

1. Find the volume of the region bounded by the circular paraboloid $x^2 + y^2 + z - 4 = 0$ and the plane $z = 0$.
2. Find the moment of inertia about its axis of a right circular cylinder of altitude h if its density is proportional to the distance from its base.
3. Let the axis of a right circular cylinder of radius 2 go through the center of a sphere of radius 3. Find the volume of the region inside the sphere and outside the cylinder.
4. If the density of a right circular cone of altitude h is proportional to the distance from its base, find its moment of inertia about its axis.
5. Find the center of mass of the solid in Exercise 4.

Use spherical coordinates in Exercises 6–10.

6. Find the volume of the solid in the first octant bounded below by the cone $x^2 + y^2 - z^2 = 0$ and above by the sphere $x^2 + y^2 + z^2 = 4$.
7. Find the volume of the solid bounded by the cone $x^2 + y^2 - z^2 = 0$ and the cylinder $x^2 + y^2 = 4$.
8. Find the moment of inertia of a spherical solid with respect to a diameter if its density is proportional to the square of the distance from the center.
9. Find the moment of inertia of a right circular cone, of altitude h , about its axis if its density is proportional to the square of the distance from its vertex.
10. Find the center of mass of a hemispherical solid if its density is proportional to the distance from its base.

Infinite Series

21.1 SEQUENCES

We recall that a function (of one variable) is a correspondence between the elements of two sets, its domain and its range, such that to each element of the domain there corresponds one and only one element of the range. Up to now, the domains of the functions we have been considering usually consisted of *all* real numbers (with possibly some exceptions) in one or more intervals, open or closed, finite or infinite.

If the domain of a function is restricted to integers, the function is called a *sequence*.

21.1.1 Definition. An infinite sequence is a function whose domain is the set of all integers greater than, or equal to, some specified integer.

As an illustration, the sequence defined by $f(n) = 1/n$, where n is any positive integer, is $\{(n, 1/n) \mid n \text{ is a positive integer}\}$; its range is $\{f(1) = 1, f(2) = \frac{1}{2}, f(3) = \frac{1}{3}, \dots, f(n) = 1/n, \dots\}$. Unless it is stated to the contrary, the domain of a sequence will be assumed to be the set of all positive integers. This enables us to write $\{1/n\}$ for the above sequence instead of $\{(n, 1/n) \mid n \text{ is a positive integer}\}$.

Another example of an infinite sequence is $\{n/2^n\}$. It is defined by $f(n) = n/2^n$ and its range is $\{\frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \dots, n/2^n, \dots\}$.

The elements of the range of a sequence f , namely $f(1), f(2), f(3), \dots$, are called the *terms* of the sequences; and $f(n)$ is called the *nth term* or *general term* of the sequence. It is customary to use subscript notation in writing the terms of a sequence. Thus we write $u_1, u_2, u_3, \dots, u_n, \dots$ instead of $f(1), f(2), f(3), \dots, f(n), \dots$ for the terms of a sequence. For example, the terms of the sequence $\{n!/10^n\}$ are $u_1 = 1/10, u_2 = 2/100, u_3 = 6/1000, \dots, u_n = n!/10^n, \dots$.

In the case of a sequence the definition 4.5.1 of the limit of a function as the independent variable approaches infinity takes on the following form:

if to each positive number ϵ there corresponds a positive integer N such that $|u_n - L| < \epsilon$ whenever $n \geq N$.

That is, a sequence has a limit if corresponding to each positive number ϵ , we can specify a term of the sequence such that this term and every following term will differ from the limit by less than ϵ .

A sequence which has a limit is said to be **convergent**, and to **converge** to that limit. A sequence which fails to have a limit is **divergent**.

The limit theorems on sums, products and quotients of functions also apply when the functions are sequences:

21.1.3 Theorem. If $\{u_n\}$ and $\{v_n\}$ are convergent sequences, and if k is a number, then

$$(a) \lim_{n \rightarrow \infty} k u_n = k \lim_{n \rightarrow \infty} u_n;$$

$$(b) \lim_{n \rightarrow \infty} (u_n + v_n) = \lim_{n \rightarrow \infty} u_n + \lim_{n \rightarrow \infty} v_n;$$

$$(c) \lim_{n \rightarrow \infty} u_n v_n = \lim_{n \rightarrow \infty} u_n \lim_{n \rightarrow \infty} v_n;$$

and, provided $\lim_{n \rightarrow \infty} v_n \neq 0$,

$$(d) \lim_{n \rightarrow \infty} u_n / v_n = \lim_{n \rightarrow \infty} u_n / \lim_{n \rightarrow \infty} v_n.$$

A proof of this theorem is analogous to the proof of 4.3.1.

Example 1. Find the limit of the sequence defined by $u_n = 3n^2/(7n^2 + 1)$.

Solution. Since $n \neq 0$,

$$u_n = \frac{3n^2}{7n^2 + 1} = \frac{3}{7 + 1/n^2}.$$

Thus

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{3}{7 + 1/n^2} = \frac{3}{7 + \lim_{n \rightarrow \infty} 1/n^2} = \frac{3}{7}.$$

The proof that $\lim_{x \rightarrow \infty} 1/x^2 = 0$, given in Example 1, Section 4.5, assumed that $1/x^2$ was defined for all positive numbers. But this also implies that $\lim_{n \rightarrow \infty} 1/n^2 = 0$ when n is restricted to positive integers.

In the same way, if $\lim_{x \rightarrow \infty} f(x) = L$ when f is defined for all real numbers greater than some number R , then $\lim_{n \rightarrow \infty} f(n) = L$ when the domain of f is restricted to integers greater than R .

Solution. The proof is trivial if $r = 0$, so it is assumed in what follows that $r \neq 0$.

We shall prove the theorem for r^x , which is defined for all numbers x , and it will follow that the theorem is true for r^n when n is restricted to positive integers.

Let $0 < r < 1$ and let ϵ be any positive number less than 1. Since $\ln r < 0$ and $\ln \epsilon < 0$ (see 12.2),

$$x > \frac{\ln \epsilon}{\ln r} \Leftrightarrow x \ln r < \ln \epsilon \Leftrightarrow \ln r^x < \ln \epsilon$$

$$\Leftrightarrow r^x < \epsilon \Leftrightarrow |r^x| < \epsilon.$$

Thus $|r^n| < \epsilon$ for all numbers x such that $x > (\ln \epsilon)/\ln r$. Therefore (by 4.5.1) $\lim_{x \rightarrow \infty} r^x = 0$ if $0 < r < 1$.

Now assume that $r > 1$, and let K be an arbitrarily chosen number greater than 1. Since $\ln r > 0$ and $\ln K > 0$,

$$x > \frac{\ln K}{\ln r} \Leftrightarrow x \ln r > \ln K \Leftrightarrow \ln r^x > \ln K$$

$$\Leftrightarrow r^x > K.$$

Therefore (by 4.5.5) $\lim_{x \rightarrow \infty} r^x = \infty$ if $r > 1$.

Example 3. Does the sequence $\{\ln n/e^n\}$ converge?

Solution. Since $\ln x$ and e^x exist for all positive numbers x , and $\lim_{x \rightarrow \infty} \ln x = \infty$, we may use l'Hôpital's rule to find

$$\lim_{x \rightarrow \infty} \frac{\ln x}{e^x} = \lim_{x \rightarrow \infty} \frac{1/x}{e^x} = 0.$$

Therefore,

$$\lim_{n \rightarrow \infty} \frac{\ln n}{e^n} = 0,$$

when n is restricted to the positive integers, and thus the sequence $\{\ln n/e^n\}$ converges to the limit 0.

The sequence $\{u_n\}$ is said to be **monotonically increasing** if $u_1 \leq u_2 \leq u_3 \leq \dots \leq u_n \leq \dots$. A sequence $\{u_n\}$ is **monotonically decreasing** if $u_1 \geq u_2 \geq u_3 \geq \dots \geq u_n \geq \dots$. A sequence which is either monotonically increasing or monotonically decreasing is said to be **monotonic**.

As an illustration, the sequence $\{(2^n - 1)/2^n\}$ is monotonically increasing, and the sequence $\{1/n\}$ is monotonically decreasing.

The number A is said to be an **upper bound** for the sequence $\{u_n\}$ if $u_n \leq A$ for $n = 1, 2, 3, \dots$, and such a sequence is said to be **bounded above**. If there is a number B such that $B \leq u_n$ for $n = 1, 2, 3, \dots$, then B is a **lower bound** for the sequence $\{u_n\}$, and the sequence is **bounded below**. The sequence

A necessary and sufficient condition for a monotonic sequence to be convergent is given in the following theorem.

21.1.4 Theorem. *A monotonic sequence is convergent if and only if it is bounded.*

Proof. Assume that the sequence $\{u_n\}$ is monotonically increasing and has an upper bound A . Since the set of real numbers $S = \{u_1, u_2, u_3, \dots\}$ has an upper bound, it has a least upper bound L , by the completeness axiom (1.4.1).

Let ϵ be a positive number. Then $L - \epsilon < L$, and $L - \epsilon$ cannot be an upper bound for S since L is the least upper bound for S . Thus there exists a term u_k of the sequence which is greater than $L - \epsilon$, and $L - \epsilon < u_n \leq L$ for all $n > k$. But this implies that $-\epsilon < u_n - L \leq 0 < \epsilon$, or $|u_n - L| < \epsilon$, for all $n > k$. Therefore $\lim_{n \rightarrow \infty} u_n = L$, and $\{u_n\}$ is convergent. This proves

that if $\{u_n\}$ is monotonically increasing and is bounded, it is convergent.

We will now assume that $\{u_n\}$ is monotonically increasing and is convergent, and will prove that it is bounded. Of course any monotonically increasing sequence is bounded below by its first term, so our task is to find an upper bound. Since $\lim_{n \rightarrow \infty} u_n = L$ (say), corresponding to each $\epsilon > 0$ there exists an integer N such that $|u_n - L| < \epsilon$ whenever $n \geq N$; that is, $-\epsilon < u_n - L < \epsilon$, or $L - \epsilon < u_n < L + \epsilon$ for all $n \geq N$. Since the sequence is monotonically increasing, this implies that $u_n < L + \epsilon$ for all positive integers n . Therefore $\{u_n\}$ is bounded above by $L + \epsilon$.

The proof of the theorem when $\{u_n\}$ is monotonically decreasing is immediate, since $\{-u_n\}$ is monotonically increasing. ■

21.1.5 Corollary. *If U is an upper bound of a monotonically increasing sequence, the sequence converges to a limit which is less than or equal to U ; if V is a lower bound of a monotonically decreasing sequence, the sequence converges to a limit which is greater than or equal to V .*

It should be noted that since $\lim_{n \rightarrow \infty} u_n$ depends only on what happens to u_n after n exceeds a certain number, any finite number of terms of the sequence can be discarded without affecting the existence of a limit or the value of the limit. It follows that 21.1.4 and 21.1.5 apply to sequences which are not monotonic, provided they become monotonic for all n greater than some integer k . If there is a number k such that for $n > k$ the sequence $\{u_n\}$ is monotonic and bounded, then $\{u_n\}$ is convergent.

Example 4. The sequence $\{10^n/2^{n^2}\}$ is not monotonic since $u_1 < u_2 > u_3 > u_4 > \dots$. But if we discard the first term and consider the sequence $u_n = 10^n/2^{n^2}$

n . Therefore it converges (by 21.1.4).

EXERCISES

Write out the first five terms of the sequences in Exercises 1–10. Determine whether the sequences converge, and if so, find the limit.

1. $\left\{\frac{n}{2n-1}\right\}$

2. $\left\{\frac{2n+1}{n+2}\right\}$

3. $\left\{\frac{n^2+1}{n^2-2n+3}\right\}$

4. $\left\{\frac{3n^2+2}{n+4}\right\}$

5. $\left\{\frac{n}{\ln(n+1)}\right\}$

6. $\left\{\frac{1}{n} \cos n\right\}$

7. $\left\{\frac{\ln n}{n^2}\right\}$

8. $\left\{\frac{e^n}{2^n}\right\}$

9. $\left\{\frac{e^n}{n^2}\right\}$

10. $\{e^{-n} \sin n\}$

Find the simplest possible expression for the n th term of the sequences indicated in Exercises 11–20. Tell whether or not the sequence converges, and if so to what limit.

11. $\left\{\frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \dots\right\}$

12. $\{2, 1, 2, 1\frac{1}{2}, 2, 1\frac{3}{4}, 2, 1\frac{7}{8}, \dots\}$

13. $\left\{1, \frac{1}{1-\frac{1}{2}}, \frac{1}{1-\frac{2}{3}}, \frac{1}{1-\frac{3}{4}}, \dots\right\}$

14. $\left\{1, \frac{1}{4}, 1, \frac{3}{16}, 1, \frac{5}{64}, 1, \frac{7}{256}, 1, \dots\right\}$

15. $\left\{0, \frac{\ln 2}{2}, \frac{\ln 3}{3}, \frac{\ln 4}{4}, \dots\right\}$

16. $\left\{0, \frac{1}{2^2}, \frac{2}{3^2}, \frac{3}{4^2}, \dots\right\}$

17. $\left\{1, \frac{2}{2^2-1^2}, \frac{3}{3^2-2^2}, \frac{4}{4^2-3^2}, \dots\right\}$

18. $\left\{\frac{1}{2-\frac{1}{2}}, \frac{2}{3-\frac{1}{3}}, \frac{3}{4-\frac{1}{4}}, \dots\right\}$

19. $\left\{2, 1, \frac{2^3}{3^2}, \frac{2^4}{4^2}, \frac{2^5}{5^2}, \dots\right\}$

20. $\left\{\sin 1^\circ, \frac{\sin 2^\circ}{2}, \frac{\sin 3^\circ}{3}, \dots\right\}$

21.2 INFINITE SERIES

21.2.1 Definition. Let $\{a_n\}$ be a sequence. Then the expression

$$\sum_{n=1}^{\infty} a_n, \text{ or } a_1 + a_2 + a_3 + \dots + a_n + \dots,$$

is called an *infinite series*.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} + \cdots$$

is an infinite series.

The numbers a_1, a_2, a_3, \dots are called the *terms* of the series.

We wish to assign a number to this infinite series and call this number the *sum* of the series. But, up to now, the word “sum” has been defined only for a finite number of terms. We seek to extend this definition so that it will apply to an infinite number of terms.

Denote by s_n the sum of the first n terms of the infinite series ($n = 1, 2, 3, \dots$). That is, let

$$s_1 = a_1,$$

$$s_2 = a_1 + a_2,$$

$$s_3 = a_1 + a_2 + a_3,$$

...

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n,$$

The sequence $\{s_n\}$ is called the *sequence of partial sums* of the series $\sum_{i=1}^{\infty} a_i$.

21.2.2 Definitions. Let $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ be an infinite series, and let $\{s_n\}$, where $s_n = a_1 + a_2 + a_3 + \cdots + a_n$ for $n = 1, 2, 3, \dots$, be the sequence of partial sums of the infinite series. If

$$\lim_{n \rightarrow \infty} s_n = S$$

exists, the series is said to be **convergent** (and to converge to the value S) and S is called the **sum** of the infinite series $\sum_{i=1}^{\infty} a_i$. If $\lim_{n \rightarrow \infty} s_n$ fails to exist, the series is **divergent** and has no sum.

If the series $\sum_{i=1}^{\infty} a_i$ has a sum S , we write

$$(1) \quad \sum_{i=1}^{\infty} a_i = S.$$

The equation (1) simply means that the sequence of partial sums, $\{s_n\}$, of the infinite series $\sum_{i=1}^{\infty} a_i$ converges to the limit S .

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}} + \cdots$$

converges, and find its sum.

Solution.

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$$

is a finite geometric series of n terms. Its sum is

$$s_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} = 2 - \frac{1}{2^{n-1}}.$$

Since

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left[2 - \frac{1}{2^{n-1}} \right] = 2 - \lim_{n \rightarrow \infty} \frac{1}{2^{n-1}} = 2,$$

the given infinite series converges and its sum is 2 (by 21.2.2).

Sometimes we wish to start the series with the term a_0 or a_2 , or some other term. If k is an arbitrarily chosen nonnegative integer, we write

$$\sum_{n=k}^{\infty} a_n = a_k + a_{k+1} + a_{k+2} + \cdots$$

When it is not important what index we assign to the first term, or when it is clear from the context, we often write $\sum a_n$ for an infinite series.

Since $\lim_{n \rightarrow \infty} (s_n - C)$, where C is a constant, exists if and only if $\lim_{n \rightarrow \infty} s_n$ exists, it follows that we can omit a finite number of terms at the beginning of an infinite series without affecting its convergence or divergence. Of course, the *value* of the sum, if any, will be affected.

21.2.3 Theorem. The two infinite series

$$\sum_{i=1}^{\infty} a_i \quad \text{and} \quad \sum_{i=k}^{\infty} a_i,$$

where k is an arbitrarily chosen positive integer, both converge or both diverge.

A test for divergence of an infinite series, which is often easy to apply, is given in the next theorem. It should be tried first when testing a series for convergence; for if the series is divergent, no further tests need be applied.

21.2.4 Theorem. A necessary condition for the series

$$\sum a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

to converge is $\lim_{n \rightarrow \infty} a_n = 0$.

where S is a number.

It follows from the definition of the limit of a sequence (21.1.2) that $\lim_{n \rightarrow \infty} s_n = S$ implies $\lim_{n \rightarrow \infty} s_{n-1} = S$. But $a_n = s_n - s_{n-1}$. Therefore

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (s_n - s_{n-1}) = \lim_{n \rightarrow \infty} s_n - \lim_{n \rightarrow \infty} s_{n-1} = S - S = 0.$$

Thus, if $\sum a_n$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$. ■

21.2.5 Corollary. If $\lim_{n \rightarrow \infty} a_n \neq 0$, then the infinite series $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ diverges.

It is important to notice that 21.2.4 gives a *necessary* condition for the convergence of an infinite series, not a *sufficient* one. That is, a series may not converge even if its n th term approaches zero. But if $\lim_{n \rightarrow \infty} a_n \neq 0$, the series *diverges*.

Example 2. The series

$$\sum_{i=1}^{\infty} ar^{i-1} = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots,$$

where $a \neq 0$ and $r \neq 0$, is called a *geometric series*. Show that the geometric series converges if $|r| < 1$ and diverges if $|r| \geq 1$.

Solution. Write $s_n = a + ar + ar^2 + \cdots + ar^{n-1}$. Then $(1 - r)s_n = a - ar^n$ and, if $r \neq 1$,

$$s_n = \frac{a - ar^n}{1 - r}.$$

Since $\lim_{n \rightarrow \infty} r^n = 0$ if $|r| < 1$ (by Example 2, 21.1), we have

$$\lim_{n \rightarrow \infty} s_n = \frac{a}{1 - r}, \quad \text{if } |r| < 1.$$

Thus the geometric series converges if $|r| < 1$.

If $|r| > 1$, then

$$\lim_{n \rightarrow \infty} |r|^n = \infty$$

(by Example 2, 21.1), and

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (ar^{n-1}) = a \lim_{n \rightarrow \infty} r^{n-1} \neq 0.$$

Therefore (by 21.2.5) the geometric series diverges when $|r| > 1$.

$\cdots + (-1)^{n-1}a + \cdots$ and $\lim_{n \rightarrow \infty} a_n \neq 0$. Thus (by 21.2.4 and 21.2.5) the geometric series diverges when $|r| = 1$.

Therefore the geometric series converges if $|r| < 1$ and diverges if $|r| \geq 1$.

Example 3. Consider the *harmonic series*

$$\sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots.$$

Its n th term is $1/n$ and $\lim_{n \rightarrow \infty} 1/n = 0$. But the series diverges. For, if $k = 2^{n-1}$,

$$\begin{aligned} s_k &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^{n-1}} \\ &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots \\ &\quad + \left(\frac{1}{2^{n-2}+1} + \frac{1}{2^{n-2}+2} + \cdots + \frac{1}{2^{n-1}}\right) \\ &\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \frac{1}{16} + \cdots + \frac{1}{16}\right) + \cdots \\ &\quad + \left(\frac{1}{2^{n-1}} + \frac{1}{2^{n-1}} + \cdots + \frac{1}{2^{n-1}}\right) \\ &\Rightarrow 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} \\ &= 1 + (n-1)\frac{1}{2} = \frac{n+1}{2}. \end{aligned}$$

Thus $s_k \geq \frac{1}{2}(n+1)$; and, since $\lim_{n \rightarrow \infty} \frac{n+1}{2} = \infty$, $\lim_{n \rightarrow \infty} s_k = \infty$. Therefore the monotonic sequence $\{s_n\}$ is unbounded and the harmonic series diverges (21.1.4).

An infinite series may be multiplied, term by term, by a nonzero constant without affecting its convergence or divergence.

21.2.6 Theorem. If c is a constant and the series $\sum a_n$ converges, then so does $\sum ca_n$, and its limit is $c \sum a_n$. If $\sum a_n$ diverges, then $\sum ca_n$ diverges, provided $c \neq 0$.

Proof. Assume that $\sum a_n$ converges, so that $\lim_{n \rightarrow \infty} s_n = S$, where $s_n = a_1 + a_2 + \cdots + a_n$. The sum of the first n terms of the series $\sum ca_n$ is cs_n . But $\lim_{n \rightarrow \infty} cs_n = c \lim_{n \rightarrow \infty} s_n = cS$. Therefore the series $\sum ca_n$ converges to the limit $cS = c \sum a_n$.

most part of the theorem, the convergence of $\sum ca_n$ would imply the convergence of $\sum (1/c)(ca_n) = \sum a_n$. ■

An infinite series is not an ordinary sum, so it is not surprising that some of the laws of operation on finite sums fail to carry over to infinite series. For example, the associative law for addition permits us to group the terms of a finite sum in any way we please by the insertion of parentheses, without affecting the sum. But consider the infinite series

$$1 - 1 + 1 - 1 + \cdots + (-1)^{n+1} + \cdots.$$

Its n th term does not approach zero as $n \rightarrow \infty$, so the series is divergent and has no sum. If we group the terms of the series so that it becomes

$$(1 - 1) + (1 - 1) + (1 - 1) + \cdots = 0 + 0 + 0 + \cdots,$$

it converges with sum zero; and if we regroup the terms another way,

$$1 - (1 - 1) - (1 - 1) - (1 - 1) - \cdots = 1 - 0 - 0 - 0 - \cdots,$$

its sum is 1.

However, if a series is convergent, it can be treated in many ways like an ordinary finite sum, as shown in the following theorems.

21.2.7 Theorem. *The terms of a convergent series may be grouped in any way and the new series will converge to the same limit as the original series.*

Proof. Let $\sum a_n$ be a convergent series whose sequence of partial sums is $\{s_n\}$ with limit S . If $\sum b_m$ is a series formed by grouping the terms of $\sum a_n$, and if $\{t_m\}$ is the sequence of partial sums of $\sum b_m$, then for each m , $t_m \in \{s_n\}$, and as m increases indefinitely, so does the corresponding n . Therefore the limit of $\{t_m\}$ exists and is equal to S . ■

Two convergent series may be added term by term, and the resulting series will converge to the sum of the limits of the original series.

21.2.8 Theorem. *If $\sum a_n$ and $\sum b_n$ are convergent infinite series, then $\sum (a_n + b_n)$ is convergent and*

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n.$$

Proof. Let $s_n = a_1 + a_2 + a_3 + \cdots + a_n$, $t_n = b_1 + b_2 + b_3 + \cdots + b_n$, and $w_n = (a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n)$. Then

$$w_n = s_n + t_n.$$

$$\lim_{n \rightarrow \infty} w_n = \lim_{n \rightarrow \infty} (s_n + t_n) = \lim_{n \rightarrow \infty} s_n + \lim_{n \rightarrow \infty} t_n.$$

Therefore

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n. \quad \blacksquare$$

21.3 TESTS FOR CONVERGENCE OF SERIES OF POSITIVE TERMS

The terms of the infinite series we have been considering so far were any real numbers, positive, negative, or zero. The definitions and theorems in the preceding section apply equally to all infinite series, regardless of the signs of the terms.

In the present section we will confine our attention to series all of whose terms are *positive numbers* (or, more generally, *nonnegative numbers*).

Notice that if all the terms of a series are nonnegative, its sequence of partial sums is monotonically increasing. Thus 21.1.4 has the following corollary.

21.3.1 Theorem. *A necessary and sufficient condition for an infinite series of nonnegative terms to converge is that its sequence of partial sums have an upper bound.*

Testing a given series for convergence would often be a difficult task if we depended on the definition of convergence alone. The trouble is that it is usually not easy to find an expression for the n th partial sum that is simple enough for us to decide whether or not it possesses a limit as $n \rightarrow \infty$. In the tests for convergence given below, it is not necessary to find an expression for the n th partial sum.

21.3.2 Definition. *The series $\sum A_n = A_1 + A_2 + A_3 + \cdots + A_n + \cdots$ is said to **dominate** the series $\sum a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ if $|a_n| \leq A_n$ for all positive integers n .*

Clearly, the terms of a dominating series must all be nonnegative, although this is not necessarily true of the series which is dominated.

21.3.3 Theorem (Comparison Test). *Let $\sum a_n$ and $\sum b_n$ be series of nonnegative terms and let $\sum b_n$ dominate $\sum a_n$, so that $a_n \leq b_n$ for all positive integers n .*

- (1) *If the dominating series $\sum b_n$ is convergent, then $\sum a_n$ is convergent.*
- (2) *If $\sum a_n$ is divergent, then the dominating series $\sum b_n$ is divergent.*

Since the nonnegative series $\sum b_n$ is convergent, its sequence of partial sums $\{t_n\}$ has an upper bound (by 21.3.1). But $0 \leq s_n \leq t_n$ for all positive integers n , because $\sum b_n$ dominates $\sum a_n$. Hence $\{s_n\}$ is bounded above and $\sum a_n$ converges.

Part (2) follows immediately, since if $\sum a_n$ is divergent it is not convergent, and therefore $\sum b_n$ cannot be convergent [by Part (1)]. ■

Example 1. Show that the series

$$\sum \frac{n}{2^{n(n+1)}}$$

converges.

Solution. Since $0 < n/(n+1) < 1$ for all positive integers n ,

$$\frac{n}{2^{n(n+1)}} < \frac{1}{2^n}.$$

But $\sum 1/2^n$ is a geometric series with $r = 1/2$, and thus is convergent. Therefore the given series is convergent, by the comparison test.

Example 2. Test for convergence

$$\sum \frac{n}{5n^2 - 4}$$

Solution.

$$\frac{n}{5n^2 - 4} > \frac{n}{5n^2} = \frac{1}{5n} = \frac{1}{5} \left(\frac{1}{n} \right).$$

Now $\sum 1/n$ is the harmonic series, which is divergent (Example 3, 21.2). Therefore $\sum 1/5n = \sum n/5n^2$ is divergent (by 21.2.6) and it follows that the given series is divergent, by the comparison test.

The next test follows from the comparison test.

21.3.4 Theorem. If $\sum a_n$ and $\sum b_n$ are series of positive terms, and if

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L > 0,$$

then either both series converge or both series diverge.

Proof. Since $\lim_{n \rightarrow \infty} a_n/b_n = L > 0$, there is a positive integer N such that

$$\left| \frac{a_n}{b_n} - L \right| < \frac{L}{2}$$

$$-\frac{L}{2} < \frac{a_n}{b_n} - L < \frac{L}{2},$$

or

$$\frac{L}{2} < \frac{a_n}{b_n} < \frac{3L}{2},$$

for $n \geq N$. Hence

$$\frac{L}{2} b_n < a_n \quad \text{and} \quad \frac{2a_n}{3L} < b_n$$

for all $n \geq N$. Thus the series $\sum a_n$ dominates the series $(L/2) \sum b_n$, and the series $\sum b_n$ dominates the series $(2/3L) \sum a_n$. It follows from the comparison test that either both the series $\sum a_n$ and $\sum b_n$ converge or both diverge. ■

It is easy to see that if $\sum a_n$ and $\sum b_n$ are series of positive terms and if $\lim_{n \rightarrow \infty} (a_n/b_n) = 0$ and $\sum b_n$ converges, then $\sum a_n$ also converges.

For $\lim_{n \rightarrow \infty} (a_n/b_n) = 0$ implies that there exists a positive integer N such that

$$\left| \frac{a_n}{b_n} \right| < 1$$

for all $n > N$. That is, $|a_n| < |b_n|$ for all $n > N$. But $a_n > 0$ and $b_n > 0$. Therefore $0 < a_n < b_n$ for all $n > N$. Thus $\sum b_n$ dominates $\sum a_n$ and, since $\sum b_n$ converges, so also does $\sum a_n$ (by 21.3.3).

Example 3. Show that the series

$$\sum \frac{1}{\sqrt{4n^2 - 7}}$$

is divergent.

Solution. The limit of the ratio of the general term of the harmonic series $\sum 1/n$ to the general term of the given series, as $n \rightarrow \infty$, is

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{\frac{1}{\sqrt{4n^2 - 7}}} &= \lim_{n \rightarrow \infty} \frac{\sqrt{4n^2 - 7}}{n} = \lim_{n \rightarrow \infty} \sqrt{\frac{4n^2 - 7}{n^2}} \\ &= \lim_{n \rightarrow \infty} \sqrt{4 - \frac{7}{n^2}} = 2. \end{aligned}$$

Since the harmonic series is divergent, the given series is divergent (by 21.3.4).

To use the two preceding tests effectively, we need some series, whose convergence or divergence we know, as bases for comparison. The next test, known as the *Maclaurin-Cauchy integral test*, will provide such series; and it is interesting and worthwhile in its own right.

monotonically decreasing function defined for all real numbers $x \geq 1$, and $a_i = f(i)$ for all positive integers i . Then the infinite series

$$\sum_{i=1}^{\infty} a_i$$

converges if and only if the improper integral

$$\int_1^{\infty} f(x) dx$$

converges.

Proof. By the mean-value theorem for integrals,

$$(1) \quad \int_i^{i+1} f(x) dx = 1 \cdot f(\xi),$$

where $i < \xi < i + 1$. Since f is monotonically decreasing

$$a_i = f(i) \geq f(\xi) \geq f(i + 1) = a_{i+1}.$$

Therefore

$$(2) \quad a_i \geq \int_i^{i+1} f(x) dx \geq a_{i+1}.$$

It follows from (2) that for every n

$$(3) \quad \sum_{i=1}^n a_i \geq \sum_{i=1}^n \int_i^{i+1} f(x) dx = \int_1^{n+1} f(x) dx \geq \sum_{i=1}^n a_{i+1} = \left(\sum_{i=1}^{n+1} a_i \right) - a_1$$

(see Fig. 414). Therefore all three of the expressions

$$\sum_{i=1}^{\infty} a_i, \quad \int_1^{\infty} f(x) dx, \quad \left(\sum_{i=1}^{\infty} a_i \right) - a_1,$$

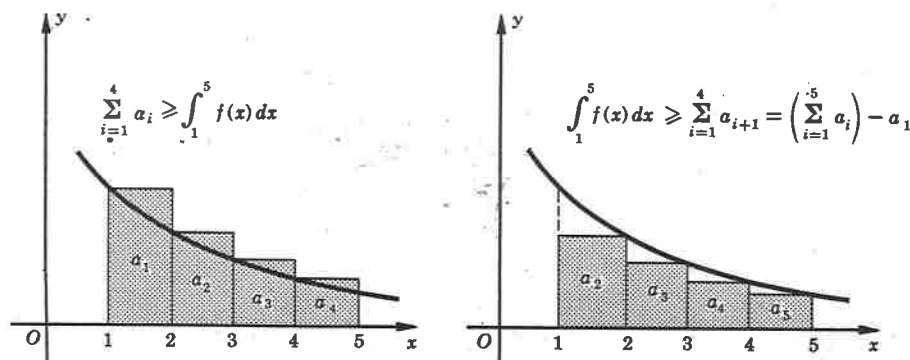


FIG. 414

then

$$S \geq \int_1^{\infty} f(x) dx \geq S - a_1. \quad \blacksquare$$

Example 4. The series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots,$$

where p is a constant, is called the p -series. Prove that

(a) the p -series converges if $p > 1$; and

(b) the p -series diverges if $p \leq 1$.

Solution. The function defined by $1/x^p$, where p is a nonnegative constant, is continuous, positive, and monotonically decreasing for $x \geq 1$. Hence, by the integral test, for $p \geq 0$, the p -series converges if and only if the improper integral

$$\int_1^{\infty} x^{-p} dx = \lim_{t \rightarrow \infty} \int_1^t x^{-p} dx \text{ exists.}$$

Now

$$\int_1^t x^{-p} dx = \frac{x^{1-p}}{1-p} \Big|_1^t = \frac{t^{1-p} - 1}{1-p} \quad \text{when } p \neq 1,$$

and

$$\int_1^t x^{-p} dx = \ln x \Big|_1^t = \ln t \quad \text{when } p = 1.$$

Since

$$\lim_{t \rightarrow \infty} t^{1-p} = 0 \quad \text{if } p > 1 \quad \text{and} \quad \lim_{t \rightarrow \infty} t^{1-p} = \infty \quad \text{if } p < 1,$$

and since

$$\lim_{t \rightarrow \infty} \ln t = \infty,$$

we have proved that the p -series converges if $p > 1$ and diverges if $0 \leq p \leq 1$.

When $p = 1$, the p -series is the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$. We have just proved that the harmonic series is divergent. (In Example 3, 21.2, we proved that the harmonic series is divergent without using the integral test.)

Now consider the p -series when $p < 0$, say $p = -q$ where q is some positive number. Then

$$\sum \frac{1}{n^p} = \sum n^q, \quad \text{where } q > 0.$$

Since $\lim_{n \rightarrow \infty} n^q = \infty$ for $q > 0$, the series $\sum n^q = \sum 1/n^p$ is divergent if $p < 0$. \blacksquare

The p -series is a useful comparison series.

$$\sum \frac{\ln n}{n^3}$$

Solution. Since $\ln n < n$ for any positive integer n , we have

$$\frac{\ln n}{n^3} < \frac{1}{n^2},$$

and the convergent p -series $\sum 1/n^2$ dominates the given series of positive terms. Hence the given series converges.

Example 6. By means of an improper integral, find an upper bound for the error in using the sum of the first five terms of the convergent series

$$\sum \frac{n}{e^{n^2}}$$

to approximate the sum of the series.

Solution. The error is

$$\sum_{n=6}^{\infty} \frac{n}{e^{n^2}}$$

and

$$\sum_{n=6}^{\infty} \frac{n}{e^{n^2}} < \int_5^{\infty} x e^{-x^2} dx = \frac{1}{2e^{25}} < \frac{1}{2(2.7)^{25}} < 0.8 \times 10^{-11}.$$

Hence the error is less than 8×10^{-12} .

21.3.6 Theorem (Ratio Test). Let $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ be a series of positive terms, and let

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \rho.$$

- (a) If $\rho < 1$, the series converges.
- (b) If $\rho > 1$, the series diverges.
- (c) If $\rho = 1$, the test is inconclusive.

Proof of (a). Let $\rho < 1$. Since $a_n > 0$ for all positive integers n , $a_{n+1}/a_n > 0$ for all n and ρ cannot be negative.

Choose a number r between ρ and 1 so that $0 \leq \rho < r < 1$ (Fig. 415);

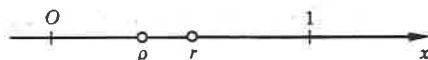


FIG. 415

such that for all $n > N$,

$$\left| \frac{a_{n+1}}{a_n} - \rho \right| < r - \rho$$

or

$$\rho - (r - \rho) < \frac{a_{n+1}}{a_n} < \rho + (r - \rho).$$

Therefore $a_{n+1} < a_n r$ for all $n > N$. Thus

$$a_{N+1} < a_N r,$$

$$a_{N+2} < a_{N+1} r < a_N r^2,$$

$$a_{N+3} < a_{N+2} r < a_N r^3,$$

...

Hence the geometric series $a_N r + a_N r^2 + a_N r^3 + \cdots$ dominates the series $a_{N+1} + a_{N+2} + a_{N+3} + \cdots$. Since $r = |r| < 1$, the geometric series converges and so the series $\sum_{n=N+1}^{\infty} a_n$ converges. Therefore the series $\sum_{n=1}^{\infty} a_n$ converges.

Proof of (b). ($\rho > 1$). As above, there is a positive integer N such that $a_{n+1}/a_n > 1$ for all $n \geq N$. That is, $a_{n+1} > a_n$ for all $n \geq N$. But $a_N > 0$, since this is a series of positive terms. Hence

$$a_n > a_N > 0$$

for all $n > N$, and $\lim_{n \rightarrow \infty} a_n$ cannot be zero. It follows that $\sum a_n$ diverges.

Proof of (c). ($\rho = 1$). For the p -series,

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots,$$

the ratio $a_{n+1}/a_n = n^p/(n+1)^p = [n/(n+1)]^p$; and

$$\lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^p = \left(\lim_{n \rightarrow \infty} \frac{n}{n+1} \right)^p = 1^p = 1.$$

But if $0 < p < 1$, the series diverges, and if $p > 1$, the series converges (Example 4, above). Hence, when $\rho = 1$, the given series $\sum a_n$ may converge or may diverge. ■

Example 7. Test for convergence the infinite series

$$\sum \frac{n^{20}}{2^n}$$

$$\frac{a_n}{a_{n+1}} = \frac{2^{n+1}}{2^n} = 2 \left(\frac{n+1}{n} \right) = 2 \left(1 + \frac{1}{n} \right)$$

and

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{1}{2} \left(\frac{n+1}{n} \right)^{20} = \frac{1}{2} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^{20} = \frac{1}{2}$$

Therefore, by the ratio test, the given series is convergent.

EXERCISES

In Exercises 1–5, a formula is given for s_n , the n th partial sum, of a series. From the relationship $s_n = s_{n-1} + a_n$ find a formula for a_n , the n th term of the series, write five terms of the series, tell whether it converges, and if so find the sum of the series. (Note: In some cases the resultant formula will not yield a_1 , the first term of the series. How can a_1 then be determined?)

$$1. s_n = \frac{n}{n+1}$$

$$2. s_n = \frac{1}{n+1}$$

$$3. s_n = \frac{n^2}{n+1}$$

$$4. s_n = \frac{n}{2n-1}$$

$$5. s_n = \frac{n^2}{n^2+1}$$

In Exercises 6–10, test the given series for convergence or divergence by means of the comparison test, or by means of theorem 21.3.4. Give the reason for your choice of test.

$$6. \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6} + \cdots$$

$$7. \frac{1}{2^2} + \frac{2}{3^2} + \frac{3}{4^2} + \frac{4}{5^2} + \frac{5}{6^2} + \cdots \quad 8. \frac{1}{2^3} + \frac{2}{3^3} + \frac{3}{4^3} + \frac{4}{5^3} + \frac{5}{6^3} + \cdots$$

$$9. \frac{2}{1 \cdot 3} + \frac{3}{2 \cdot 4} + \frac{4}{3 \cdot 5} + \frac{5}{4 \cdot 6} + \frac{6}{5 \cdot 7} + \cdots$$

$$10. \frac{2}{1 \cdot 3 \cdot 4} + \frac{3}{2 \cdot 4 \cdot 5} + \frac{4}{3 \cdot 5 \cdot 6} + \frac{5}{4 \cdot 6 \cdot 7} + \frac{6}{5 \cdot 7 \cdot 8} + \cdots$$

In Exercises 11–15, test for convergence by means of the integral test.

$$11. \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6} + \cdots$$

$$12. \frac{1}{1^2+1} + \frac{2}{2^2+1} + \frac{3}{3^2+1} + \frac{4}{4^2+1} + \frac{5}{5^2+1} + \cdots$$

$$13. \frac{1}{2 \ln 2} + \frac{1}{3 \ln 3} + \frac{1}{4 \ln 4} + \frac{1}{5 \ln 5} + \frac{1}{6 \ln 6} + \cdots$$

$$15. \frac{1}{\sqrt{1^3+1}} + \frac{1}{\sqrt{2^3+1}} + \frac{1}{\sqrt{3^3+1}} + \frac{1}{\sqrt{4^3+1}} + \frac{1}{\sqrt{5^3+1}} + \cdots$$

(Note: The integral involved in Exercise 15 cannot be evaluated directly. Can you show, however, that it is dominated by an integral which can be evaluated?)

In Exercises 16–20, estimate the error involved in taking the first five terms as the sum of the series.

$$16. \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6} + \cdots$$

$$17. \frac{1}{2^2-1} + \frac{1}{3^2-1} + \frac{1}{4^2-1} + \frac{1}{5^2-1} + \frac{1}{6^2-1} + \cdots$$

$$18. \frac{1}{e} + \frac{2}{e^2} + \frac{3}{e^3} + \frac{4}{e^4} + \frac{5}{e^5} + \cdots \quad 19. \frac{1}{e} + \frac{2^2}{e^2} + \frac{3^2}{e^3} + \frac{4^2}{e^4} + \frac{5^2}{e^5} + \cdots$$

$$20. 1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \cdots$$

In Exercises 21–25, test for convergence by means of the ratio test.

$$21. 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots \quad 22. \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \frac{5}{2^5} + \cdots$$

$$23. 1 + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \frac{1}{4^{3/2}} + \frac{1}{5^{3/2}} + \cdots$$

$$24. 3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \frac{3^5}{5!} + \cdots$$

$$25. \frac{\ln 2}{2^2} + \frac{\ln 3}{2^3} + \frac{\ln 4}{2^4} + \frac{\ln 5}{2^5} + \frac{\ln 6}{2^6} + \cdots$$

21.4 ALTERNATING SERIES. ABSOLUTE CONVERGENCE.

An infinite series of the form

$$\sum_{i=1}^{\infty} (-1)^{i+1} a_i = a_1 - a_2 + a_3 - \cdots + (-1)^{n+1} a_n + \cdots,$$

where $a_i > 0$, for $i = 1, 2, 3, \dots$, is called an *alternating series*.

21.4.1 Theorem. *The alternating series*

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - \cdots$$

converges if $0 < a_{n+1} < a_n$ for all positive integers n and if $\lim_{n \rightarrow \infty} a_n = 0$.

$$(1) \quad s_{2n} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-1} - a_{2n})$$

and

$$(2) \quad s_{2n+1} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n} - a_{2n+1}),$$

the quantities in parentheses are all positive numbers because, by hypothesis, $a_n > a_{n+1}$ for all positive integers, n . Hence the sequence $\{s_{2n}\}$ is monotonically increasing and the sequence $\{s_{2n+1}\}$ is monotonically decreasing.

The fact that all the quantities in parentheses in (1) and (2) are positive numbers also implies that $s_{2n} > 0$ and $s_{2n+1} < a_1$, for all positive integers n . Since $s_{2n+1} = s_{2n} + a_{2n+1}$, then $s_{2n} < s_{2n+1}$. Combining these results, we have

$$(3) \quad 0 < s_{2n} < s_{2n+1} < a_1$$

for all positive integers n . Therefore the monotonic sequences $\{s_{2n}\}$ and $\{s_{2n+1}\}$ are bounded below by 0 and above by a_1 , and are therefore convergent.

Since $\lim_{n \rightarrow \infty} a_{2n+1} = 0$ by hypothesis and $s_{2n+1} = s_{2n} + a_{2n+1}$,

$$\lim_{n \rightarrow \infty} s_{2n+1} = \lim_{n \rightarrow \infty} (s_{2n} + a_{2n+1}) = \lim_{n \rightarrow \infty} s_{2n} + \lim_{n \rightarrow \infty} a_{2n+1} = \lim_{n \rightarrow \infty} s_{2n}.$$

This is, both the sequences $\{s_{2n}\}$ and $\{s_{2n+1}\}$ have the same limit, which we call S .

Therefore $\lim_{m \rightarrow \infty} s_m = S$ whether m takes on odd or even values or both,

and $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges to the value S . ■

Example 1. The alternating series

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^{n+1} \frac{1}{n} + \cdots$$

converges since $a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$ for all positive integers n , and $\lim_{n \rightarrow \infty} a_n =$

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

21.4.2 Theorem. If

$$\sum (-1)^{n+1} a_n = a_1 - a_2 + a_3 - \cdots + (-1)^{n+1} a_n + \cdots$$

is an alternating series such that $a_n > a_{n+1} > 0$ and $\lim_{n \rightarrow \infty} a_n = 0$, the error made by using the sum of the first k terms as an approximation for the sum of the series is less than a_{k+1} , the absolute value of the first neglected term.

$$\sum_{i=1}^{\infty} (-1)^{i+1} a_i = a_1 - a_2 + a_3 - \cdots + (-1)^{k+1} a_k + R_k,$$

where $R_k = \sum_{i=k+1}^{\infty} (-1)^{i+1} a_i$ is the remainder of the given series after the first k terms. The error in approximating the sum of the given series by the sum of its first k terms is the sum of the series of terms neglected, that is, R_k . But $R_k = \sum_{i=k+1}^{\infty} (-1)^{i+1} a_i$ is an alternating series with the characteristics assumed in 21.4.1, so the absolute value of each of the partial sums of the series $R_k = (-1)^{k+2} a_{k+1} + (-1)^{k+3} a_{k+2} + (-1)^{k+4} a_{k+3} + \cdots$ is less than the term a_{k+1} , as shown in the inequalities (3) in the proof of 21.4.1. ■

This characteristic of convergent alternating series of the type we have been discussing makes them very valuable for computation. For example, we shall soon see that

$$\ln 1.2 = 0.2 - \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3} - \frac{(0.2)^4}{4} + \frac{(0.2)^5}{5} - \cdots$$

By 21.4.2, the error in using the sum of the first four terms of this series to approximate the value of $\ln 1.2$ is less than $a_5 = (0.2)^5/5 = 0.000064$.

21.4.3 Definition. The infinite series

$$\sum a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is said to **converge absolutely** if

$$\sum |a_n| = |a_1| + |a_2| + |a_3| + \cdots + |a_n| + \cdots$$

converges.

Example 2. The series

$$1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots + \frac{(-1)^{n+1}}{3^{n-1}} + \cdots$$

converges absolutely because the series

$$\begin{aligned} |1| + \left| -\frac{1}{3} \right| + \left| \frac{1}{9} \right| + \left| -\frac{1}{27} \right| + \cdots + \left| \frac{(-1)^{n+1}}{3^{n-1}} \right| + \cdots \\ = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^{n-1}} + \cdots \end{aligned}$$

converges since it is a geometric series with $r = \frac{1}{3} < 1$.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n} + \cdots$$

does not converge absolutely, since

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$$

is the harmonic series which we found to be divergent.

21.4.4 Definition. If a series is convergent but not absolutely convergent, it is said to be **conditionally convergent**.

Thus the alternating series $\sum (-1)^{n+1}/n$ in Example 1 is conditionally convergent.

21.4.5 Theorem. If a series converges absolutely, it converges.

Proof. Assume that the series $\sum |a_n|$ converges and let $b_n = a_n + |a_n|$. Notice that for each positive integer n , either $b_n = 0$ or $b_n = 2|a_n|$. Thus

$$0 \leq b_n \leq 2|a_n|,$$

and if we let

$$A = \sum_{i=1}^{\infty} |a_i|, \quad A_n = \sum_{i=1}^n |a_i| \quad \text{and} \quad B_n = \sum_{i=1}^n b_i,$$

then $0 \leq B_n \leq 2A_n \leq 2A$. Therefore (by 21.3.1) $\lim_{n \rightarrow \infty} B_n$ exists and $\sum_{i=1}^{\infty} b_i$ converges.

Since $\sum |a_n|$ and $\sum b_n$ both converge, then (by 21.2.8)

$$\sum_{i=1}^{\infty} (b_i - |a_i|) = \sum_{i=1}^{\infty} a_i$$

converges. ■

Of course all convergent series of nonnegative terms are absolutely convergent. Our tests for convergence of series of positive terms (21.3) may be used to determine whether a series containing an infinite number of positive terms and an infinite number of negative terms is absolutely convergent. It is often easier to establish the absolute convergence of such a series and then infer its convergence by 21.4.5, than to prove convergence directly.

Example 3. Determine whether the series

$$\sum (10 \sin \frac{1}{2} n\pi) / n^{1.1}$$

converges or diverges.

values of its terms is not monotonic since sometimes a later term is greater than an earlier term and sometimes it is less. But the absolute convergence of the series is easily established.

Since $-1 \leq \sin \frac{1}{2} n\pi \leq 1$ for all positive integers n , $|(10 \sin \frac{1}{2} n\pi) / n^{1.1}| \leq 10/n^{1.1}$. Thus the series

$$\sum |(10 \sin \frac{1}{2} n\pi) / n^{1.1}|$$

is dominated by the series $\sum 10/n^{1.1} = 10 \sum 1/n^{1.1}$. Since $\sum 1/n^{1.1}$ is a convergent p -series (with $p = 1.1 > 1$), the given series converges absolutely. Therefore the given series converges.

21.4.6 Theorem. For any series of nonzero terms, $\sum u_n$, let

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \rho.$$

- (1) If $\rho < 1$, the series converges absolutely.
- (2) If $\rho > 1$, the series diverges.

Proof. Part (1) follows immediately from 21.3.6 and the definition of absolute convergence. As to (2), we showed in the proof of part (b) of 21.3.6 that $\lim_{n \rightarrow \infty} |u_n|$ cannot be zero. Hence $\lim_{n \rightarrow \infty} u_n$ cannot be zero and the series $\sum u_n$ diverges. ■

We saw, in 21.2, that a convergent series behaves in some ways like a finite sum. In particular, we showed in 21.2.6 and 21.2.7 that convergent series obey laws resembling the associative and distributive laws for finite sums. If a series is *absolutely* convergent, the commutative law of addition also holds; that is, any rearrangement of the terms of an absolutely convergent series will result in a series which is also absolutely convergent to the same sum. We state this, without proof, in the following theorem.

21.4.7 Theorem. If a series is absolutely convergent, its terms may be rearranged without affecting the absolute convergence of the series or its sum.

It was proved in 21.2.8, that if any two series are convergent (conditionally or absolutely), they can be added term by term and the resulting series also will be convergent, with a sum equal to the sum of the sums of the original series.

If two series are *absolutely* convergent, the series formed by multiplying the terms of one series by the terms of the other series as indicated in the next theorem, will converge absolutely and its sum will be the product of the sums of the original series.

$\sum b_n = b_1 + b_2 + b_3 + \cdots$ are absolutely convergent with sums A and B , then the series

$$a_1b_1 + a_1b_2 + a_2b_1 + a_1b_3 + a_2b_2 + a_3b_1 + \cdots$$

converges absolutely and its sum is AB .

Notice that the terms of the product series consist of all possible products of the form a_ib_j taken in the order for which first $i + j = 2$, then $i + j = 3$, then $i + j = 4$, etc.

A proof of this theorem may be found in books on advanced calculus.

EXERCISES

In each of Exercises 1–10, determine whether the given series converges absolutely, converges conditionally, or diverges.

1. $\frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} - \frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6} - \cdots$

2. $\frac{1}{2 \ln 2} - \frac{1}{3 \ln 3} + \frac{1}{4 \ln 4} - \frac{1}{5 \ln 5} + \frac{1}{6 \ln 6} - \cdots$

3. $\frac{1}{2\sqrt{2}} - \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} - \frac{1}{5\sqrt{5}} + \frac{1}{6\sqrt{6}} - \cdots$

4. $\frac{1}{\sqrt{2^2-1}} - \frac{1}{\sqrt{3^2-1}} + \frac{1}{\sqrt{4^2-1}} - \frac{1}{\sqrt{5^2-1}} + \frac{1}{\sqrt{6^2-1}} - \cdots$

5. $1 - \frac{2}{2^2} + \frac{3}{2^3} - \frac{4}{2^4} + \frac{5}{2^5} - \cdots$

6. $1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \cdots$

7. $\frac{1}{1^2+1} - \frac{2}{2^2+1} + \frac{3}{3^2+1} - \frac{4}{4^2+1} + \frac{5}{5^2+1} - \cdots$

8. $\frac{1}{2!} - \frac{2}{3!} + \frac{3}{4!} - \frac{4}{5!} + \frac{5}{6!} - \cdots$

9. $\frac{1}{e} - \frac{2^2}{e^2} + \frac{2^3}{e^3} - \frac{2^4}{e^4} + \frac{2^5}{e^5} - \cdots$

10. $1 - \frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \cdots$

21.5 POWER SERIES

If $\{a_n\}$ is a sequence of constants, the expression

$$(1) \quad \sum_{i=0}^{\infty} a_i x_i = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

is called a **power series in x** .

studying. Clearly, the series (1) must converge if $x = 0$. But are there any other numbers x for which the power series converges?

Example 1. Find all numbers x for which the following power series converges:

$$\sum_{n=0}^{\infty} \frac{x^n}{(n+1)2^n} = 1 + \frac{1}{2 \cdot 2}x + \frac{1}{3 \cdot 4}x^2 + \frac{1}{4 \cdot 8}x^3 + \cdots + \frac{1}{(n+1)2^n}x^n + \cdots$$

Solution. Applying the ratio test (21.4.6), we find

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+2)2^{n+1}} \div \frac{x^n}{(n+1)2^n} \right| = \frac{|x|}{2} \lim_{n \rightarrow \infty} \left(\frac{n+1}{n+2} \right) = \frac{|x|}{2}$$

Thus the power series converges absolutely when x is a number which makes $\rho = |x|/2 < 1$, and diverges when $\rho = |x|/2 > 1$. The series is absolutely convergent when $|x| < 2$ and diverges when $|x| > 2$.

If $x = 2$ or $x = -2$, the ratio test fails. But when $x = 2$, the power series becomes the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$, which diverges; and when $x = -2$, the power series is the convergent alternating series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$.

Hence the set of numbers for which this power series converges is the half-open interval $[-2, 2)$. It is absolutely convergent for all numbers x in the open interval $(-2, 2)$ and converges conditionally for $x = -2$; it diverges for all other numbers.

The set of numbers for which a power series converges is called the **convergence set for the power series**.

The foregoing example suggests that any power series in x to which we can apply the ratio test will have for its convergence set one of the following:

- The origin alone;
- All points in an open interval $(-r, r)$ whose center is the origin, and possibly one or both endpoints.
- All real numbers.

In 21.5.2 and 21.5.3 we shall prove that this is true for *all* power series in x , even for those to which we cannot apply the ratio test.

In case (b), the series converges for $|x| < r$ and diverges for $|x| > r$, and we call r the **radius of convergence** and the open interval $(-r, r)$ the **interval of convergence** (Fig. 416).

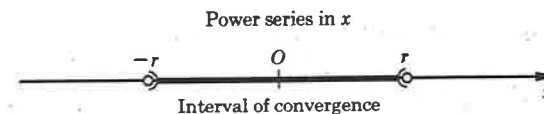


FIG. 416

vergence to be $r = 0$. In case (c), the interval of convergence is

A power series converges *absolutely* at all points in its interval of convergence. It follows from the meaning of absolute convergence that if the power series converges at both endpoints of its interval of convergence, the convergence there must be absolute; but if it converges at only one endpoint, the convergence at that endpoint must be conditional.

Example 2. Find the convergence set for the power series

$$\sum n! x^n = 1 + x + 2x^2 + 6x^3 + \cdots + n! x^n + \cdots$$

Solution. Using the ratio test, we have

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{(n+1)! x^{n+1}}{n! x^n} \right| = \lim_{n \rightarrow \infty} |(n+1)x|.$$

For $x = 0$, $\rho = 0$. But for any number $x \neq 0$, $\rho = \infty$. Thus this power series in x converges only for $x = 0$. For any other number, it diverges.

Example 3. Show that the power series

$$\sum \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots$$

converges for all real numbers x .

Solution. The limit of the absolute value of the ratio of the $(n+1)$ st term to the n th term, as $n \rightarrow \infty$, is

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \div \frac{x^n}{n!} \right| = |x| \lim_{n \rightarrow \infty} \frac{1}{n+1} = |x| \cdot 0 = 0,$$

for all numbers x . Since $\rho < 1$, the series converges (by the ratio test). Hence the given power series converges for all real numbers x .

A useful result follows from Example 3, above. Since $\sum x^n/n!$ converges for all real numbers, the limit of its n th term as $n \rightarrow \infty$ must be zero. Therefore

$$\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0,$$

for all real numbers x .

Example 4. Prove that if k is an arbitrary real number and $-1 < x < 1$, then

$$\lim_{n \rightarrow \infty} \frac{k(k-1)(k-2) \cdots (k-n)}{n!} x^n = 0.$$

$$\sum_{n=1}^{\infty} \frac{k(k-1)(k-2) \cdots (k-n)}{n!} x^n,$$

we find

$$\begin{aligned} \rho &= \lim_{n \rightarrow \infty} \left| \frac{k(k-1) \cdots (k-n)(k-n-1)x^{n+1}}{(n+1)!} \cdot \frac{n!}{k(k-1) \cdots (k-n)x^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{k-n-1}{n+1} \right| |x| = |x|. \end{aligned}$$

When $|x| < 1$, we have $\rho < 1$ and the series converges. Thus the limit of its n th term as $n \rightarrow \infty$ is zero; that is,

$$\lim_{n \rightarrow \infty} \frac{k(k-1)(k-2) \cdots (k-n)}{n!} x^n = 0,$$

if $-1 < x < 1$.

21.5.2 Theorem. If the power series $\sum a_n x^n$ converges for a number $x_1 \neq 0$, then it converges absolutely for all numbers x such that $|x| < |x_1|$.

Proof. Assume that $\sum a_n x_1^n$ converges. Then $\lim_{n \rightarrow \infty} a_n x_1^n = 0$. Thus there is a positive integer N such that $|a_n x_1^n| < 1$ for all $n \geq N$. For any number x such that $|x| < |x_1|$,

$$|a_n x^n| = |a_n x_1^n| \left| \frac{x}{x_1} \right|^n < \left| \frac{x}{x_1} \right|^n,$$

when $n \geq N$. Thus the series $\sum_{n=N}^{\infty} |a_n x^n|$ is dominated by the series

$$\sum_{n=N}^{\infty} \left| \frac{x}{x_1} \right|^n.$$

But this latter series is a convergent geometric series because $|x| < |x_1|$ and $|x/x_1| < 1$. Therefore the series $\sum_{n=N}^{\infty} |a_n x^n|$ converges and so the given power series converges absolutely for all numbers x such that $|x| < |x_1|$. ■

21.5.3 Corollary. If the power series $\sum a_n x^n$ diverges for a number x_2 , then it diverges for all numbers x such that $|x| > |x_2|$.

Proof. If the series converges for some number x such that $|x| > |x_2|$, it must converge for x_2 (by 21.5.2), which is contrary to hypothesis. Therefore the series diverges for all x such that $|x| > |x_2|$. ■

From these theorems it follows that if a power series converges at some point other than the origin, either the series converges everywhere or there is

interval $(-r, r)$ and diverges everywhere outside the closed interval $[-r, r]$. These theorems say nothing about the convergence or divergence of the power series at the endpoints of the interval of convergence.

More generally, if $\{a_n\}$ is a sequence, the expression

$$\sum a_n(x-b)^n = a_0 + a_1(x-b) + a_2(x-b)^2 + \cdots + a_n(x-b)^n + \cdots$$

is called a **power series in $(x-b)$** .

Since this power series in $(x-b)$ can be obtained from a power series in x by the translation $x = x' - b$, all that we have said about power series in x applies equally to power series in $(x-b)$. The interval of convergence is now $(b-r, b+r)$, with center at the point b (Fig. 417). If the power series

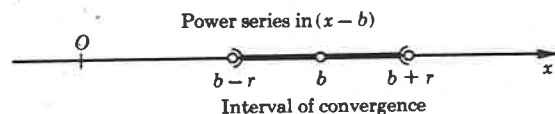


FIG. 417

in $(x-b)$ converges only for $x = b$, we say that $r = 0$. If it converges for all real numbers, we say that $r = \infty$.

Example 5. Find the interval of convergence and the convergence set of the power series in $(x-2)$:

$$\sum_{n=0}^{\infty} (-1)^n \frac{(x-2)^n}{n+1} = 1 - \frac{x-2}{2} + \frac{(x-2)^2}{3} - \cdots + \frac{(-1)^n (x-2)^n}{n+1} + \cdots$$

Solution. Again we use the ratio test.

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{(x-2)^{n+1}}{n+2} \cdot \frac{n+1}{(x-2)^n} \right| = |x-2| \lim_{n \rightarrow \infty} \left(\frac{n+1}{n+2} \right) = |x-2|.$$

$$\text{Now } \rho < 1 \Leftrightarrow |x-2| < 1 \Leftrightarrow -1 < x-2 < 1 \Leftrightarrow 1 < x < 3.$$

Hence the interval of convergence of this power series in $(x-2)$ is $(1, 3)$.

When $x = 1$, the series is $\sum 1/(n+1)$, which is the (divergent) harmonic series.

When $x = 3$, the series is $\sum (-1)^n 1/(n+1) = 1 - \frac{1}{2} + \frac{1}{3} - \cdots$, which is a convergent alternating series.

Therefore the convergence set of this power series in $(x-2)$ is $(1, 3]$.

EXERCISES

In Exercises 1–20, find the convergence set of the given power series.

1. $\frac{x}{1 \cdot 2} - \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4} - \frac{x^4}{4 \cdot 5} + \frac{x^5}{5 \cdot 6} - \cdots$

4. $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots$

5. $x + 2x^2 + 3x^3 + 4x^4 + \cdots$

6. $x + 2^2x^2 + 3^2x^3 + 4^2x^4 + \cdots$ 7. $1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \cdots$

8. $1 + x + \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} + \frac{x^4}{\sqrt{4}} + \frac{x^5}{\sqrt{5}} + \cdots$

9. $1 - \frac{x}{1 \cdot 3} + \frac{x^2}{2 \cdot 4} - \frac{x^3}{3 \cdot 5} + \frac{x^4}{4 \cdot 6} - \cdots$

10. $\frac{x}{2^2 - 1} + \frac{x^2}{3^2 - 1} + \frac{x^3}{4^2 - 1} + \frac{x^4}{5^2 - 1} + \cdots$

11. $1 - \frac{x}{2} + \frac{x^2}{2^2} - \frac{x^3}{2^3} + \frac{x^4}{2^4} - \cdots$

12. $1 + 2x + 2^2x^2 + 2^3x^3 + 2^4x^4 + \cdots$

13. $1 + 2x + \frac{2^2x^2}{2!} + \frac{2^3x^3}{3!} + \frac{2^4x^4}{4!} + \cdots$

14. $\frac{x}{2} + \frac{2x^2}{3} + \frac{3x^3}{4} + \frac{4x^4}{5} + \frac{5x^5}{6} + \cdots$

15. $\frac{(x-1)}{1} + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \frac{(x-1)^4}{4} + \cdots$

16. $1 + (x+2) + \frac{(x+2)^2}{2!} + \frac{(x+2)^3}{3!} + \frac{(x+2)^4}{4!} + \cdots$

17. $1 + \frac{(x+1)}{2} + \frac{(x+1)^2}{2^2} + \frac{(x+1)^3}{2^3} + \frac{(x+1)^4}{2^4} + \cdots$

18. $\frac{(x-2)}{1^2} + \frac{(x-2)^2}{2^2} + \frac{(x-2)^3}{3^2} + \frac{(x-2)^4}{4^2} + \cdots$

19. $\frac{(x+5)}{1 \cdot 2} + \frac{(x+5)^2}{2 \cdot 3} + \frac{(x+5)^3}{3 \cdot 4} + \frac{(x+5)^4}{4 \cdot 5} + \cdots$

20. $(x+3) - 2(x+3)^2 + 3(x+3)^3 - 4(x+3)^4 + \cdots$

21.6 FUNCTIONS DEFINED BY POWER SERIES

Since the power series

$$\sum a_n(x-b)^n = a_0 + a_1(x-b) + a_2(x-b)^2 + \cdots + a_n(x-b)^n + \cdots$$

has a unique sum at each point in its convergence set, a function f is defined by

$$f(x) = a_0 + a_1(x-b) + a_2(x-b)^2 + \cdots + a_n(x-b)^n + \cdots$$

at any point x in the domain of f .
We state, without proof, the following important theorem about this function f :

21.6.1 Theorem. Let $\sum_{n=0}^{\infty} a_n(x-b)^n$ be a power series in $(x-b)$, and let f be the function defined by

$$(1) \quad f(x) = a_0 + a_1(x-b) + a_2(x-b)^2 + \cdots + a_n(x-b)^n + \cdots,$$

where x is any number in the convergence set of the series. Then

(i) f is continuous at every point in the convergence set of the power series;

$$(ii) \quad f'(x) = \sum_{n=0}^{\infty} D_x[a_n(x-b)^n] = \sum_{n=1}^{\infty} n a_n(x-b)^{n-1} \\ = a_1 + 2a_2(x-b) + 3a_3(x-b)^2 + \cdots + n a_n(x-b)^{n-1} + \cdots$$

is valid at every point inside the interval of convergence of the original power series;

$$(iii) \quad \int_b^x f(t) dt = \sum_{n=0}^{\infty} \int_b^x a_n(t-b)^n dt = \sum_{n=0}^{\infty} \frac{a_n(x-b)^{n+1}}{n+1} \\ = a_0(x-b) + \frac{a_1(x-b)^2}{2} + \frac{a_2(x-b)^3}{3} + \cdots \\ + \frac{a_n(x-b)^{n+1}}{n+1} + \cdots$$

is valid at every point x inside the interval of convergence of the original power series (1).

From parts (ii) and (iii) of this theorem we see that a power series can be differentiated or integrated term by term and that the resulting series represent the derivative and integral, respectively, of the function defined by the original power series at every number x inside the interval of convergence of the original power series.

Example 1. We know that the sum of the convergent geometric series $1 + x + x^2 + \cdots + x^n + \cdots$, where $|x| < 1$, is $1/(1-x)$. That is,

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \quad \text{for } |x| < 1.$$

$$\int_0^x \frac{1}{1-t} dt = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^{n+1}}{n+1} + \cdots,$$

But

$$\int_0^x \frac{1}{1-t} dt = -\ln(1-t) \Big|_0^x = -\ln(1-x),$$

for $|x| < 1$. Hence

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^{n+1}}{n+1} + \cdots$$

is valid for all numbers x such that $-1 < x < 1$.

If we let $x = \frac{1}{2}$, say, we obtain

$$-\ln\left(1 - \frac{1}{2}\right) = -\ln\left(\frac{1}{2}\right) = \ln 1 - \ln\left(\frac{1}{2}\right) = \ln\left(1 \div \frac{1}{2}\right) = \ln 1.5 \\ = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2^2} + \cdots$$

The sum of the first three terms of this latter series is 0.40 to two decimal places, and the value of $\ln 1.5$ (from a table) is 0.41 to two decimal places. By using enough terms of this series we can find $\ln 1.5$ correct to any number of decimal places we like.

Consider any power series in $(x-a)$,

$$\sum a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots,$$

and denote its interval of convergence by $(a-r, a+r)$, where $0 \leq r \leq \infty$; and let f be the function defined by

$$f(x) = a_0 + a_1(x-b) + a_2(x-b)^2 + \cdots + a_n(x-b)^n + \cdots$$

for all x in $(b-r, b+r)$. By part (ii) of the preceding theorem,

$$f'(x) = a_1 + 2a_2(x-b) + 3a_3(x-b)^2 + \cdots$$

$$f''(x) = 2a_2 + 2 \cdot 3 a_3(x-b) + 3 \cdot 4 a_4(x-b)^2 + \cdots$$

$$f'''(x) = 2 \cdot 3 a_3 + 2 \cdot 3 \cdot 4 a_4(x-b) + 3 \cdot 4 \cdot 5 a_5(x-b)^2 + \cdots$$

...

$$f^{(n)}(x) = n! a_n + (n+1)! a_{n+1}(x-b) + \frac{(n+2)!}{2!} a_{n+2}(x-b)^2 + \cdots$$

...

If we let $x = b$ in these equations, we get $f(b) = a_0, f'(b) = a_1, f''(b) = 2! a_2, f'''(b) = 3! a_3, \dots, f^{(n)}(b) = n! a_n, \dots$, and if we solve these latter equations for the coefficients $a_0, a_1, a_2, \dots, a_n, \dots$ and substitute in (1), we obtain

$$f(x) = f(b) + f'(b)(x-b) + \frac{f''(b)}{2!}(x-b)^2 \\ + \cdots + \frac{f^{(n)}(b)}{n!}(x-b)^n + \cdots$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(b)(x-b)^n}{n!} = f(b) + f'(b)(x-b) + \frac{f''(b)(x-b)^2}{2!} + \dots$$

$$+ \frac{f^{(n)}(b)(x-b)^n}{n!} + \dots$$

This is called the **Taylor's series** in $(x-b)$ of the function f .

21.6.2 Theorem. Consider any power series in $(x-b)$,

$$(1) \quad a_0 + a_1(x-b) + a_2(x-b)^2 + \dots + a_n(x-b)^n + \dots,$$

and denote its interval of convergence by $(b-r, b+r)$, where $0 \leq r \leq \infty$. Denote by f the function defined by

$$(2) \quad f(x) = a_0 + a_1(x-b) + a_2(x-b)^2 + \dots + a_n(x-b)^n + \dots,$$

where x is any number in $(b-r, b+r)$. Then the power series (1) is the Taylor's series in $(x-b)$ of the function f , namely,

$$f(x) = f(b) + f'(b)(x-b) + \frac{f''(b)(x-b)^2}{2!} + \dots + \frac{f^{(n)}(b)(x-b)^n}{n!} + \dots$$

when $b-r < x < b+r$.

Thus there is only one power series in $(x-b)$ which defines a function f , and that is the Taylor's series in $(x-b)$ of the function f .

When $b=0$, the Taylor's series in x is often called a **Maclaurin's series**:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \dots + \frac{f^{(n)}(0)x^n}{n!} + \dots$$

All of the derivatives which appear in the coefficients of a Maclaurin's series are evaluated at the origin, and the interval of convergence has the origin for center.

We sometimes say that the Taylor's series in $(x-b)$ of a function f is the **Taylor's expansion of f about the point b** , and the Maclaurin's series of a function f is the **Maclaurin's expansion of f about the origin**.

EXERCISES

1. By algebraic division, find the power series representation for $1/(1+x)$. What is its interval of convergence?
2. Integrate the power series of Exercise 1 term by term to secure the power series for $\ln(1+x)$. What will the interval of convergence of this series be?
3. Add the series of Example 1 (21.6) and Exercise 2 above to secure the series for $\ln(1+x)/(1-x)$. What will the interval of convergence of this series be?

conclude that the natural logarithm of any positive number can be found by means of this series.

5. By algebraic division, find the series expansion for $2/(1-x^2)$, and secure the same result as that of Exercise 3 by integrating this series term by term.

6. By algebraic division, find the series expansion for $1/(1+x^2)$. What is the interval of convergence of this series?

7. Integrate the series of Exercise 6 term by term to secure the series expansion of $\tan^{-1}x$. What is the interval of convergence of this series?

8. Use the series of Exercise 7 to secure a famous series expression for $\frac{1}{2}\pi$. From Theorem 21.4.2, how many terms of this series would be necessary to compute π accurately to 6 decimal places?

9. Integrate the series for $\ln(1+x)$ term by term, and then show by manipulating the series for $\ln(x+1)$ that

$$\int \ln(1+x) dx = (1+x) \ln(1+x) - x.$$

10. Show that if $f(x) = P_n(x)$, an n th degree polynomial in x , then the Maclaurin series terminates after $n+1$ terms, and is the polynomial itself.

21.7 TAYLOR'S FORMULA

We stated that a function f defined by a power series $\sum a_n(x-b)^n$ (so that the sum of the power series at any number x in its convergence set is the value $f(x)$ of the function there) has derivatives of every order, and that the value of the n th derivative at b is $f^{(n)}(b) = n!a_n$.

Now consider a function F which is defined in some way other than by a power series (for example, by a closed expression, such as $F(x) = e^{x-1}$, or $F(x) = \sin x$, etc.). If such a function possesses derivatives of all orders at some particular point b , then we can certainly write down a power series in the form of a Taylor's series,

$$f(x) = F(b) + F'(b)(x-b) + \frac{F''(b)(x-b)^2}{2!} + \dots + \frac{F^{(n)}(b)(x-b)^n}{n!} + \dots$$

This series, the **Taylor's series** (or **Maclaurin's series** if $b=0$) generated by the function F , may or may not converge for numbers other than b . Even if it does converge for all points in some interval about b , say $(b-r, b+r)$, and thus defines some function f there, is the function f the same as the function F that we started with?

Certainly, from what we have seen, both f and F have equal derivatives of all orders at b , since

$$F^{(n)}(b) = n! a_n = f^{(n)}(b),$$

the above question can be yes or no, depending on the function F .

Example. Consider the function F defined by

$$F(x) = \begin{cases} e^{-1/x^2} & \text{when } x \neq 0, \\ 0 & \text{when } x = 0. \end{cases}$$

By using the definition of the derivative, one can show that $F^{(n)}(0) = 0$ for all positive integers n . Thus the Taylor's series about the origin, which is generated by F , is

$$F(0) + F'(0)x + \frac{F''(0)x^2}{2!} + \cdots + \frac{F^{(n)}(0)x^n}{n!} + \cdots = 0 + 0 + 0 + \cdots + 0 + \cdots$$

The sum of this Taylor's series is zero for all real numbers x , yet the value of the function F is different from zero for any $x \neq 0$.

Hence the Taylor's series generated by the function F is convergent for all real numbers, yet it fails to represent the function F at any point except the origin.

For each particular function, the question as to whether its Taylor's expansion (the Taylor's series generated by the function) represents the function may be answered by the use of a remarkable formula known as *Taylor's formula with remainder*.

21.7.1 Theorem. (Taylor's Theorem). Let f be a function which is defined in some open interval $(b-r, b+r)$, $0 \leq r < \infty$, and whose first $n+1$ derivatives exist and are continuous throughout the interval. Then for all numbers x in $(b-r, b+r)$,

$$f(x) = f(b) + f'(b)(x-b) + \frac{f''(b)(x-b)^2}{2!} + \cdots + \frac{f^{(n)}(b)(x-b)^n}{n!} + R_n(x),$$

$$\text{where } R_n(x) = \frac{1}{n!} \int_b^x (x-t)^n f^{(n+1)}(t) dt.$$

Proof. For any particular number x in $(b-r, b+r)$,

$$\int_b^x f'(t) dt = f(x) - f(b);$$

and this may be written

$$(1) \quad f(x) = f(b) + \int_b^x f'(t) dt.$$

Let us apply the method of integration by parts to the integral on the right

$$\begin{aligned} \int_b^x f'(t) dt &= \left[f'(t)(t-x) - \int (t-x)f''(t) dt \right]_b^x \\ &= f'(b)(x-b) - \int_b^x (t-x)f''(t) dt. \end{aligned}$$

If we substitute this result in (1), we get

$$(2) \quad f(x) = f(b) + f'(b)(x-b) + \int_b^x (x-t)f''(t) dt.$$

Let us again apply integration by parts, this time to the integral in (2). If $u = f''(t)$, $du = f'''(t) dt$, $dv = (x-t) dt$, and $v = -\frac{1}{2}(x-t)^2$, we obtain

$$\begin{aligned} \int_b^x (x-t)f''(t) dt &= \left[f''(t) \left\{ -\frac{(x-t)^2}{2} \right\} + \int \frac{(x-t)^2}{2} f'''(t) dt \right]_b^x \\ &= f''(b) \frac{(x-b)^2}{2} + \int_b^x \frac{(x-t)^2}{2} f'''(t) dt. \end{aligned}$$

By substituting this in (2), we get

$$(3) \quad f(x) = f(b) + f'(b)(x-b) + \frac{f''(b)(x-b)^2}{2!} + \int_b^x \frac{(x-t)^2}{2} f'''(t) dt.$$

By mathematical induction we can show that if we apply this process n times, we obtain

$$f(x) = f(b) + f'(b)(x-b) + \frac{f''(b)(x-b)^2}{2!} + \cdots + \frac{f^{(n)}(b)(x-b)^n}{n!} + R_n(x),$$

$$\text{where } R_n(x) = \frac{1}{n!} \int_b^x (x-t)^n f^{(n+1)}(t) dt. \quad \blacksquare$$

The expression

$$R_n(x) = \frac{1}{n!} \int_b^x (x-t)^n f^{(n+1)}(t) dt$$

is called the *integral form of the remainder in Taylor's formula*.

Since

$$f(x) = \sum_{i=0}^n \frac{f^{(i)}(b)(x-b)^i}{i!} + R_n(x),$$

then

$$f(x) - \sum_{i=0}^n \frac{f^{(i)}(b)(x-b)^i}{i!} = R_n(x).$$

terms of the Taylor's series in $(x - b)$ generated by f . Therefore, a necessary and sufficient condition for the Taylor's series in $(x - b)$, generated by a function f , to represent f at all points in some interval $(b - r, b + r)$ is that $\lim_{n \rightarrow \infty} R_n(x) = 0$ whenever $|x - b| < r$.

21.7.2 Theorem. Let f be a function having derivatives of all orders in some interval $(b - r, b + r)$. A necessary and sufficient condition that the Taylor's series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(b)(x-b)^n}{n!}$$

represent the function f at all points in the interval is that, whenever $|x - b| < r$,

$$\lim_{n \rightarrow \infty} R_n(x) = 0,$$

where $R_n(x)$ is the remainder after $(n + 1)$ terms in Taylor's formula (21.7.1).

It is often not easy in practice to apply this theorem, because of the difficulty in proving that $\lim_{n \rightarrow \infty} R_n(x) = 0$. Two other forms of $R_n(x)$ will be given in the next section. For some functions one of the three forms is the most convenient to use, while for other functions another form is preferable. We usually try to find bounds for $R_n(x)$ and then prove that the bound approaches zero as $n \rightarrow \infty$.

Example 1. Prove that the Taylor's series in x , generated by e^x , converges for all real numbers x and represents the function defined by e^x at all real numbers x .

Solution. Since the n th derivative of e^x is e^x for all real numbers x and $e^0 = 1$, the Taylor's series about 0 generated by e^x is

$$(1) \quad 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

From Taylor's formula, we have

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x),$$

where

$$R_n(x) = \frac{1}{n!} \int_0^x (x-t)^n e^t dt.$$

Choose an arbitrary number x , different from zero, and hold it fixed. Then, by

$$R_n(x) = \frac{1}{n!} \int_0^x (x-t)^n e^t dt = \frac{x}{n!} (x - \xi_n)^n e^{\xi_n}.$$

Thus

$$|R_n(x)| = \left| \frac{x}{n!} \right| \cdot |(x - \xi_n)^n| \cdot |e^{\xi_n}|.$$

But $\left| \frac{x}{n!} \right| = \frac{|x|}{n!}$, and, since $0 < |x - \xi_n| < |x|$, $|(x - \xi_n)^n| = |x - \xi_n|^n < |x|^n$.

Also, $0 < e^{\xi_n} < e^{|x|}$ and thus $|e^{\xi_n}| = e^{\xi_n} < e^{|x|}$. Hence

$$|R_n(x)| < \frac{|x|}{n!} \cdot |x|^n e^{|x|} = C \frac{|x|^{n+1}}{n!},$$

where C is a constant. Thus (using 21.5.1)

$$\lim_{n \rightarrow \infty} |R_n(x)| \leq C \lim_{n \rightarrow \infty} \frac{|x|^{n+1}}{n!} = C \cdot 0 = 0,$$

which implies that $\lim_{n \rightarrow \infty} R_n(x) = 0$ for every real number $x \neq 0$.

Therefore the Taylor's series (1) converges and represents e^x for all real numbers x .

Example 2. Compute the value of e correct to five decimal places.

Solution. Letting $x = 1$ in Example 1, above, we have

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + R_n(1).$$

We showed in Example 1 that

$$R_n(1) < \frac{e}{n!}.$$

In 12.2 we found $\ln 2 \doteq 0.7$ by the trapezoidal rule. By again using the trapezoidal rule we could find that $\ln 3 \doteq 1.1$. Since $\ln x$ was shown to be continuous for $x > 0$, and e was defined to be that number for which $\ln e = 1$, it follows from the intermediate value theorem that $2 < e < 3$. Therefore

$$R_n(1) < \frac{e}{n!} < \frac{3}{n!}.$$

We wish to take enough terms of the series so that the error in neglecting the remaining ones will be less than 0.000 005. Thus we seek n so that

$$R_n(1) < \frac{3}{n!} < 0.000\,005,$$

or

$$n! > \frac{3}{0.000\,005} = 600,000.$$

$$e \doteq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{10!}$$

In performing this computation we use the fact that $1/n!$ can be found by dividing $1/(n-1)!$ by n . The work may be conveniently arranged as follows:

$$\begin{array}{rcl} 1.000\ 000 & \leq & 1 \leq 1.000\ 000 \\ 1.000\ 000 & \leq & 1 \leq 1.000\ 000 \quad (\text{divide by } 2) \\ 0.500\ 000 & \leq & 1/2! \leq 0.500\ 000 \quad (\text{divide by } 3) \\ 0.166\ 666 & \leq & 1/3! \leq 0.166\ 667 \quad (\text{divide by } 4) \\ 0.041\ 666 & \leq & 1/4! \leq 0.041\ 667 \quad (\text{divide by } 5) \\ 0.008\ 333 & \leq & 1/5! \leq 0.008\ 334 \\ 0.001\ 388 & \leq & 1/6! \leq 0.001\ 389 \\ 0.000\ 198 & \leq & 1/7! \leq 0.000\ 199 \\ 0.000\ 024 & \leq & 1/8! \leq 0.000\ 025 \quad (\text{divide by } 9) \\ 0.000\ 002 & \leq & 1/9! \leq 0.000\ 003 \\ \hline 2.718\ 277 & \leq & \text{sum} \leq 2.718\ 284 \end{array}$$

Thus the value of e , correct to 5 decimal places, is 2.718 28.

EXERCISES

1. Derive the formula $D_x^n \ln(1+x) = (n-1)!(1+x)^{-n}(-1)^{n+1}$ and use it to find the Maclaurin expansion for $\ln(1+x)$, with the integral form of the remainder.

2. As in Exercise 1, find the Maclaurin expansion for $\ln(1-x)$.

3. If an attempt is made to secure a general formula for the n th derivative of $\tan^{-1}x$ the result is unworkably complicated. Show, however, that if the first derivative of $\tan^{-1}x$ is expanded in a power series by algebraic division, a simple formula can be found for $f^n(0)$ by differentiating this series term by term successively. Give this formula.

4. Use the result of Exercise 3 to find the Maclaurin expansion for $\tan^{-1}x$ to five nonzero terms. Do not attempt to find a form for the remainder.

5. Find the Maclaurin expansion for e^{-x} , with the integral form of the remainder.

6. Find the Maclaurin expansion for 2^x with the integral form of the remainder.

In Exercises 7–10, find the first four nonvanishing terms of the Maclaurin expansion of the given function, but do not attempt to find the remainder.

7. $\sec x$.

8. $\tan x$.

9. $\sin^{-1}x$.

10. $\sqrt{1+x}$.

In Exercises 11–15, find the first four nonvanishing terms of the power series for the given functions by performing the indicated algebraic divisions.

11. Find the series for e^{-x} by dividing 1 by the series for e^x .

14. Find the series for $\frac{1}{\sqrt{1+x}}$ by dividing 1 by the series for $\sqrt{1+x}$.

15. Find the series for $\frac{\sin x}{x}$ by dividing the series for $\sin x$ by x .

In Exercises 16–20, find the first four terms of the Taylor series for the given function about the given point. In Exercises 16–18, show the integral form of the remainder.

16. e^x , about $x = 1$.

17. $\sin x$, about $x = \frac{1}{2}\pi$.

18. $\cos x$, about $x = \frac{1}{3}\pi$.

19. $\tan x$, about $x = \frac{1}{4}\pi$.

20. $\sec x$, about $x = \frac{1}{4}\pi$.

21.8 OTHER FORMS OF THE REMAINDER IN TAYLOR'S THEOREM

In the preceding section, we proved Taylor's theorem with the remainder in the integral form. Two other frequently used forms of the remainder will now be derived. With some functions one form of the remainder is more convenient, with other functions another form is preferable.

The integral form of the remainder is

$$(1) \quad R_n(x) = \frac{1}{n!} \int_b^x (x-t)^n f^{(n+1)}(t) dt,$$

where $f^{(n+1)}$ is assumed to be continuous in $(b-r, b+r)$. Let x be an arbitrary number in $(b-r, b+r)$. By the mean value theorem for integrals (9.6.2), there is a number ξ_n between b and x such that

$$R_n(x) = \frac{(x-b)(x-\xi_n)^n}{n!} f^{(n+1)}(\xi_n).$$

This is called the *Cauchy form* of the remainder in Taylor's theorem.

21.8.1 Theorem (Cauchy's Form of the Remainder). *If $f^{(n+1)}$ is continuous in the open interval $(b-r, b+r)$, where $0 \leq r < \infty$, and if x is a number in this interval, then the Cauchy form of the remainder in Taylor's theorem (21.7.1) is*

$$R_n(x) = \frac{(x-b)(x-\xi_n)^n}{n!} f^{(n+1)}(\xi_n),$$

where ξ_n is some number between b and x .

To derive the third form of the remainder, we go back to the integral form (1). Let x be any particular number in $(b-r, b+r)$. Since $f^{(n+1)}$ is continuous in the closed interval I whose endpoints are b and x , $f^{(n+1)}$ has

tively. Thus

$$m \leq f^{(n+1)}(t) \leq M$$

for all t in I .

If $b < x$, then $(x - t)^n$ is nonnegative for all numbers t in $I = [b, x]$, and

$$m(x - t)^n \leq f^{(n+1)}(t)(x - t)^n \leq M(x - t)^n.$$

Therefore

$$m \int_b^x (x - t)^n dt \leq \int_b^x f^{(n+1)}(t)(x - t)^n dt \leq M \int_b^x (x - t)^n dt,$$

or, since $\int_b^x f^{(n+1)}(t)(x - t)^n dt = n! R_n(t)$ [by (1)],

$$m \leq \frac{n! R_n(t)}{\int_b^x (x - t)^n dt} \leq M.$$

If $x < b$, the above inequalities are reversed. In either case, let

$$(2) \quad H = \frac{n! R_n(t)}{\int_b^x (x - t)^n dt}$$

Then H is a number between the minimum and the maximum values of the continuous function $f^{(n+1)}$ on I and, by the intermediate value theorem (9.6.1), there exists a number μ_n between b and x such that $f^{(n+1)}(\mu_n) = H$. Substituting this in (2), we get

$$R_n(x) = \frac{f^{(n+1)}(\mu_n)}{n!} \int_b^x (x - t)^n dt,$$

or, since $\int_b^x (x - t)^n dt = \frac{(x - b)^{n+1}}{(n + 1)}$,

$$R_n(x) = \frac{(x - b)^{n+1}}{(n + 1)!} \cdot f^{(n+1)}(\mu_n).$$

This is called the *Lagrange form* of the remainder in Taylor's theorem.

21.8.2 Theorem. (Lagrange's Form of the Remainder). If $f^{(n+1)}$ is continuous in $(b - r, b + r)$, where $0 \leq r \leq \infty$, and x is in this interval, then the Lagrange form of the remainder in Taylor's theorem (21.7.1) is

$$R_n(x) = \frac{(x - b)^{n+1}}{(n + 1)!} \cdot f^{(n+1)}(\mu_n),$$

where μ_n is some number between b and x .

tion that $f^{(n+1)}$ exist in $(b - r, b + r)$ and not necessarily be continuous there. When $n = 0$, Taylor's theorem with the remainder in the Lagrange form is simply the theorem of mean value (7.8.2):

$$f(x) = f(b) + f'(\xi)(x - b),$$

where ξ is a number between b and x . For this reason, Taylor's theorem, with the remainder in Lagrange's form, is sometimes called the *extended theorem of mean value*.

Example 1. Find the Maclaurin's series generated by $\sin x$ and show that the series represents the function for all real numbers x .

Solution.

$$\begin{aligned} f(x) &= \sin x, & f(0) &= 0, \\ f'(x) &= \cos x, & f'(0) &= 1, \\ f''(x) &= -\sin x, & f''(0) &= 0, \\ f'''(x) &= -\cos x, & f'''(0) &= -1, \\ f^{(iv)}(x) &= \sin x, & f^{(iv)}(0) &= 0 \end{aligned}$$

Hence, by Taylor's theorem,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!} + R_n(x).$$

Using the Cauchy form of the remainder, we have

$$R_n(x) = \frac{x(x - \xi)^n}{n!} \cdot f^{(n+1)}(\xi),$$

where ξ is some number between 0 and x .

Now $f^{(n+1)}(\xi)$ is $\pm \sin \xi$ or $\pm \cos \xi$, depending on n . Thus, for all n , $|f^{(n+1)}(\xi)| \leq 1$. Hence

$$|R_n(x)| = \left| \frac{x(x - \xi)^n}{n!} \right| \cdot |f^{(n+1)}(\xi)| \leq \left| \frac{x(x - \xi)^n}{n!} \right|,$$

and

$$\lim_{n \rightarrow \infty} |R_n(x)| = |x| \lim_{n \rightarrow \infty} \frac{|(x - \xi)^n|}{n!} = 0$$

for all real numbers x (by 21.5.1). Therefore $\lim_{n \rightarrow \infty} R_n(x) = 0$, and

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!} + \cdots$$

for all real numbers x .

the function for all real numbers x .

Solution. Since

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!} + \cdots$$

for all numbers x , we can differentiate both members, term by term, and get

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^{n+1} x^{2n-2}}{(2n-2)!} + \cdots,$$

which is true for all numbers x (by 21.6.1).

Example 3. Show that the binomial theorem

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots + \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}x^n + \cdots$$

is valid for all real numbers k if $-1 < x < 1$.

Solution. By substitution we see that the formula is valid for $x = 0$. We therefore assume in what follows that $0 < |x| < 1$.

For the function f defined by $f(x) = (1+x)^k$, we have

$$\begin{aligned} f(x) &= (1+x)^k, & f(0) &= 1, \\ f'(x) &= k(1+x)^{k-1}, & f'(0) &= k, \\ f''(x) &= k(k-1)(1+x)^{k-2}, & f''(0) &= k(k-1), \\ f'''(x) &= k(k-1)(k-2)(1+x)^{k-3}, & f'''(0) &= k(k-1)(k-2), \\ &\vdots & & \end{aligned}$$

$$f^{(n)}(x) = k(k-1)(k-2)\cdots(k-n+1)(1+x)^{k-n},$$

$$f^{(n)}(0) = k(k-1)(k-2)\cdots(k-n+1),$$

$$f^{(n+1)}(x) = k(k-1)\cdots(k-n)(1+x)^{k-n-1}.$$

Thus, by Taylor's theorem,

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots + \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}x^n + R_n(x).$$

Of course, if k is a positive integer or zero, the Taylor's series has but a finite number of nonzero terms, and $R_n(x) = 0$ for some n ; this is the binomial theorem of elementary algebra. We will consider the situation where k is any real number

$$R_n(x) = \frac{k(k-1)(k-2)\cdots(k-n)(1+\xi)^{k-n-1}}{n!}x^{n+1},$$

where ξ is a number between 0 and x .

This remainder can be rewritten

$$R_n(x) = \frac{k(k-1)(k-2)\cdots(k-n)}{n!} [x(1+\xi)^{k-1}] \left(\frac{x-\xi}{1+\xi} \right)^n.$$

We will first show that for $-1 < x < 1$,

$$\left| \frac{x-\xi}{1+\xi} \right| \leq |x|.$$

If $0 < \xi < 1$,

$$\left| \frac{x-\xi}{1+\xi} \right| = \left| \frac{x(1-\xi/x)}{1+\xi} \right| < |x|.$$

If $-1 < x < \xi < 0$,

$$\frac{x-\xi}{1+\xi} = \frac{-[-x+\xi]}{1+\xi} = \frac{-[|x| - |\xi|]}{1-|\xi|} = \frac{-|x|(1-|\xi/x|)}{1-|\xi|}.$$

Since $|\xi| < |x| < 1$,

$$1 < \frac{|x|}{|\xi|} < \frac{1}{|\xi|} \quad \text{and} \quad 1 > \frac{|\xi|}{|x|} > |\xi|.$$

Therefore

$$0 > \frac{|\xi|}{|x|} - 1 > |\xi| - 1, \quad 1 - |\xi| > 1 - \frac{|\xi|}{|x|} > 0.$$

Hence

$$\left| \frac{-|x|(1-|\xi/x|)}{1-|\xi|} \right| < |x|$$

and thus

$$\left| \frac{x-\xi}{1+\xi} \right| < |x| \quad \text{if} \quad 0 < |\xi| < |x|.$$

Thus

$$\begin{aligned} (1) \quad |R_n(x)| &= |x(1+\xi)^{k-1}| \cdot \left| \frac{k(k-1)(k-2)\cdots(k-n)}{n!} \left(\frac{x-\xi}{1+\xi} \right)^n \right| \\ &\leq |x(1+\xi)^{k-1}| \cdot \left| \frac{k(k-1)(k-2)\cdots(k-n)}{n!} \right| |x|^n \end{aligned}$$

for $-1 < x < 1$. In this latter expression, $|x(1+\xi)^{k-1}|$ does not depend on n , and we have previously shown (in Example 4, Section 21.5) that

$$\lim_{n \rightarrow \infty} \left| \frac{k(k-1)(k-2)\cdots(k-n)}{n!} \right| |x|^n = 0$$

when $-1 < x < 1$. Therefore

$$\lim_{n \rightarrow \infty} R_n(x) = 0 \quad \text{if} \quad -1 < x < 1.$$

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \dots + \frac{k(k-1)(k-2)\dots(k-n+1)}{n!}x^n + \dots$$

for any real number k , if $-1 < x < 1$.

Example 4. Find the Maclaurin's series for the function f defined by $f(x) = \sin^{-1} x$, and prove that it represents the function for $-1 < x < 1$.

Solution. $D_x \sin^{-1} x = (1-x^2)^{-1/2}$. By the binomial theorem,

$$(1-x^2)^{-1/2} = 1 + \frac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 2} \frac{x^4}{2!} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 2} \frac{x^6}{3!} + \dots + \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2^n} \frac{x^{2n}}{n!} + \dots,$$

or

$$(1-x^2)^{-1/2} = 1 + \frac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 4}x^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \dots + \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot 2n}x^{2n} + \dots,$$

which is valid for $-1 < x < 1$. Integrating this, term by term, we get

$$\sin^{-1} x = x + \frac{1}{2} \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{x^7}{7} + \dots + \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot 2n} \frac{x^{2n+1}}{2n+1} + \dots,$$

which is valid for $-1 < x < 1$. The constant of integration is zero since $\sin^{-1} 0 = 0$.

Example 5. To how many decimal places of accuracy may $\sin 10^\circ$ be computed by using the first two nonvanishing terms of the Maclaurin expansion for $\sin x$?

Solution. The answer to this and to many similar computational problems is best found by using the Lagrange form of the remainder. Since the angles 10° and 0.17453 radians are approximately equal, the series in which we are interested is

$$\sin x = 0 + x + 0 \cdot \frac{x^2}{2} - \frac{1}{3!}x^3 + 0 \cdot x^4 + \frac{1}{5!}x^5 + \dots,$$

where the real number x is approximately equal to 0.17453 . In using the Lagrange form of the remainder after the first two nonvanishing terms of the series we have the option of using either a 4th degree remainder or a 5th degree remainder. Since the latter will, in general, yield a better estimate of the accuracy than the former, we choose the 5th degree form,

$$R_4(x) = \frac{\cos \xi}{5!}x^5, \quad 0 < \xi < x,$$

where $x \doteq 0.17453$.

5!

$$= \frac{1.619 \times 10^{-4}}{120}$$

$$= 1.35 \times 10^{-6}.$$

Hence, the error is less than 2×10^{-6} , and the computation of $\sin 10^\circ$ using two nonvanishing terms of the Maclaurin series for $\sin x$ is good to five decimal places.

Since the series for $\sin x$ is an alternating series, we could also have used Theorem 21.4.2 to achieve the same result.

Had we elected to use the 4th degree form of the remainder term, the remainder would have been given by

$$R_3(x) = \frac{\sin \xi}{4!}x^4, \quad 0 < \xi < x.$$

The best that could have been done in bounding $\sin \xi$ in the interval $0 < \xi < x$, without introducing computational difficulties that would have been needlessly complicated, would have been to say that $\sin 10^\circ$ was less than $\sin 30^\circ$, and therefore $\sin \xi$ was less than $\frac{1}{2}$.

Then

$$R_3(0.17453) < \frac{1}{2} \frac{(0.17453)^4}{4!}$$

or

$$R_3(0.17453) < \frac{9.279 \times 10^{-4}}{48} = 1.933 \times 10^{-4}.$$

In other words, this remainder term would only assure us that the first two nonvanishing terms of the series for $\sin x$ would yield three-decimal-place accuracy, whereas, as was seen above, the accuracy is actually good to five decimal places.

EXERCISES

1. To what accuracy may $\sin 1^\circ$ be computed by using the approximation $\sin x \doteq x$?
2. To what accuracy may $\sin 1^\circ$ be computed by using the first two nonvanishing terms of the Maclaurin expansion for $\sin x$?
3. How many nonvanishing terms of the Maclaurin expansion for $\sin x$ must be used to secure six-decimal-place accuracy in computing $\sin 28^\circ$?
4. If we expand $\sin x$ about $\frac{1}{4}\pi$, as in Exercise 17, 21.7, how many terms of this series must be used to secure six-decimal-place accuracy in the computation of $\sin 28^\circ$?
5. How many terms of the series

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

suffice to compute e accurately to four decimal places?

7. Expand e^x about $x = 1$. How many terms of this series must be used to secure six-decimal-place accuracy in the computation of $e^{1.1}$?

8. How accurate is the computation of $e^{1.1}$ if we use only the first two terms of the series of Exercise 7?

9. From the Maclaurin series for e^x , state an approximation formula for e^x when $|x|$ is small.

10. From the series of Exercise 7, state an approximation formula for e^{1+x} when $|x|$ is small.

11. Give an approximation formula for $\sin(\frac{1}{2}\pi + x)$ when $|x|$ is small.

12. From Exercise 18, 21.7, deduce an approximation formula for $\cos(\frac{1}{2}\pi + x)$ when $|x|$ is small.

13. Show that the approximation formulas of Exercises 9 to 12 can be deduced simply by using differentials.

14. From the Maclaurin expansion for $\ln(1+x)$, compute $\ln(1.1)$ to three-decimal-place accuracy.

15. For what values of x does the approximation $\ln(1+x) \doteq x$ give two-decimal-place accuracy?

16. Use the series for $\ln\left(\frac{1+x}{1-x}\right)$ to compute $\ln 2$ to three decimal places.

17. Use the series for e^x and the relation $e^{\ln 2} = 2$ to check your answer to Exercise 16.

18. Use the series for $\sin^{-1}x$ to compute $\sin^{-1}0.2$ to three decimal places. (Even though an expression for the remainder term is not given, can you tell from the series itself how many terms must be taken?)

19. Put the answer to Exercise 18 back in the series for $\sin x$ to check the work.

20. From the series for $\sin^{-1}x$, derive an approximation for $\sin^{-1}x$ good to two-decimal-place accuracy when $-\frac{1}{2} \leq x \leq \frac{1}{2}$.

21.9 COMPLEX VARIABLE

We discussed the development of the real number system in Chapter 1 but we did not go on to study the imaginary numbers. Up to now, the word "number" without a qualifier has meant real number.

21.9.1 Definition. A complex number is an expression that can be put in the form $a + bi$, where a and b are real numbers and $i^2 = -1$.

It is assumed that the reader is familiar with the application of algebraic operations to complex numbers.

If $b = 0$, the complex number $a + bi$ is *real*. If $b \neq 0$, the number is *imaginary*. If $a = 0$ and $b \neq 0$ the complex number $a + bi$ is sometimes called a *pure imaginary*.

great imagination. But custom dictates these names and we will use them.

Example 1. The numbers 3 , -17 , π , $\sqrt{2}$, $\frac{1}{4}$ and 0.0003 are real numbers. The numbers $\sqrt{-4}$, $7 - 3i$ and $1.03 + \sqrt{-0.2}$ are imaginary numbers. All the numbers in this example are complex numbers.

Complex numbers may be plotted on an *Argand diagram* (Fig. 418). The number $x + iy$, where x and y are real numbers, is located x directed units from the y -axis and y directed units from the x -axis. The x -axis in an

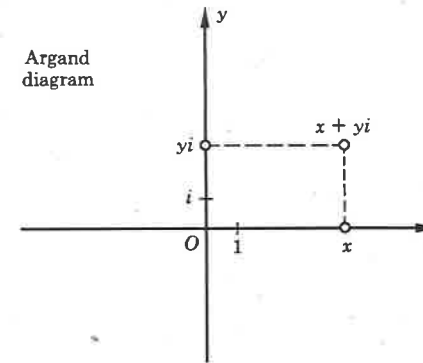


FIG. 418

Argand diagram is called the *axis of reals* since a complex number $x + iy$ is real if and only if $y = 0$. Similarly, the y -axis is called the *axis of (pure) imaginaries* because $x + iy$ is on the y -axis if and only if $x = 0$.

Example 2. In an Argand diagram (Fig. 419), the imaginary number $3 + 2i$

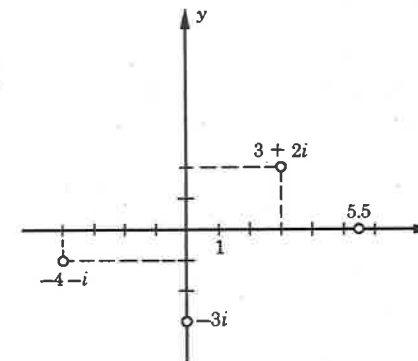


FIG. 419

19.8 Page 616

1. None.
5. Rel. min. is $-3\sqrt{3}/2$ at $(5\pi/3, 7\pi/6)$;
rel. max. is $3\sqrt{3}/2$ at $(\pi/3, 11\pi/6)$.
9. $32\sqrt{3}$.
3. Rel. min. is $-\frac{1}{8}$, at $(\pm\frac{1}{2}, 0)$.
7. $6\frac{2}{3}$, $6\frac{2}{3}$ and $6\frac{2}{3}$.
11. $\frac{2}{3}\pi$; 4 inches.

19.9 Page 619

1. (a) $\frac{100}{3}$; (b) 60.
3. 336.
5. 180.
7. $\frac{\partial P}{\partial y} = 2xy = \frac{\partial Q}{\partial x}$; 10.
9. $\frac{\partial P}{\partial y} = -1 = \frac{\partial Q}{\partial x}$; $\frac{13}{6}$.
11. $\frac{\partial P}{\partial y} = 3x^2 + 1 = \frac{\partial Q}{\partial x}$; $\frac{207}{4}$.

19.10 Page 625

1. 15795.
3. $-\pi/2$.
5. πab .
7. $f(x, y) = -k(x^2 + y^2)^{-3/2}(xi + yj)$, where k is a positive constant of proportionality.

20.2 Page 631

1. $-\frac{3}{8}$.
3. $\frac{928}{8}$.
5. $\frac{107}{21}$.
7. 240.
9. $-\ln \sqrt{2}$.
11. $(e^{20} - e^5 - 15)/10$.
13. $(2 - \pi)/8$.

20.3 Page 634

1. 10.
3. $\frac{146}{15}$.
5. 10.
7. $81\pi/8$.
9. $\ln \sqrt{\sec 1}$.
11. $(e^2 - 3e + 2e^{1/2})/3$.
13. $a^3/90$.
15. $4\pi a^3/35$.

20.4 Page 638

1. $\frac{858}{1215}$.
3. $\sqrt{3}/2 - 1 + 65\pi^4/5184$.
5. $\sqrt{3} - \frac{1}{2}$.
7. $\frac{1}{2} + \pi^2/32 - \pi^3/192$.
9. $(\bar{x}, \bar{y}) = (2, 9/5)$; $I_x = 117$, $I_y = 160$; $I_0 = 277$.
11. $(\bar{x}, \bar{y}) = (\frac{1}{2}\frac{9}{8}, \frac{1}{3})$; $I_x = \frac{5}{28}$, $I_y = \frac{2}{5}$; $I_0 = \frac{285}{4}$.
13. $(\bar{x}, \bar{y}) = (\frac{2}{3}\frac{4}{5}, \frac{1}{3}\frac{5}{2})$; $I_x = \frac{1}{3}\frac{2}{4}\frac{4}{5}$, $I_y = \frac{7}{2}\frac{6}{7}$; $I_0 = \frac{4024}{10385}$.

20.5 Page 643

1. $2\sqrt{3} + 4\pi/3$.
3. $\frac{1}{8}\pi a^2$.
5. $\frac{3}{2}\pi a^2$.
7. $\pi(b^2 - a^2)$.
9. $\pi(a^2 + \frac{1}{2}b^2)$.
13. $x = 99\sqrt{3}/(45\sqrt{3} - 5\pi)$, $y = 0$.
15. $x = 21a/20$, $y = 0$.
17. $x = 0$, $y = 0$.
19. $I_x = 8k(1251\sqrt{3} - 140\pi)/525$; $I_y = 8k(6981\sqrt{3} - 140\pi)/525$.
21. $I_x = 33k\pi a^5/40$; $I_y = 93k\pi a^5/40$.

20.6 Page 648

1. -40.
3. 1620.
5. $\frac{2}{3}$.
7. $-\pi/8$.
9. $24 - 27\pi/2$.
11. 62.
13. $3(e^5 - 5e^3 + 5e^2 - 1)/5$.
15. 8π .

20.7 Page 652

1. $32\sqrt{2}/5$.
3. $\frac{128}{15}$.
5. 12π .
7. 144.
9. 12π .
11. $x = 0$, $y = 0$, $\bar{z} = 16/15$.
13. $x = 0$, $y = 0$, $\bar{z} = \frac{8}{3}$.
15. $x = 0$, $y = 16/7$, $\bar{z} = 2$.
17. $\frac{1}{180}$.
19. $126,976k/315$.

20.8 Page 658

1. 8π .
3. $20\sqrt{5}\pi/3$.
5. On its axis, $\frac{2}{3}h$ from base.
7. $32\pi/3$.
9. $\pi kh(3s^5 - 5h^2s^3 + 2h^5)/45$, where s is the slant height.

21.1 Page 663

1. $\{1, \frac{2}{3}, \frac{3}{8}, \frac{4}{7}, \frac{5}{9}, \dots\}$. Converges. Limit = $\frac{1}{2}$.
3. $\{1, \frac{5}{8}, \frac{5}{9}, \frac{1}{11}, \frac{1}{8}, \dots\}$. Converges. Limit = 1.
5. $\left\{\frac{1}{\ln 2}, \frac{2}{\ln 3}, \frac{3}{\ln 4}, \frac{4}{\ln 5}, \frac{5}{\ln 6}, \dots\right\}$. Diverges.
7. $\left\{0, \frac{\ln 2}{4}, \frac{\ln 3}{9}, \frac{\ln 4}{16}, \frac{\ln 5}{25}, \dots\right\}$. Converges. Limit = 0.

11. $u_n = \frac{n}{n+1}$ Converges. Limit = 1. 13. $u_n = n$. Diverges.

15. $u_n = \frac{\ln n}{n}$ Converges. Limit = 0. 17. $u_n = \frac{n}{2n-1}$ Converges. Limit = $\frac{1}{2}$.

19. $u_n = \frac{2^n}{n^2}$ Diverges.

21.3 Page 676

1. $a_n = \frac{1}{n(n+1)}$; $\sum a_k = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6} + \dots$ Sum = 1.

3. $a_n = \frac{n^2 + n - 1}{n(n+1)}$; $\sum a_k = \frac{1}{1 \cdot 2} + \frac{5}{2 \cdot 3} + \frac{11}{3 \cdot 4} + \frac{19}{4 \cdot 5} + \frac{29}{5 \cdot 6} + \dots$ Diverges.

5. $a_n = \frac{2n-1}{(n^2+1)(n^2-2n+2)}$; $\sum a_k = \frac{1}{2} + \frac{3}{5 \cdot 2} + \frac{5}{10 \cdot 5} + \frac{7}{17 \cdot 10} + \frac{9}{26 \cdot 17} + \dots$ Converges. Sum = 1.

7. Divergent. $\lim_{n \rightarrow \infty} \left[\frac{n}{(n+1)^2} \right] \left[\frac{1}{n} \right] = 1$.

9. Divergent. $\lim_{n \rightarrow \infty} \left[\frac{n+1}{n(n+2)} \right] \left[\frac{1}{n} \right] = 1$.

11. Converges.

13. Diverges.

15. Converges.

17. Error $< \int_5^{\infty} \frac{dx}{x^2 + 2x} < .169$.

19. Error $< \int_5^{\infty} x^2 e^{-x} dx < .248$.

21. Converges.

23. No test. But note that this is a convergent p -series.

25. Converges.

21.4 Page 682

1. Absolutely convergent.

3. Absolutely convergent.

5. Absolutely convergent.

7. Conditionally convergent.

9. Absolutely convergent.

21.5 Page 686

1. $[-1, 1]$.

3. $(-\infty, \infty)$.

5. $(-1, 1)$.

7. $(-1, 1]$.

9. $[-1, 1]$.

11. $(-2, 2)$.

13. $(-\infty, \infty)$.

15. $[0, 2)$.

17. $(-3, 1)$.

19. $[-6, -4]$.

21.6 Page 690

1. (a) $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots$ (b) $(-1, 1)$.

7. (a) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$ (b) $(-1, 1)$.

9. $\int \ln(1+x) dx = \frac{x^2}{2} - \frac{x^3}{2 \cdot 3} + \frac{x^4}{3 \cdot 4} - \frac{x^5}{4 \cdot 5} + \dots$

21.7 Page 696

1. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n+1} \int_0^x \frac{(x-t)^n}{(1+t)^{n+1}} dt$.

3. $f^n(0) = 0$, if n is even; $f^n(0) = (-1)^{(n-1)/2}(n-1)!$, if n is odd.

5. $1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots + \frac{(-1)^{n+2}}{n!} \int_0^x (x-t)^n e^{-t} dt$.

7. $\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \dots$

9. $\sin^{-1} x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \dots$

11. $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

13. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \dots$

15. $\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots$

17. $\sin x = \frac{1}{2} + \frac{\sqrt{3}}{2} \left(x - \frac{\pi}{6} \right) - \frac{1}{2 \cdot 2!} \left(x - \frac{\pi}{6} \right)^2 - \frac{\sqrt{3}}{2 \cdot 3!} \left(x - \frac{\pi}{6} \right)^3 + \frac{1}{3!} \int_{\pi/6}^x (x-t)^3 \sin t dt$.

19. $\tan x = 1 + 2 \left(x - \frac{\pi}{4} \right) + \frac{4}{2!} \left(x - \frac{\pi}{4} \right)^2 + \frac{16}{3!} \left(x - \frac{\pi}{4} \right)^3 + \dots$

21.8 Page 703

1. $\sin 1^\circ$ can be computed to five decimal place accuracy using the approximation $\sin x = x$.

3. Four nonvanishing terms of the Maclaurin expansion of $\sin x$ yield $\sin 28^\circ$ accurate to eight decimal places.

5. Eight terms of the series for e yield three decimal place accuracy.

7. $e^x = e + e(x-1) + \frac{e}{2!}(x-1)^2 + \dots + \frac{e^{n+1}(\mu)}{(n+1)!}(x-1)^{n+1}$, $1 < \mu < x$.

Five terms of this series yield $e^{1.1}$ accurate to six decimal places.

9. $e^x \approx 1 + x$, for small x .

11. $\sin\left(\frac{\pi^2}{6} + x\right) \approx \frac{1}{2} + \frac{\sqrt{3}}{2}x$, for small $|x|$.

15. $-0.1 < x < 0.1$.

17. $e^{.693} = 2.04$.

19. $\sin .201 = .200$.

$$(d) \mathbf{r}(t) = (\sin t)\mathbf{i} + [\ln(1+t)]\mathbf{j}, \quad (e) \mathbf{r}(t) = (t-3)^{1/2}\mathbf{i} + (t-4)^{-1}\mathbf{j}; \quad (f) \mathbf{r}(t) = (e^t - 2)\mathbf{i} + e^{-t}\mathbf{j}.$$

2. For what values of t is each of the functions in Exercise 1 continuous?

3. Find $D_t \mathbf{r}(t)$ and $D_t^2 \mathbf{r}(t)$ for each of the following.

$$(a) \mathbf{r}(t) = (\ln t)\mathbf{i} - 3t^2\mathbf{j};$$

$$(b) \mathbf{r}(t) = 2(3-t^2)^{-1}\mathbf{i} + (\tan^{-1} t)\mathbf{j}.$$

$$(c) \mathbf{r}(t) = (\sin t)\mathbf{i} + (\cos 2t)\mathbf{j}; \quad (d) \mathbf{r}(t) = (\ln t)\mathbf{i} + e^{2t}\mathbf{j}.$$

$$(e) \mathbf{r}(t) = (\tan t)\mathbf{i} - t^4\mathbf{j}; \quad (f) \mathbf{r}(t) = (e^t - e^{-t})\mathbf{j}.$$

In each of Exercises 4–12 the position of a moving particle at time t is given by $\mathbf{r}(t)$. Find the velocity and acceleration vectors, $\mathbf{v}(t)$ and $\mathbf{a}(t)$, and their values at the given time $t = t_1$, and the speed of the particle then. Sketch a portion of the graph of $\mathbf{r}(t)$ containing the position P of the particle when $t = t_1$, and draw $\mathbf{v}(t_1)$ and $\mathbf{a}(t_1)$ with their initial points at P .

$$4. \mathbf{r}(t) = (3t^2 - 1)\mathbf{i} + t\mathbf{j}; t_1 = \frac{1}{2}. \quad 5. \mathbf{r}(t) = e^{-t}\mathbf{i} + e^t\mathbf{j}; t_1 = 1.$$

$$6. \mathbf{r}(t) = (\tan t)\mathbf{i} + (\sin t)\mathbf{j}; t_1 = \frac{1}{8}\pi.$$

$$7. \mathbf{r}(t) = 2(\cos t)\mathbf{i} - 3(\sin^2 t)\mathbf{j}; t = \frac{1}{3}\pi.$$

$$8. \mathbf{r}(t) = a(\sin t)\mathbf{i} + b(\cos t)\mathbf{j}; t = \frac{1}{4}\pi.$$

$$9. \mathbf{r}(t) = 3t^2\mathbf{i} + t^3\mathbf{j}; t = 2.$$

$$10. \mathbf{r}(t) = (a \sinh bt)\mathbf{i} + (a \cosh bt)\mathbf{j}; t = 0.$$

$$11. \mathbf{r}(t) = 4(1 - \sin t)\mathbf{i} + 4(t - \cos t)\mathbf{j}; t = \frac{2}{3}\pi.$$

$$12. \mathbf{r}(t) = \frac{3}{t}\mathbf{i} - \frac{t}{3}\mathbf{j}; t = 2.$$

In Exercises 13–18, find the length of the graph of $\mathbf{r}(t)$ from the point on the curve corresponding to $t = t_1$ to the point on the curve corresponding to $t = t_2$.

$$13. \mathbf{r}(t) = 2t^2\mathbf{i} + (3t + 1)\mathbf{j}; t_1 = -1, t_2 = 1.$$

$$14. \mathbf{r}(t) = 2 \cos t\mathbf{i} + (2 \sin t + 5)\mathbf{j}; t_1 = \frac{1}{8}\pi, t_2 = \pi.$$

$$15. \mathbf{r}(t) = e^t \sin t\mathbf{i} + e^t \cos t\mathbf{j}; t_1 = 2, t_2 = 6.$$

$$16. \mathbf{r}(t) = \sin t\mathbf{i} + \sin^2 t\mathbf{j}; t_1 = 0, t_2 = \frac{1}{2}\pi.$$

$$17. \mathbf{r}(t) = 4t^{3/2}\mathbf{i} + 3t\mathbf{j}; t_1 = 0, t_2 = 2.$$

$$18. \mathbf{r}(t) = (5 - \cos 2t)\mathbf{i} + 2 \sin t\mathbf{j}; t_1 = 0, t_2 = \frac{1}{2}\pi.$$

19. If $\mathbf{r}(t)$ gives the position at time t of a moving particle of mass m , the vector $\mathbf{F}(t) = m\mathbf{a}(t)$ is the force acting on the particle at time t . Show that if a particle is moving on a circle with constant speed, then the force acting on the particle is directed toward the center. (Hint: Parametric equations of a circle with center at the origin and radius r are $x = r \cos \theta$, $y = r \sin \theta$. Since the speed of the particle is constant, $D_t \theta$ is constant.)

20. The position of a moving particle at time t is given by $\mathbf{r}(t) = (t^2 - 6t)\mathbf{i} + 5t\mathbf{j}$. Find when the speed of the particle is a minimum.

Improper Integrals. Indeterminate Forms

16.1 INFINITE LIMITS OF INTEGRATION

In the definition of the definite integral $\int_a^b f(x) dx$, the interval $[a, b]$ was assumed to be finite. Is it possible to extend the definition of the definite integral so that the interval of integration is infinite, and if so, how is the value of such an “integral” to be computed?

Consider the definite integral

$$(1) \quad \int_1^b \frac{dx}{x^2}, \quad \text{where } b > 1.$$

We have at once

$$\int_1^b \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^b = 1 - \frac{1}{b}.$$

Clearly, for each value of b greater than 1, the definite integral (1) exists, and we can make the value of (1) as close as we please to 1 by taking b sufficiently large. This is expressed by writing

$$\int_1^\infty \frac{dx}{x^2} = 1.$$

16.1.1 Definition. If f is continuous on the infinite interval $[a, \infty)$, then

$$\int_a^\infty f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx,$$

provided this limit exists. If this limit does exist, the integral is said to be **convergent**; otherwise it is **divergent**.

Definite integrals with infinite limits of integration are included in what are called **improper integrals**.

Example 1. Determine whether $\int_5^\infty \frac{dx}{x}$ diverges.

Solution. This improper integral diverges because (by 16.1.1)

$$\int_5^\infty \frac{dx}{x} = \lim_{t \rightarrow \infty} \int_5^t \frac{dx}{x} = \lim_{t \rightarrow \infty} [\ln x]_5^t = \lim_{t \rightarrow \infty} (\ln t - \ln 5) = \lim_{t \rightarrow \infty} \ln t - \ln 5 = \infty.$$

Example 2. If possible, evaluate the improper integral $\int_0^\infty xe^{-x^2} dx$.

Solution.

$$\begin{aligned} \int_0^\infty xe^{-x^2} dx &= \lim_{t \rightarrow \infty} \int_0^t xe^{-x^2} dx = \lim_{t \rightarrow \infty} -\frac{1}{2} \int_0^t e^{-x^2} (-2x) dx \\ &= \lim_{t \rightarrow \infty} -\frac{1}{2} (e^{-t^2} - 1) = \frac{1}{2}. \end{aligned}$$

Similar definitions apply when the lower limit of integration is infinite and when both limits of integration are infinite.

16.1.2 Definition. If f is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x) dx = \lim_{t \rightarrow -\infty} \int_t^b f(x) dx,$$

provided this limit exists. If this limit exists, the improper integral is **convergent**; otherwise it is **divergent**.

16.1.3 Definition. If f is everywhere continuous, then

$$\int_{-\infty}^\infty f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^\infty f(x) dx,$$

where c is an arbitrarily chosen number, provided both of the improper integrals in the right-hand member are convergent.

Example 3. If the improper integral

$$\int_{-\infty}^\infty \frac{dx}{(x^2 + 1)^2}$$

is convergent, evaluate it.

Solution. By means of the trigonometric substitution $x = \tan \theta$, we find

$$\int \frac{dx}{(x^2 + 1)^2} = \frac{x}{2(x^2 + 1)} + \frac{1}{2} \tan^{-1} x.$$

$$\begin{aligned} \int_{-\infty}^\infty \frac{dx}{(x^2 + 1)^2} &= \int_{-\infty}^0 \frac{dx}{(x^2 + 1)^2} + \int_0^\infty \frac{dx}{(x^2 + 1)^2} \\ &= \lim_{t \rightarrow -\infty} \int_t^0 \frac{dx}{(x^2 + 1)^2} + \lim_{s \rightarrow \infty} \int_0^s \frac{dx}{(x^2 + 1)^2} \\ &= \lim_{t \rightarrow -\infty} \left[\frac{x}{2(x^2 + 1)} + \frac{1}{2} \tan^{-1} x \right]_t^0 + \lim_{s \rightarrow \infty} \left[\frac{x}{2(x^2 + 1)} + \frac{1}{2} \tan^{-1} x \right]_0^s \\ &= \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}. \end{aligned}$$

It is instructive to interpret an improper integral as an area. As an illustration, consider two integrals discussed above:

$$\int_1^\infty \frac{dx}{x} \quad \text{and} \quad \int_1^\infty \frac{dx}{x^2}$$

The graphs of their integrands are shown in Fig. 312. Both graphs have the positive x -axis as an asymptote; yet the shaded area in (a) becomes greater than any preassigned number as it extends indefinitely far to the right, while the shaded area in (b) is always less than 1, no matter how far to the right it goes. Neither graph ever intersects the x -axis, but as x increases the graph of the second approaches the x -axis more rapidly than the graph of the first. The student should show that

$$\int_a^\infty \frac{dx}{x^n}, \quad (a > 0),$$

converges for all numbers $n > 1$ and diverges for $n \leq 1$.

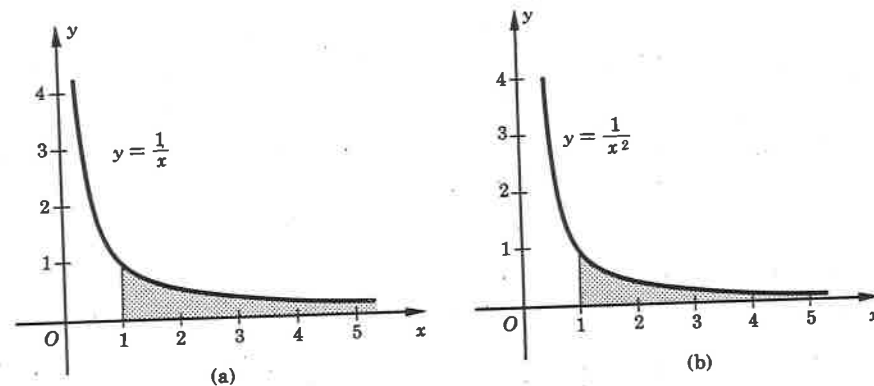


FIG. 312

The area A of the region between the curve $y = f(x)$ and the x -axis, and to the right of $x = a$ is

$$A = \int_a^{\infty} |f(x)| dx,$$

if this improper integral converges.

Similar interpretations hold for other convergent integrals with infinite limits of integration.

Example 4. Find the area of the region under the curve $y = 1/(x^2 + 1)$ and above the x -axis.

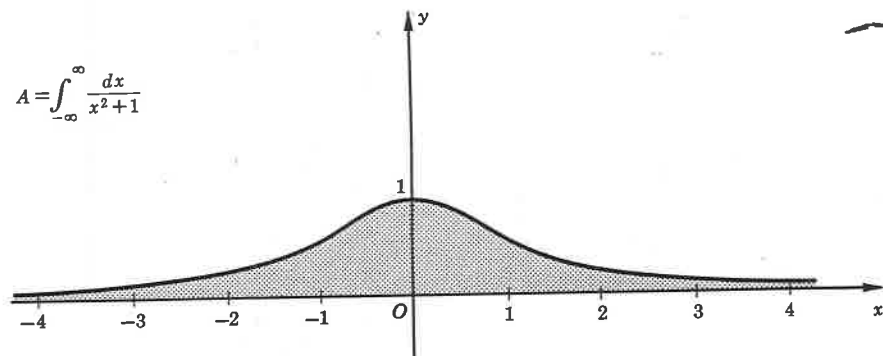


FIG. 313

Solution. The region whose area is wanted extends to left and right indefinitely (Fig. 313). Thus the area A is given by

$$A = \int_{-\infty}^{\infty} \frac{dx}{x^2 + 1}.$$

We evaluate this improper integral as follows:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dx}{x^2 + 1} &= \int_{-\infty}^0 \frac{dx}{x^2 + 1} + \int_0^{\infty} \frac{dx}{x^2 + 1} \\ &= \lim_{t \rightarrow -\infty} \int_t^0 \frac{dx}{x^2 + 1} + \lim_{t \rightarrow \infty} \int_0^t \frac{dx}{x^2 + 1} \\ &= \lim_{t \rightarrow -\infty} (-\tan^{-1} t) + \lim_{t \rightarrow \infty} \tan^{-1} t \\ &= \frac{\pi}{2} + \frac{\pi}{2} = \pi. \end{aligned}$$

Hence $A = \pi$ square units.

In each of Exercises 1–20, evaluate the given improper integral or show that it is divergent.

✓ 1. $\int_2^{\infty} \frac{dx}{e^x}$

✓ 3. $\int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}}$

✓ 5. $\int_2^{\infty} \frac{dx}{\sqrt{x}}$

✓ 7. $\int_{-\infty}^0 \frac{dx}{1 + x^2}$

✓ 9. $\int_2^{\infty} \frac{dx}{(1 - x)^{2/3}}$

✓ 11. $\int_{-\infty}^{\infty} \frac{x dx}{(x^2 + 4)^2}$

13. $\int_{-\infty}^0 e^{3x} dx$

✓ 15. $\int_{-\infty}^{-1} \frac{dx}{x^4}$

✓ 17. $\int_2^{\infty} e^{-x} \sin x dx$

✓ 19. $\int_1^{\infty} \frac{\ln x}{x} dx$

2. $\int_1^{\infty} \frac{dx}{x^{5/4}}$

4. $\int_3^{\infty} \frac{dx}{x^2}$

6. $\int_1^{\infty} e^x dx$

8. $\int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 4}$

10. $\int_3^{\infty} \frac{x dx}{\sqrt{9 + x^2}}$

12. $\int_0^{\infty} x e^{-x^2} dx$

14. $\int_3^{\infty} \frac{dx}{x(\ln x)^2}$

16. $\int_1^{\infty} \frac{x dx}{(2 + x^2)^{3/2}}$

18. $\int_{-\infty}^{\infty} \frac{dx}{2x^2 + 2x + 1}$

20. $\int_1^{\infty} \frac{x^2 dx}{(2 + x^2)^{3/2}}$

21. Show that if $a > 0$, then $\int_a^{\infty} \frac{dx}{x^n}$ converges for all numbers $n > 1$ and diverges for $n \leq 1$.

22. Find the area of the region to the right of the line $x = 3$ and between the curve $y = \frac{8}{4x^2 - 1}$ and the x -axis. Make a sketch.

23. Find the area of the region in the first quadrant and below the curve $y = e^{-x}$.

24. Extend the definition of volume of a solid of revolution to find the volume of the solid generated by revolving about the x -axis the region to the right of the line $x = 1$ and between the curve $y = x^{-3/2}$ and the x -axis. Make a sketch.

16.2 INFINITE INTEGRANDS

Another type of improper integral has finite limits of integration but an integrand which becomes infinite at one or more points in the interval of integration.

$$\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx,$$

provided this limit exists.

Example 1. Evaluate, if possible, the improper integral

$$\int_0^2 \frac{dx}{\sqrt{4-x^2}}.$$

Solution. The integrand is continuous on $[0, 2)$ but $\lim_{x \rightarrow 2^-} [1/\sqrt{4-x^2}] = \infty$.

By 16.2.1,

$$\begin{aligned} \int_0^2 \frac{dx}{\sqrt{4-x^2}} &= \lim_{t \rightarrow 2^-} \int_0^t \frac{dx}{\sqrt{4-x^2}} = \lim_{t \rightarrow 2^-} \left[\sin^{-1} \frac{x}{2} \right]_0^t \\ &= \lim_{t \rightarrow 2^-} \left(\sin^{-1} \frac{t}{2} - \sin^{-1} 0 \right) = \frac{\pi}{2}. \end{aligned}$$

This improper integral may be interpreted as the area of the region bounded by the curve $y = 1/\sqrt{4-x^2}$, the coordinate axes, and the line $x = 2$ (Fig. 314).

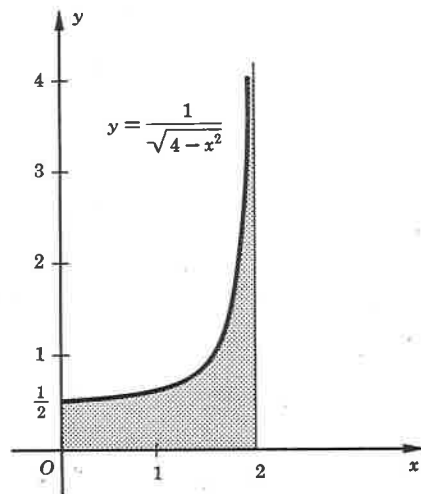


FIG. 314

A similar definition applies when the integrand becomes infinite at the lower limit of integration.

then

$$\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx,$$

provided this limit exists.

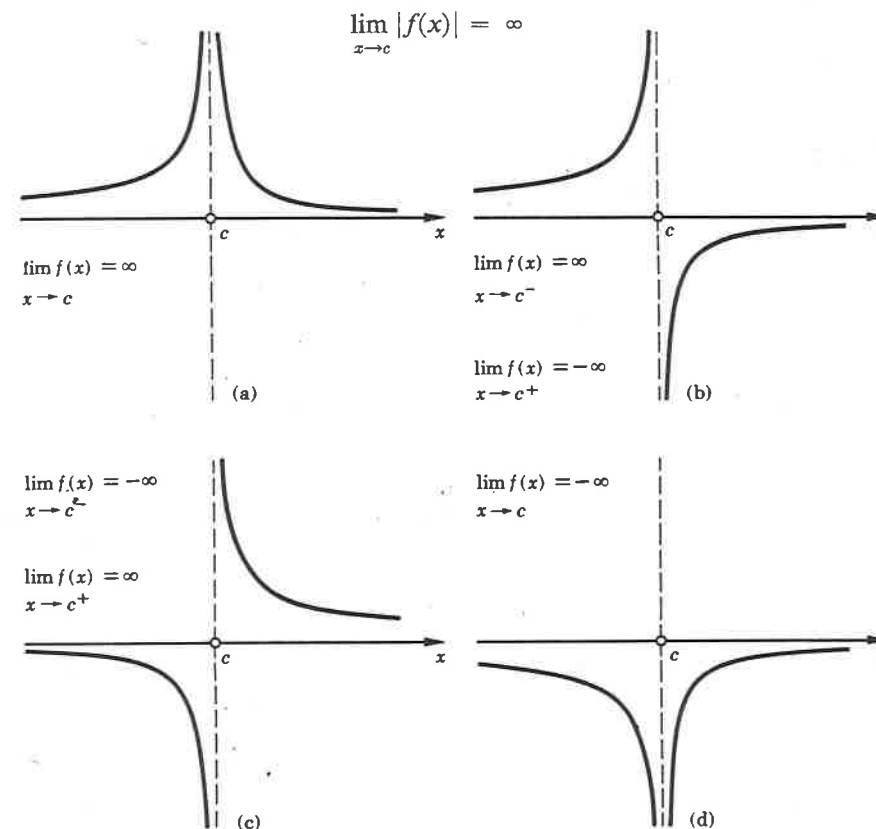


FIG. 315

If the integrand, $f(x)$, is continuous throughout its interval of integration except at an interior point c where $\lim_{x \rightarrow c} |f(x)| = \infty$, the following definition applies. Notice that the symbol $\lim_{x \rightarrow c} |f(x)| = \infty$ includes the four cases shown in Fig. 315, (a), (b), (c) and (d).

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx,$$

provided both of the integrals in the right member exist.

This definition (16.2.3) is readily modified to apply to infinite intervals of integration by replacing a by $-\infty$, b by ∞ , or both (see Exercise 16).

Example 2. Evaluate, if possible, the improper integral

$$\int_0^3 \frac{dx}{(x-1)^{2/3}}.$$

Solution. The integrand is continuous for all values of x in $[0, 3]$ except $x = 1$ where $\lim_{x \rightarrow 1} [1/(x-1)^{2/3}] = \infty$. Applying 16.2.3, we have

$$\begin{aligned} \int_0^3 \frac{dx}{(x-1)^{2/3}} &= \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}} \\ &= \lim_{t \rightarrow 1^-} \int_0^t \frac{dx}{(x-1)^{2/3}} + \lim_{s \rightarrow 1^+} \int_s^3 \frac{dx}{(x-1)^{2/3}} \\ &= \lim_{t \rightarrow 1^-} \left[3(x-1)^{1/3} \right]_0^t + \lim_{s \rightarrow 1^+} \left[3(x-1)^{1/3} \right]_s^3 \\ &= 3 \lim_{t \rightarrow 1^-} [(t-1)^{1/3} + 1] + 3 \lim_{s \rightarrow 1^+} [2^{1/3} - (s-1)^{1/3}] \\ &= 3 + 3(2^{1/3}) = 3(1 + \sqrt[3]{2}) \doteq 6.78. \end{aligned}$$

This result may be interpreted as the area of the shaded region shown in Fig. 316.

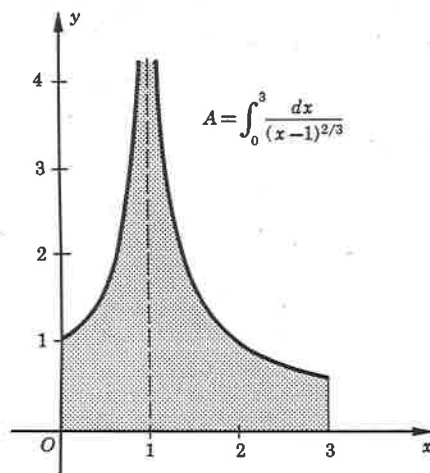


FIG. 316

$$\int_0^4 \frac{dx}{(x-2)^2}$$

Solution. If we failed to notice that the integrand is discontinuous at $x = 2$, we might be tempted to say that since an antiderivative of the integrand is $-1/(x-2)$, the value of this integral is

$$\left. \frac{-1}{x-2} \right|_0^4 = -\frac{1}{2} - \frac{1}{2} = -1.$$

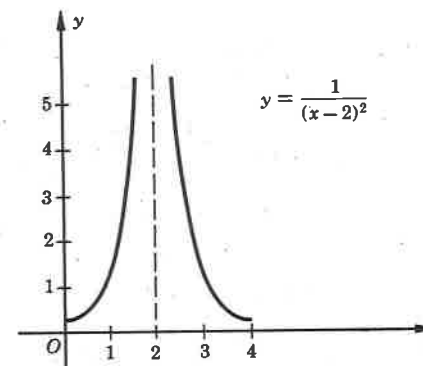


FIG. 317

Yet a glance at Fig. 317 tells us that it is impossible for this integral to have a negative value since the curve is everywhere above the x -axis.

But if we proceed correctly, using 16.2.3, we find that

$$\begin{aligned} \int_0^4 \frac{dx}{(x-2)^2} &= \int_0^2 \frac{dx}{(x-2)^2} + \int_2^4 \frac{dx}{(x-2)^2} \\ &= \lim_{t \rightarrow 2^-} \left[\frac{-1}{x-2} \right]_0^t + \lim_{s \rightarrow 2^+} \left[\frac{-1}{x-2} \right]_s^4 \\ &= \lim_{t \rightarrow 2^-} \frac{-1}{t-2} - \frac{1}{2} - \frac{1}{2} - \lim_{s \rightarrow 2^+} \frac{-1}{s-2} \\ &= \infty - \frac{1}{2} - \frac{1}{2} + \infty. \end{aligned}$$

Therefore the given integral diverges and has no value.

In Exercises 1–11, evaluate the given improper integral or show that it is divergent.

✓✓ 1. $\int_1^4 \frac{dx}{\sqrt{x-1}}$

✓✓ 3. $\int_{1/2}^2 \frac{dx}{x(\ln x)^{1/5}}$

✓ 5. $\int_3^5 \frac{dx}{(4-x)^{2/3}}$

✓ 7. $\int_{-2}^0 \frac{dx}{2x+3}$

✓ 9. $\int_0^2 \frac{x dx}{(x^2-1)^{2/3}}$

11. $\int_0^2 \frac{dx}{\sqrt{2x-x^2}}$

2. $\int_0^2 \frac{3 dx}{x^2+x-2}$

4. $\int_1^2 \frac{dx}{(x-1)^{1/3}}$

6. $\int_0^3 \frac{x dx}{\sqrt{9-x^2}}$

8. $\int_{-3}^3 \frac{dx}{\sqrt{9-x^2}}$

10. $\int_{-2}^{-1} \frac{dx}{(x+1)^{4/3}}$

12. Find the area of the region bounded by the curve $y = (x-8)^{-2/3}$, the x -axis, and the lines $x = 0$ and $x = 8$. Make a sketch.

13. Find the area of the region in the first quadrant which is under the curve $y = 1/(2x-6)^{1/4}$ and between the lines $x = 3$ and $x = 5$. Make a sketch.

14. Find the area of the region between the curves $y = 1/x$ and $y = 1/(x^3+x)$, from $x = 0$ to $x = 1$. Make a sketch.

15. Show that $\int_0^1 \frac{dx}{x^n}$ converges for all numbers $n < 1$ and diverges for $n \geq 1$.

16. Let f be a function that is continuous on the infinite interval $[a, \infty)$ except at one interior point c , $a < c$, and let $\lim_{x \rightarrow c} |f(x)| = \infty$. Combine the ideas in 16.1.1

and 16.2.3 to formulate a definition of $\int_a^\infty f(x) dx$.

16.3 EXTENDED MEAN VALUE THEOREM

The mean value theorem (7.8.2), which dealt with one function f , was extended to two functions f and g by the French mathematician A. L. Cauchy (1789–1857). It is one of those basic theorems in calculus which enable us to prove other, more immediately “practical,” theorems.

16.3.1 Cauchy’s Mean Value Theorem. *If f and g are functions which are continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , and if $g'(x) \neq 0$ for all x in (a, b) , then there exists a number z between a and b such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(z)}{g'(z)}$$

$$H(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x).$$

Then

$$H(a) = f(b)g(a) - f(a)g(b) = H(b).$$

Since f and g are continuous on $[a, b]$, H is continuous on $[a, b]$. Moreover,

$$(1) \quad H'(x) = [f(b) - f(a)]g'(x) - [g(b) - g(a)]f'(x)$$

exists on (a, b) because $f'(x)$ and $g'(x)$ exist there.

Thus the function H satisfies the conditions for Rolle’s theorem (7.8.1), and so there exists a number z between a and b such that $H'(z) = 0$. By substituting this in (1), we obtain

$$(2) \quad [f(b) - f(a)]g'(z) = [g(b) - g(a)]f'(z).$$

Now $g'(x) \neq 0$ for all x in (a, b) by hypothesis, and thus $g(a) \neq g(b)$ since otherwise Rolle’s theorem would insure the existence of a number z in (a, b) such that $g'(z) = 0$. Therefore (2) can be rewritten

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(z)}{g'(z)},$$

where z is some number such that $a < z < b$. ■

Notice that if we let $g(x) = x$, Cauchy’s extended mean value theorem becomes our former mean value theorem (7.8.2). Thus 7.8.2 is a special case of the present 16.3.1.

16.4 INDETERMINATE FORMS

The function F , defined by

$$(1) \quad F(x) = \frac{x^2 - 3x + 2}{x^2 + 3x - 10},$$

is defined for all numbers x except $x = 2$ and $x = -5$ where the denominator is zero. But at $x = 2$ the numerator also is zero, and $F(x)$ is said to have the *indeterminate form* $0/0$ at $x = 2$. We cannot “find the value” of this indeterminate form at $x = 2$ because F has no value there.

But $F(x)$ may have a limit as $x \rightarrow 2$. Some functions which have the indeterminate form $0/0$ at a particular number do have a limit there while others do not. The function F , defined in (1), has a limit at 2 which is easy to find. Thus

$$\lim_{x \rightarrow 2} F(x) = \lim_{x \rightarrow 2} \frac{x^2 - 3x + 2}{x^2 + 3x - 10} = \lim_{x \rightarrow 2} \frac{(x-2)(x-1)}{(x-2)(x+5)}.$$

Now

$$F(x) = \frac{(x-2)(x-1)}{(x-2)(x+5)} = \frac{x-1}{x+5} \quad \text{if } x \neq 2 \text{ and } x \neq -5;$$

$x \rightarrow 2$

$$\lim_{x \rightarrow 2} F(x) = \lim_{x \rightarrow 2} \frac{(x-2)(x-1)}{(x-2)(x+5)} = \lim_{x \rightarrow 2} \frac{x-1}{x+5} = \frac{1}{7}.$$

In the last step we used the limit of a quotient theorem (4.3.1). Recall that this theorem requires that the limit of the denominator be a number different from zero and thus cannot be used to find the limit of (1) directly.

On the other hand the function G defined by

$$(2) \quad G(x) = \frac{x^2 + 3x - 10}{x^2 - 4x + 4},$$

which also has the indeterminate form $0/0$ at 2 , fails to have a limit as $x \rightarrow 2$. For

$$\begin{aligned} \lim_{x \rightarrow 2} G(x) &= \lim_{x \rightarrow 2} \frac{x^2 + 3x - 10}{x^2 - 4x + 4} = \lim_{x \rightarrow 2} \frac{x+5}{x-2} \\ &= \lim_{x \rightarrow 2} \left(1 + \frac{7}{x-2} \right) = 1 + \lim_{x \rightarrow 2} \frac{7}{x-2} = \infty. \end{aligned}$$

Finding the limit of an indeterminate form is really not new to us. In 12.8 we showed that

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0.$$

Moreover, every time we established a formula for differentiation we found the limit of $\Delta y / \Delta x$ as $\Delta x \rightarrow 0$ and a necessary condition for the limit to exist is that Δy approach zero as $\Delta x \rightarrow 0$.

16.4.1 Definition. If f and g are functions such that $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$, the function defined by $f(x)/g(x)$ is said to have the **indeterminate form $0/0$** at a .

There is another type of indeterminate form which is symbolized by ∞/∞ . As an illustration, $\lim_{x \rightarrow 0} (-\ln |x|) = \infty$ and $\lim_{x \rightarrow 0} \cot |x| = \infty$; for this reason the function G defined by

$$G(x) = \frac{-\ln |x|}{\cot |x|}$$

is said to have the indeterminate form ∞/∞ at $x = 0$. Although $G(x)$ does not exist at $x = 0$, it so happens that

$$\lim_{x \rightarrow 0} G(x) = \lim_{x \rightarrow 0} \frac{-\ln |x|}{\cot |x|} = 0,$$

but we cannot find this by the methods already familiar to us.

$\lim_{x \rightarrow a} g(x) = \pm \infty$, then the function defined by $f(x)/g(x)$ is said to have the **indeterminate form $\pm \infty/\infty$** at a .

The rest of this chapter will be devoted to a powerful method, called *l'Hôpital's rule*, for finding the limits of functions at points where they have an indeterminate form.

16.5 L'HÔPITAL'S RULES

The French mathematician G. F. A. de l'Hôpital (1661–1704) wrote the first calculus textbook. In it he published a method for finding the limit, if any, of a quotient of functions when both the numerator and the denominator approach zero. It came to him from his teacher, Johann Bernoulli.

16.5.1 Theorem (l'Hôpital's Rule). Let f and g be functions which are differentiable in an open interval I containing the point a , except possibly at a itself; and let $g'(x) \neq 0$ for all $x \neq a$ in I . If $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$, and if

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L,$$

then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = L.$$

Proof. We did not assume that the functions f and g were defined at a , so we now define new functions F and G as follows:

$$(1) \quad \begin{aligned} F(x) &= f(x) \quad \text{for } x \neq a, \quad \text{and} \quad F(a) = 0; \\ G(x) &= g(x) \quad \text{for } x \neq a, \quad \text{and} \quad G(a) = 0. \end{aligned}$$

Then F and G satisfy the hypotheses of Cauchy's mean value theorem (16.3.1), and if we let $b = x$, where x is a point of I different from a , the formula in 16.3.1 becomes

$$\frac{F(x) - F(a)}{G(x) - G(a)} = \frac{F'(z)}{G'(z)}$$

or, using (1),

$$(2) \quad \frac{f(x)}{g(x)} = \frac{f'(z)}{g'(z)},$$

where z is some number between a and x . Therefore

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(z)}{g'(z)}.$$

and x , it follows that $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$.

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

Example 1. Find

$$\lim_{x \rightarrow 0} \frac{\tan 2x}{\ln(1+x)}$$

Solution. $\lim_{x \rightarrow 0} \tan 2x = \lim_{x \rightarrow 0} \ln(1+x) = 0$. If we apply l'Hôpital's rule, we get

$$\lim_{x \rightarrow 0} \frac{\tan 2x}{\ln(1+x)} = \lim_{x \rightarrow 0} \frac{2 \sec^2 2x}{\frac{1}{1+x}} = \lim_{x \rightarrow 0} [2(1+x) \sec^2 2x] = 2.$$

Sometimes when we apply l'Hôpital's rule to $f(x)/g(x)$ at a point a we find that $f'(x)/g'(x)$ also has the indeterminate form $0/0$ at a . In that case we apply l'Hôpital's rule again, this time to $f'(x)/g'(x)$.

Example 2. Find $\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3}$.

Solution. By l'Hôpital's rule,

$$\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3} = \lim_{x \rightarrow 0} \frac{\cos x - 1}{3x^2}$$

But $(\cos x - 1)/3x^2$ is also indeterminate, of the form $0/0$, at $x = 0$. If we again apply l'Hôpital's rule, we get

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{3x^2} = \lim_{x \rightarrow 0} \frac{-\sin x}{6x}$$

and $\frac{-\sin x}{6x}$ is also indeterminate at $x = 0$. But a third application of l'Hôpital's rule yields

$$\lim_{x \rightarrow 0} \frac{-\sin x}{6x} = \lim_{x \rightarrow 0} \frac{-\cos x}{6} = -\frac{1}{6}$$

Therefore

$$\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3} = -\frac{1}{6}$$

Several variations on l'Hôpital's rule hold true. All limits can be right-hand limits, or all limits can be left-hand limits, in 16.5.1 and the theorem remains valid. Also, a or L can be replaced by $\pm \infty$ in 16.5.1 without affecting the validity of the theorem. The proofs of all these variations on l'Hôpital's rule are analogous to that of 16.5.1.

16.5.2 Theorem (l'Hôpital's Second Rule). Let f and g be functions which are differentiable in an open interval I containing the point a , except at a itself, and let $g'(x) \neq 0$ for all $x \neq a$ in I . If $\lim_{x \rightarrow a} |f(x)| = \lim_{x \rightarrow a} |g(x)| = \infty$, and if

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L,$$

then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = L.$$

Again, all the limits in 16.5.1 and 16.5.2 can be replaced by right-hand limits, or by left-hand limits, without affecting the validity of the theorem. Also, a or L can be $+\infty$ or $-\infty$ and both of these rules remain true.

Example 3. Find $\lim_{x \rightarrow 0} \frac{-\ln |x|}{\cot |x|}$.

Solution. Since $\lim_{x \rightarrow 0} (-\ln |x|) = \lim_{x \rightarrow 0} \cot |x| = \infty$, the given quotient has the indeterminate form ∞/∞ at 0. By differentiating numerator and denominator and applying l'Hôpital's second rule, we find

$$\lim_{x \rightarrow 0} \frac{-\ln |x|}{\cot |x|} = \lim_{x \rightarrow 0} \frac{-1/|x|}{-\csc^2 |x|} = \lim_{x \rightarrow 0} \left[-\sin |x| \left(\frac{\sin |x|}{|x|} \right) \right] = 0(1) = 0.$$

Example 4. Find $\lim_{x \rightarrow \infty} \frac{2^x}{x^2}$, if it exists.

Solution. Since $\lim_{x \rightarrow \infty} 2^x = \lim_{x \rightarrow \infty} x^2 = \infty$, we apply l'Hôpital's second rule (twice):

$$\lim_{x \rightarrow \infty} \frac{2^x}{x^2} = \lim_{x \rightarrow \infty} \frac{2^x \ln 2}{2x} = \lim_{x \rightarrow \infty} \frac{2^x (\ln 2)^2}{2} = \infty.$$

Therefore $\lim_{x \rightarrow \infty} \frac{2^x}{x^2}$ does not exist.

EXERCISES

Find the following limits.

1. $\lim_{x \rightarrow 0} \frac{\sin x - 2x}{x}$

3. $\lim_{x \rightarrow \infty} \frac{\ln x}{x^{10}}$

2. $\lim_{x \rightarrow 0^+} \frac{\ln x}{1/x}$

4. $\lim_{x \rightarrow 0} \frac{\tan x}{3x}$

7. $\lim_{x \rightarrow 3} \frac{2x^2 - x - 15}{3x^2 - 8x - 3}$
9. $\lim_{x \rightarrow 0} \frac{\sin^{-1} x}{3 \tan^{-1} x}$
11. $\lim_{x \rightarrow 0} \frac{x^2}{\ln \cos x}$
13. $\lim_{x \rightarrow \infty} \frac{10^x}{x^{10}}$
15. $\lim_{x \rightarrow \infty} \frac{\ln x}{a^x}$
17. $\lim_{x \rightarrow 0} \frac{3 - \csc x}{7 + \cot x}$
19. $\lim_{x \rightarrow 0} \frac{\tan x - x}{\sin x - x}$
21. $\lim_{x \rightarrow 0} \frac{x - \ln(1+x)}{\cos x - 1}$
23. $\lim_{x \rightarrow 0} \frac{x e^{2x} + 7x}{1 - \cos x}$
25. $\lim_{x \rightarrow 0} \frac{\tan^{-1} x - x}{8x^3}$
27. $\lim_{x \rightarrow 0} \frac{x \tan x - \ln(x+1) + x}{x^2}$
29. $\lim_{x \rightarrow 0^+} \frac{\cot x}{\ln x}$
8. $\lim_{x \rightarrow 0^+} \frac{2 \sin x}{\sqrt{x}}$
10. $\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{4 \sin x}$
12. $\lim_{x \rightarrow \infty} \frac{\ln x}{x}$
14. $\lim_{x \rightarrow \infty} \frac{x^{250}}{e^x}$
16. $\lim_{x \rightarrow \pi/2} \frac{\ln \sin x}{\frac{1}{2}\pi - x}$
18. $\lim_{x \rightarrow \pi/2} \frac{\sin 4x}{\sin 2x}$
20. $\lim_{x \rightarrow 9^+} \frac{\sqrt{x} - 3}{\sqrt{x} - 9}$
22. $\lim_{x \rightarrow 0^+} \frac{10^{\sqrt{x}} - 1}{2^{\sqrt{x}} - 1}$
24. $\lim_{x \rightarrow \pi/2^-} \frac{\cos x}{\sqrt{\frac{1}{2}\pi - x}}$
26. $\lim_{x \rightarrow 0^+} \frac{\ln \csc x}{\ln \cot x}$
28. $\lim_{\theta \rightarrow \pi^-} \frac{2 \sec \frac{1}{2}\theta}{\tan \frac{1}{2}\theta}$
30. $\lim_{x \rightarrow \pi/4} \frac{1 - \tan x}{\frac{1}{4}\pi - x}$

16.6 OTHER INDETERMINATE FORMS

If $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = \infty$, then the function defined by $f(x) \cdot g(x)$ is said to have the *indeterminate form* $0 \cdot \infty$ at a .

If $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = \infty$, then the function defined by $f(x) - g(x)$ has the *indeterminate form* $\infty - \infty$ at a .

When $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$, the function defined by $f(x)^{g(x)}$ has the *indeterminate form* 0^0 at a . Similar definitions apply to the *indeterminate forms* ∞^0 and 1^∞ .

All of these indeterminate forms can be reduced to the indeterminate forms $0/0$ or ∞/∞ by algebraic manipulation so that l'Hôpital's rules can be tried.

Solution. Since $\lim_{x \rightarrow \pi/2} \tan x = \infty$ and $\lim_{x \rightarrow \pi/2} \ln \sin x = 0$, this is an $\infty \cdot 0$ indeterminate form. Rewriting the given expression as the quotient $(\ln \sin x)/\cot x$, and applying l'Hôpital's rule, we obtain

$$\begin{aligned} \lim_{x \rightarrow \pi/2} (\tan x \cdot \ln \sin x) &= \lim_{x \rightarrow \pi/2} \frac{\ln \sin x}{\cot x} = \lim_{x \rightarrow \pi/2} \frac{\frac{1}{\sin x} \cdot \cos x}{-\csc^2 x} \\ &= \lim_{x \rightarrow \pi/2} (-\cos x \cdot \sin x) = 0. \end{aligned}$$

Example 2. Find $\lim_{x \rightarrow 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right)$.

Solution. This is an $\infty - \infty$ indeterminate form. By combining the two fractions we have

$$\lim_{x \rightarrow 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right) = \lim_{x \rightarrow 1} \frac{x \ln x - x + 1}{(x-1) \ln x},$$

which has the indeterminate form $0/0$ at 1. If we apply l'Hôpital's rule twice, we get

$$\lim_{x \rightarrow 1} \frac{x \ln x - x + 1}{(x-1) \ln x} = \lim_{x \rightarrow 1} \frac{x \ln x}{x-1 + x \ln x} = \lim_{x \rightarrow 1} \frac{1 + \ln x}{2 + \ln x} = \frac{1}{2}.$$

Example 3. Find $\lim_{x \rightarrow 0} (1+x)^{1/x}$.

Solution. Since $\lim_{x \rightarrow 0} (1+x) = 1$ and $\lim_{x \rightarrow 0} \frac{1}{x} = \infty$, the given function has the indeterminate form 1^∞ at 0.

Let $y = (1+x)^{1/x}$ and take the natural logarithm of both members; we get

$$\ln y = \frac{\ln(1+x)}{x}.$$

Since the right-hand member of this latter equation has the indeterminate form $0/0$ at $x = 0$, we apply l'Hôpital's rule and obtain

$$\lim_{x \rightarrow 0} \ln y = \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = \lim_{x \rightarrow 0} \frac{1}{1+x} = 1.$$

That is, $\lim_{x \rightarrow 0} \ln y = 1$. Since $\ln y$ and its inverse are continuous functions, $\ln(\lim_{x \rightarrow 0} y) = \lim_{x \rightarrow 0} \ln y = 1 = \ln e$. Therefore $\lim_{x \rightarrow 0} y = e$, and

$$\lim_{x \rightarrow 0} (1+x)^{1/x} = e,$$

EXERCISES

Evaluate.

1. $\lim_{x \rightarrow 0^+} x^x$.
3. $\lim_{x \rightarrow \pi/2} (\sec x - \tan x)$.
5. $\lim_{x \rightarrow 0} (x + e^{x/2})^{2/x}$.
7. $\lim_{x \rightarrow \infty} x^{1/x}$.
9. $\lim_{x \rightarrow 0^+} x \ln x$.
11. $\lim_{x \rightarrow 0} \csc 2x \tan^{-1} x$.
13. $\lim_{x \rightarrow 0} (\cos x - \sin x)^{1/x}$.
15. $\lim_{x \rightarrow 0} (x^2 + 3x + 1)^{2/(3x)}$.
17. $\lim_{x \rightarrow 1} \left(\frac{1}{x-1} - \frac{x}{\ln x} \right)$.
19. $\lim_{x \rightarrow 0} (\sin x)^{\sin x}$.
21. $\lim_{x \rightarrow 0} (e^{3x} - 2x)^{-3/x}$.
23. $\lim_{x \rightarrow \pi/2} \left(\frac{1}{1 - \sin x} - \frac{2}{\cos^2 x} \right)$.
24. Evaluate the improper integral $\int_0^1 \ln x \, dx$ or show that it is divergent.
25. Evaluate the improper integral $\int_0^{\pi} \csc x \, dx$ or show that it is divergent.
26. Find the area of the region between the negative x -axis and the curve $y = 2xe^x$. Make a sketch.
2. $\lim_{x \rightarrow 0} (\csc x - \cot x)$.
4. $\lim_{x \rightarrow 0} x^2 \csc x$.
6. $\lim_{x \rightarrow 0} \left[\csc^2 x - \frac{1}{x^2} \right]$.
8. $\lim_{x \rightarrow 0} (\cos x)^{\cot x}$.
10. $\lim_{x \rightarrow \pi/2} (\cos x)^{(\pi/2)-x}$.
12. $\lim_{x \rightarrow 0} (\cos x)^{1/x^2}$.
14. $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x$.
16. $\lim_{x \rightarrow 0} \left[\csc x - \frac{1}{x} \right]$.
18. $\lim_{x \rightarrow 0^+} (\sin x)^x$.
20. $\lim_{x \rightarrow 0} (\cos x)^{1/x}$.
22. $\lim_{x \rightarrow 0^+} \tan x \ln \sin x$.

Analytic Geometry of Three-Dimensional Space

17.1 CARTESIAN COORDINATES IN THREE-SPACE

The position of a point in plane analytic geometry was established by its directed distances from two mutually perpendicular lines. In three-dimensional space, the position of a point is fixed by its distances from each of three mutually perpendicular planes.

For example, any point P in a room can be located if we know its perpendicular distances from two adjacent walls and from the floor (Fig. 318).

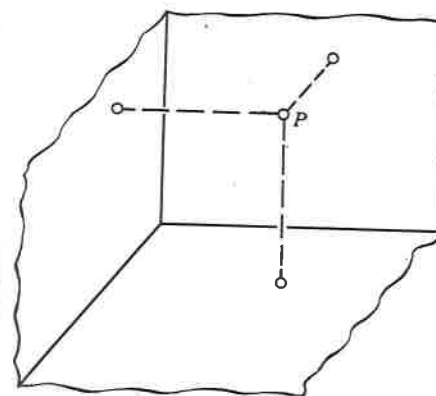


FIG. 318

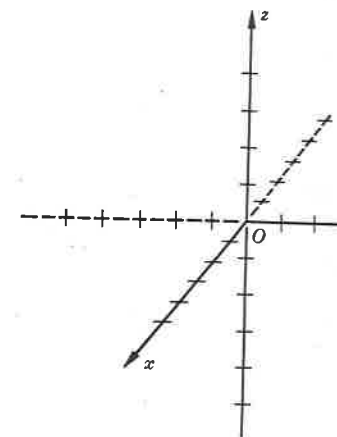


FIG. 319

Let Ox , Oy , and Oz be three mutually perpendicular directed lines in space (Fig. 319). They are called the x -axis, y -axis, and z -axis, respectively, and O is the *origin*. All lines parallel to a coordinate axis have the same positive directions as that axis.

7. Tangent: $3x - y - 4 = 0$; normal: $x + 3y - 28 = 0$.
 9. Tangent: $2x + 3y + (6\sqrt{2} - 7) = 0$; normal: $3x - 2y + (22 + 5\sqrt{2}/2) = 0$.
 11. $-x^3 + 4x^2 - 20x + 17 + C$. 13. $\frac{1}{3}x^3 + C$.
 15. 8. 17. $3\pi a^2$.

15.4 Page 469

1. $8(37^{3/2} - 1)/27$. 3. $6a$.
 5. $\sqrt{17} + \frac{1}{4} \ln(\sqrt{17} + 4)$. 7. $p[2\sqrt{3} + \ln(2 + \sqrt{3})]$.
 9. $3\pi a/2$. 11. $(1 + a^2)^{1/2}(e^{a\pi} - 1)/a$.
 13. $8a$.

15.5 Page 473

3. $(-2, -3)$; $(4, -5)$; $(-5, 4)$; $(2, 4)$; $(0, -6)$; $(5, 0)$; $(\sqrt{2}, -3)$; $(-\sqrt{3}, e)$; $(-\frac{3}{2}\pi, -6)$; $(0, 1)$.
 5. (a) $(-8, -4)$; (b) $(6, 3)$; (c) $(\sqrt{2} - 2, 1)$; (d) $(3, 3)$.
 7. (a) $(3, -3)$; (b) $(1, 17)$; (c) $(7, 2\sqrt{2})$; (d) $(6, -10)$.
 11. Velocity, acceleration, weight, force, momentum.
 13. $(-20 + 20\sqrt{17}, -20 - 20\sqrt{17}) \div (62.46, -102.46)$.

15.6 Page 478

1. (a) $4i - 17j$; (b) -3 ; (c) -15 ; (d) -234 ; (e) -36 ; (f) 30 .
 3. (a) $10i + 2j$; (b) $-6i - 6j$; (c) $-\frac{1}{2}i + 4\sqrt{2}j$; (d) $3i - \pi j$.
 9. $k = 2$, $m = 3$. 11. $a + b + c = 0$.
 15. $\frac{1}{2}(a + b)$.

15.8 Page 483

1. (a) $\{t | -\sqrt{3} \leq t \leq \sqrt{3}, t \neq 0\}$; (b) R ; (c) \emptyset ; (d) $(-1, 1]$; (e) $\{t | t \geq 3, t \neq 4\}$; (f) R .
 3. (a) $t^{-1}i - 6tj$, $-t^{-2}i - 6j$;
 (b) $4t(3 - t^2)^{-2}i + (1 + t^2)^{-1}j$, $(12t^2 + 12)(3 - t^2)^{-3}i - 2t(1 + t^2)^{-2}j$;
 (c) $(\cos t)i - (2 \sin 2t)j$, $(\sin t)i - (4 \cos 2t)j$;
 (d) $t^{-1}i + 2e^{2t}j$, $-t^{-2}i + 4e^{2t}j$;
 (e) $(\sec^2 t)i - 4t^3j$, $2(\sec^2 t \tan t)i - 12t^2j$;
 (f) $(e^t + 2e^{-2t})j$, $(e^t - 4e^{-2t})j$.
 5. $v(t) = -e^{-t}i + e^tj$; $a(t) = e^{-t}i + e^tj$; $v(1) = -e^{-1}i + ej$; $a(1) = e^{-1}i + ej$;
 $|v(1)| = e^{-1}\sqrt{e^4 + 1}$.
 7. $v(t) = (-2 \sin t)i - (3 \sin 2t)j$; $a(t) = (-2 \cos t)i - (6 \cos 2t)j$; $v(\frac{1}{3}\pi) = -\sqrt{3}i - \frac{3}{2}\sqrt{3}j$; $a(\frac{1}{3}\pi) = -i + 3j$; $|v(\frac{1}{3}\pi)| = \sqrt{39}/2$.
 9. $v(t) = 6ti + 3t^2j$; $a(t) = 6i + 6tj$; $v(2) = 12i + 12j$; $a(2) = 6i + 12j$;
 $|v(2)| = 12\sqrt{2}$.
 11. $v(t) = (-4 \cos t)i + (4 + 4 \sin t)j$; $a(t) = (4 \sin t)i + (4 \cos t)j$; $v(\frac{3}{2}\pi) = 2i + (4 + 2\sqrt{3})j$; $a(\frac{3}{2}\pi) = 2\sqrt{3}i - 2j$; $|v(\frac{3}{2}\pi)| = 4\sqrt{2 + \sqrt{3}}$.
 13. $5 + (9 \ln 3)/4$. 15. $e^2(e^4 - 1)\sqrt{2}$.

16.1 Page 489

1. $1/e^2$. 3. $\frac{1}{2}\pi$.
 5. Divergent. 7. $\frac{1}{2}\pi$.
 9. Divergent. 11. 0.
 13. $\frac{1}{3}$. 15. $\frac{1}{3}$.
 17. $(\sin 2 + \cos 2)/(2e^2)$. 19. Divergent.
 23. 1.

16.2 Page 494

1. $2\sqrt{3}$. 3. 0.
 5. 6. 7. Divergent.
 9. $3(1 + \sqrt[3]{3})/2$. 11. π .
 13. $4\sqrt{2}/3$.

16.5 Page 499

1. -1 . 3. 0. 5. $\frac{1}{6} \ln 5$.
 7. $\frac{1}{10}$. 9. $\frac{1}{3}$. 11. -2 .
 13. Does not exist. 15. 0. 17. -1 .
 19. -2 . 21. -1 . 23. Does not exist.
 25. $-\frac{1}{24}$. 27. $\frac{3}{2}$. 29. Does not exist.

16.6 Page 502

1. 1. 3. 0. 5. e^3 .
 7. 1. 9. 0. 11. $\frac{1}{2}$.
 13. $1/e$. 15. e^2 . 17. $-\frac{8}{3}$.
 19. 1. 21. $1/e^3$. 23. $-\frac{1}{2}$.
 25. Divergent.

17.1 Page 506

3. The yz -plane. 5. The z -axis.
 7. A plane parallel to the xz -plane, and 2 units to the left of the origin.
 9. A plane containing the z -axis, and bisecting the first octant.

17.2 Page 508

1. Yes, the z -axis; $PQ = 6$. 5. -7 .
 3. 3.
 7. (a) $5\sqrt{3}$; (b) 7; (c) $\sqrt{83}$; (d) $\sqrt{149}$.
 9. $\sqrt{(x-2)^2 + (y-3)^2 + (z-1)^2} = 5$; a sphere with center at $(2, 3, 1)$ and radius 5.
 11. A sphere with center at $(\frac{1}{2}, -4, 2)$ and radius 3.
 13. $x^2 + y^2 + z^2 = 49$.