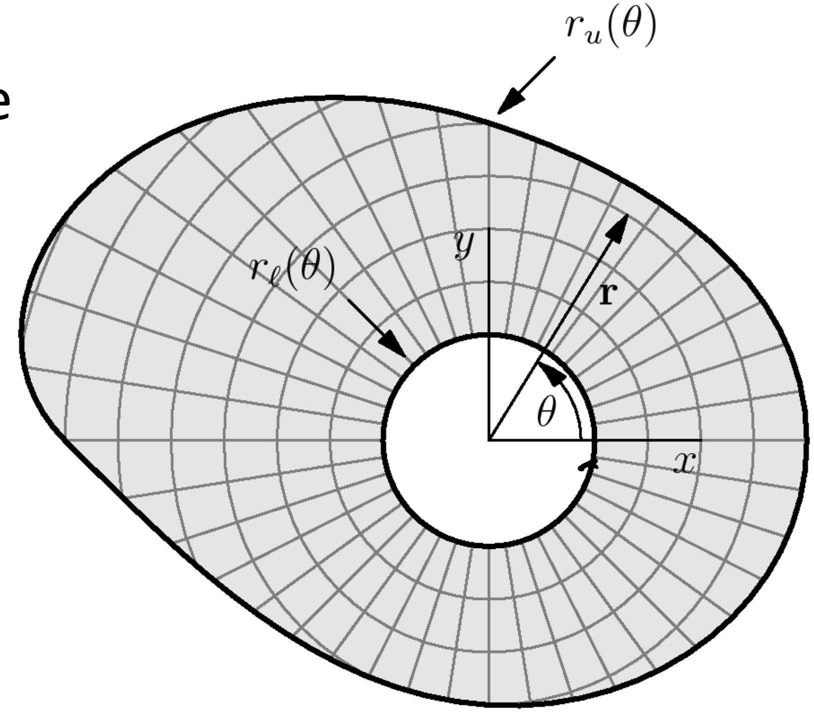
y=rsinθ

Length of small element l= dr

Width of small element w= rdθ

The area of a small element A=rdrdθ

To go from (x,y) to (r, θ), we make the transformation



dxdy ⟹ rdrdθ

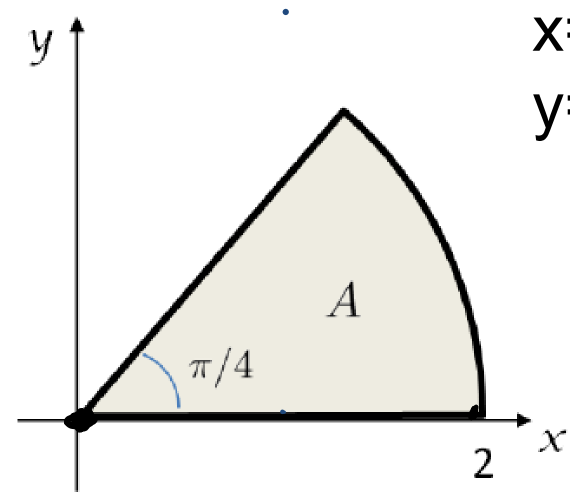
And then integrate, picking one coordinate to integrate over first:

$$\iint_D f(x,y)dydx = \int_{\boldsymbol{\theta}_1}^{\boldsymbol{\theta}_2} \int_{r_l(\boldsymbol{\theta})}^{r_u(\boldsymbol{\theta})} f(r\cos\boldsymbol{\theta}, r\sin\boldsymbol{\theta}) r dr d\boldsymbol{\theta}$$

Example: Integrate

$$\int\int_A x\mathrm{d}x\ \mathrm{d}y$$

where A is the area shown:



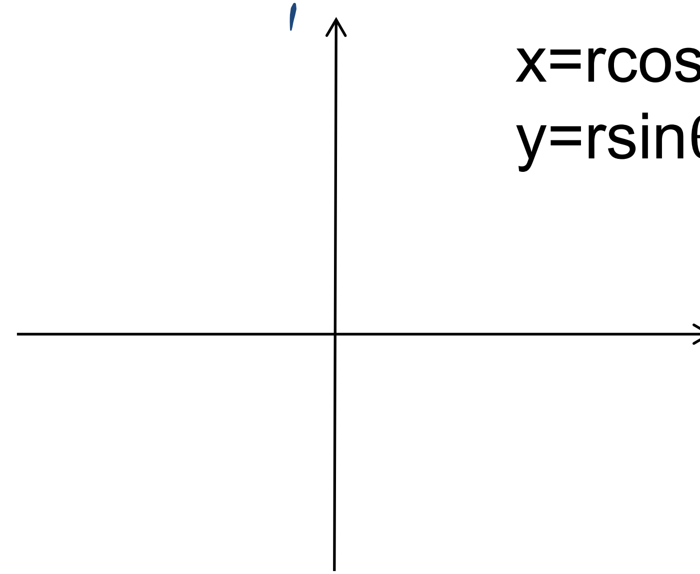
x=rcosθ
y=rsinθ

$$A = \iint_D x dx dy = \int_0^{\pi/4}\int_0^2 r\cos\theta\ r dr d\theta = \int_0^2 r^2 dr \int_0^{\pi/4} \cos\theta d\theta =$$

$$= \left.\frac{r^3}{3}\right|_0^2 \left(\sin\frac{\pi}{4} - \sin 0\right) = \frac{8}{3}\frac{\sqrt{2}}{2} = \frac{4\sqrt{2}}{3}$$

Example: evaluate

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}(x^2+y^2)\,dx\,dy$$



x=rcosθ
y=rsinθ

$$A=\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}(x^2+y^2)dxdy=\int_{0}^{2\pi}\int_{0}^{1}r^2\ rdrd\theta=\int_{0}^{1}r^3dr\int_{0}^{2\pi}d\theta=$$

$$=\left.\frac{r^4}{4}\right|_0^1 2\pi=2\pi\frac{1}{4}=\frac{\pi}{2}$$

Example (important for quantum electrodynamics, statistics):

Use polar coordinates to evaluate:

x=rcosθ
y=rsinθ

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx$$

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2} dxdy = \int_{0}^{\infty}\int_{0}^{2\pi} e^{-r^2} rdrd\theta =$$

$$= \int_{0}^{2\pi} d\theta \int_{0}^{\infty} e^{-r^2} rdr = \frac{2\pi}{2}\int_{0}^{\infty} e^{-u} du = -\pi e^{-u}\Big|_{0}^{\infty} = \pi \Rightarrow I = \sqrt{\pi}$$

$$u = r^2 \quad du = 2rdr$$

Example: Show that the area of a circle is πR^2

$$A = \iint_D dxdy$$

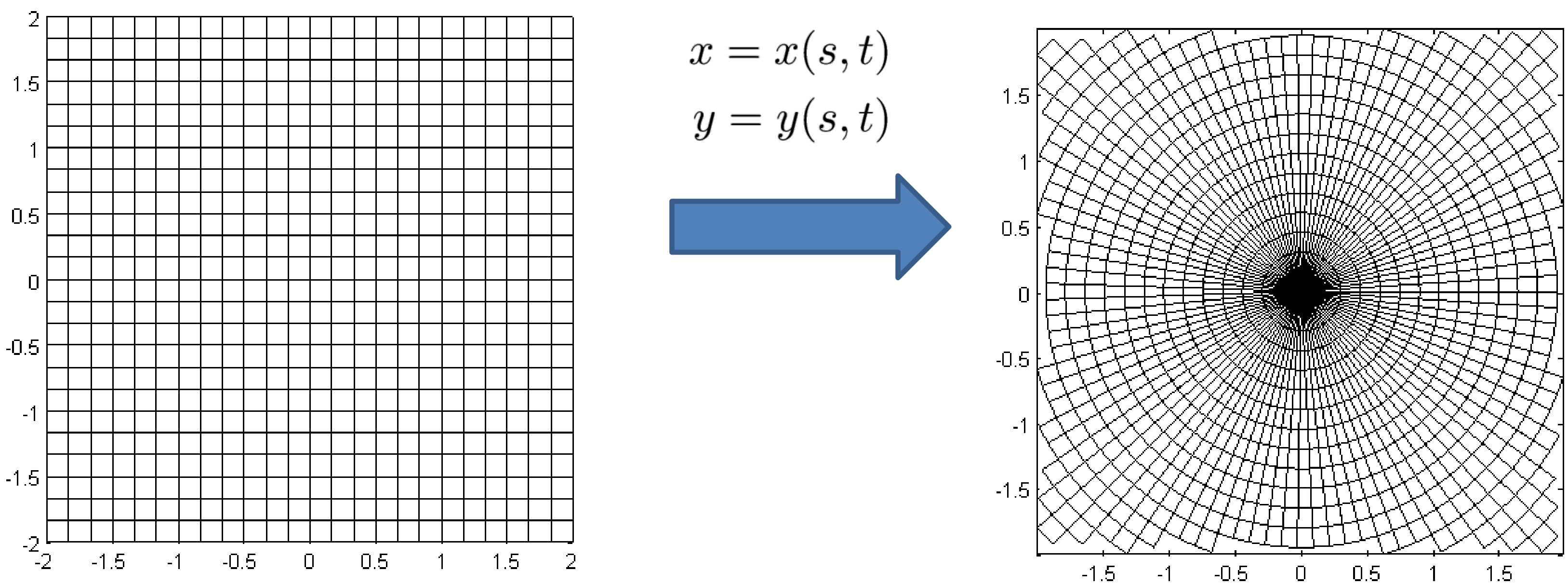
$$A = \iint_D dxdy = \int_0^{2\pi}\int_0^{R} r dr d\theta = \int_0^{R} r dr \int_0^{2\pi} d\theta =$$

$$= \frac{r^2}{2}\Bigg|_0^R 2\pi = 2\pi\frac{R^2}{2} = \pi R^2$$

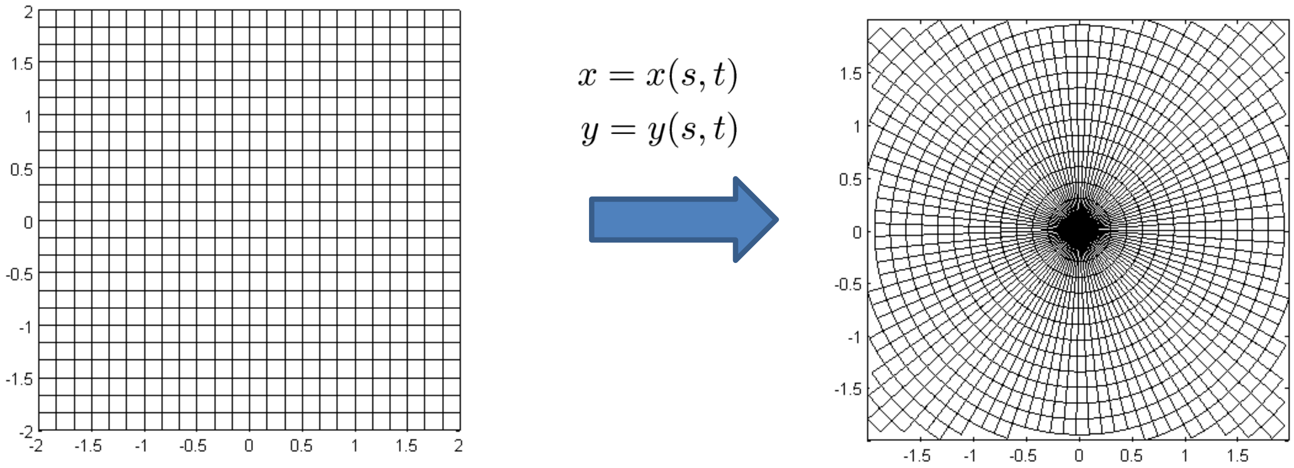
General change of coordinates:

We can write a new coordinate system in terms of the old as

$$x = x(s,t)$$
$$y = y(s,t)$$



We would like a way of writing the area element dA = dx dy in terms of the new coordinates.



To change the area element from (x,y) to (s,t) we use the <u>Jacobian:</u>

$$dxdy = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} dsdt = \left| \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} \right| \text{dsdt}$$

This is often written $$dxdy = \left| \frac{\partial(x,y)}{\partial(s,t)} \right| dsdt$$

with $$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix}$$

Example: Cartesian to polar coordinates:

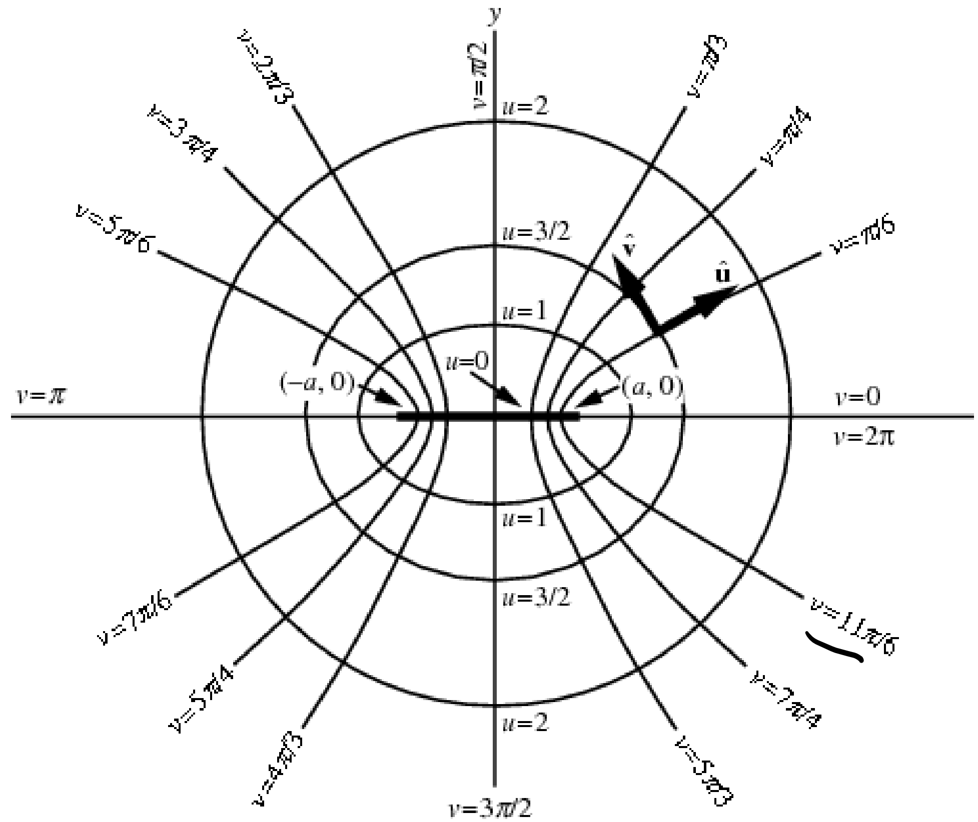
x=rcosθ
y=rsinθ

$$dxdy = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \boldsymbol{\theta}} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \boldsymbol{\theta}} \end{vmatrix} drd\boldsymbol{\theta} = \begin{vmatrix} cos\boldsymbol{\theta} & -rsin\boldsymbol{\theta} \\ sin\boldsymbol{\theta} & rcos\boldsymbol{\theta} \end{vmatrix} drd\boldsymbol{\theta} = rdrd\boldsymbol{\theta}$$

Example: Cartesian to Elliptic coordinates

$x = a \cosh u \cos v$

$y = a \sinh u \sin v$



$$dxdy = a^2(\sinh^2 u + \sin^2 v)dudv$$