

## Sample Questions

### Q1.

Use the definition of a limit to prove:

$$\lim_{n \rightarrow \infty} \frac{n^2 + 2n + 1}{2n^2 + 1} = \frac{1}{2} \quad (1)$$

Establish the right hand side of the identity by using L'Hôpital's rule.

### Q2.

Find the following limits:

a)

$$\lim_{x \rightarrow 0} \frac{\sin^2 x}{2x^2} \quad (2)$$

b)

$$\lim_{x \rightarrow \infty} \frac{(x - e^{-x})}{(2x + e^{-x} \sin x)} \quad (3)$$

c)

$$\lim_{x \rightarrow 0^+} x \ln x \quad (4)$$

d)

$$\lim_{x \rightarrow \infty} \frac{a^x}{x^n} \quad (5)$$

where  $a > 1$  and  $n \in \mathbb{N}$ .

### Q3.

a) State the Mean Value Theorem.

b) Given the definition of an integral via

$$F(b) - F(a) = \int_a^b f(y) dy \quad (6)$$

show that there exists a value  $c \in [a, b]$  such that:

$$f(c) = \frac{1}{b-a} \int_a^b f(y) dy \quad (7)$$

**Q4.**

Take the sequence defined by:

$$\{a_n\}_{n=1}^{\infty} = \frac{(2+n)^{2n} + (-2)^n}{(2+n)^{2n} - 2^n} \quad (8)$$

a) Explain why the sequence is bounded, and apply the Bolzano-Weierstrass theorem.

b) Find a convergent subsequence

c) Evaluate the limit as  $n$  approaches infinity.

**Q5.**

a) Use the ratio test to prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad (9)$$

converges for  $p > 1$ .

b) Show that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^n} \quad (10)$$

converges absolutely. Does the series converge?

c) State and prove the type of convergence exhibited by:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \quad (11)$$

**Q6.**

For the recursion defined by:

$$x_{n+1} = a + \frac{b}{x_n} \quad (12)$$

where  $a, b > 0$ ,

a) Write the recursion as a continued fraction.

b) Show that the sequence converges.

c) Show that the sequence is Cauchy:

$$\lim_{m, n \rightarrow \infty} |x_m - x_n| = 0 \quad (13)$$

d) Find the limit of the sequence. Is it a limit superior or a limit inferior? Give reasons.

**Q7.**

Using mathematical induction, prove that:

$$\sum_{k=1}^N (-1)^k k^2 = \frac{(-1)^N N(N+1)}{2} \quad (14)$$

**Q8.**

Use partial fractions to prove:

$$\frac{1}{n^3 - n} = \frac{1}{2} \left( \frac{1}{n+1} - \frac{1}{n-1} - \frac{2}{n} \right) \quad (15)$$

Hence show that the infinite sum has limit:

$$\sum_{n=2}^{\infty} \frac{1}{n^3 - n} = \frac{1}{4} \quad (16)$$

**Q9.**

Find the convergence set for the following power series:

a)

$$f(x) = \sum_{n=0}^{\infty} \frac{(-x)^{2n}}{(2n)!} \quad (17)$$

b)

$$f(x) = \sum_{n=0}^{\infty} \frac{(x+1)^n}{2^n} \quad (18)$$

c) Prove the sum formula:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \quad (19)$$

What is the radius of convergence?

**Q10.**

a) Evaluate a power series expansion of

$$f(x) = \frac{1}{1+x^2} \quad (20)$$

in powers of  $x$ . What is the radius of convergence of this series?

b) Integrate this series term by term to find the power series of  $\tan^{-1}(x)$ .

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \quad (21)$$

c) Prove that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4} \quad (22)$$

Estimate this value to 6 decimal places.

### Q11.

a) Prove that the function  $f(x) = \frac{\sin x}{x}$  is uniformly continuous on the interval  $[0,1]$ .

b) Prove that the function  $f(x) = \frac{1}{x}$  is not uniformly continuous on the interval  $[0,1]$ , but it is continuous.

### Q12.

a) Show that the function given by:

$$f(x) = \frac{1}{(x-1)^2} \quad (23)$$

decreases monotonically on the interval  $(1, \infty)$ .

b) Let  $\mathbf{x} = (x_1, x_2, \dots, x_n)$ ,  $\mathbf{y} = (y_1, y_2, \dots, y_n)$  and define

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{k=1}^n |x_k - y_k|^2} \quad (24)$$

$$d_{\infty}(\mathbf{x}, \mathbf{y}) = \max(|x_i - y_i|) \quad (25)$$

Show that

$$d_{\infty}(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{y}) \leq \sqrt{n} d_{\infty}(\mathbf{x}, \mathbf{y}) \quad (26)$$

What does this mean for the equivalence of these two metrics?

### Q13.

Consider the sequence of functions defined by:

$$f_n(x) = n^2 x(1-x^2)^{2n} \quad (27)$$

with pointwise limit  $f$ .

a) Find  $f$  and show it is integrable over  $[0,1]$ .

b) Show that we have the following relationship:

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx \neq \int_0^1 f(x) dx \quad (28)$$

**Q14.**

a) Consider the sequence of functions on  $[0,1]$ :

$$g_n^k(x) = n^k x^n (1-x) \quad (29)$$

i) Show that for fixed  $k$ , the pointwise limit as  $n \rightarrow \infty$  is given by the zero function on  $[0,1]$ .

ii) By finding the critical point(s) of each function in the sequence, show that the sequence  $\{g_n^k(x)\}_{n=1}^\infty$  converges uniformly to the zero function if and only if  $k < 1$ .

iii) By using an appropriate integral, show that  $\{g_n^k(x)\}_{n=1}^\infty$  converges in the  $L^2$  sense if and only if  $k < 3/2$ .

**Q15.**

Show, by integration, that if  $\frac{d^2x}{dt^2} = f(x)$ ,  $x(0) = a$ ,  $x'(0) = b$ , then

$$x(u) = a + bu + \int_0^u (u-s)f(x(s))ds \quad (30)$$

**Q16.**

Consider the curve  $Ax^2 + 2Bxy + Cy^2 = 1$  where  $A > 0$  and  $B^2 < AC$ . Let  $M$  denote the distance from the origin to the furthest point on the curve. Show that:

$$M^2 = \frac{(A+C) + \sqrt{(A-C)^2 + 4B^2}}{2(AC - B^2)} \quad (31)$$

**Q17.**

Prove from first principles the following theorem:

$$\lim_{x \rightarrow 0} f(x) = L \Leftrightarrow \lim_{y \rightarrow \infty} f\left(\frac{1}{y}\right) = L \quad (32)$$

**Q18.**

Let  $f(x) = \int_0^1 \cos(x^3 t^2) dt$  for  $x > 0$ .

a) Show that this function can be expressed in integral form as:

$$f(x) = \frac{1}{x^{3/2}} \int_0^{x^{3/2}} \cos(u^2) du \quad (33)$$

b) By writing  $\cos(u^2) = \frac{1}{2u} \frac{d}{du}(\sin(u^2))$ , show that for any  $b > 1$ ,

$$\int_1^b \cos(u^2) du = \frac{\sin(b^2)}{2b} - \frac{\sin(1)}{2} + \frac{1}{2} \int_1^b \frac{\sin(u^2)}{u^2} du \quad (34)$$

c) Show that for  $b > 1$ ,

$$\left| \int_1^b \frac{\sin(u^2)}{u^2} du \right| \leq 1 - \frac{1}{b} \quad (35)$$

and prove that

$$\left| \int_1^b \frac{\cos(u^2)}{u^2} du \right| \leq 1 \quad (36)$$

d) Show that for  $x > 1$

$$|f(x)| \leq \frac{2}{x^{3/2}} \quad (37)$$

e) Show that the improper integral

$$\int_1^\infty f(x) dx \quad (38)$$

is convergent.

### Q19.

Assume the convexity property:

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2) \quad (39)$$

Show that the functions  $f(x) = x^2, x^4, |x|$  are convex. Prove that only  $x^2, x^4$  are strictly convex.

b) Use the convexity property to prove Jensen's inequality for convex functions:

$$f\left(\frac{1}{n} \sum_{n=1}^\infty x_n\right) \leq \frac{1}{n} \sum_{n=1}^\infty f(x_n) \quad (40)$$

### Q20.

Let  $a_n = \frac{1}{n} + (-1)^n$ . Find:

- a)  $\sup_n a_n$
- b)  $\inf_n a_n$
- c)  $\limsup_n a_n$
- d)  $\liminf_n a_n$

Does  $\lim_{n \rightarrow \infty} a_n$  exist? Give a reason for your answer.

**Q21.**

a) Find the Taylor series for

$$f(s) = \frac{1}{\sqrt{1-s}} \quad (41)$$

about  $s = 0$ , where  $|s| < 1$ .

b) Hence, or otherwise prove the identity:

$$\sin^{-1}(x) = \sum_{n=1}^{\infty} \frac{3 \cdot 5 \cdot 7 \cdots (2n-1)}{2^n n!} \frac{x^{2n+1}}{2n+1} \quad (42)$$

c) Use this identity to conclude that:

$$\sum_{n=1}^{\infty} \frac{3 \cdot 5 \cdot 7 \cdots (2n-1)}{2^n n! (2n+1)} = \frac{\pi}{2} \quad (43)$$

**Q22.**

Let  $f_{\alpha}(x) = x^{\alpha}$ .

a) Prove that

$$\frac{d^n}{dx^n} f_{\alpha}(x) = p(\alpha, n) x^{\alpha-n} \quad (44)$$

$$p(\alpha, n) = \prod_{k=1}^n (\alpha - k + 1) = \alpha(\alpha-1) \cdots (\alpha-n+1) \quad (45)$$

b) Evaluate a Taylor series around  $x = 2$  to prove that

$$x^{\alpha} = \sum_{k=0}^{\infty} 2^{\alpha-k} \binom{\alpha}{k} (x-2)^k \quad (46)$$

c) Prove that:

$$\sqrt{x+2} = \sqrt{2} \sum_{k=0}^{\infty} \binom{1/2}{k} \left[ \frac{x}{2} \right]^k \quad (47)$$

d) Using the gamma function values  $\Gamma(3/2) = \frac{\sqrt{\pi}}{2}$ , and the expression for the binomial coefficient:

$$\binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)} \quad (48)$$

show that:

$$\sqrt{x+2} = \sqrt{\frac{\pi}{2}} \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(3/2-k)} \left[ \frac{x}{2} \right]^k \quad (49)$$

Provide an approximation to  $\sqrt{x+2}$  valid to  $\mathcal{O}(x^3)$ . You may assume  $\Gamma(n+1) = n\Gamma(n)$ .

**Q23.**

Prove that if  $f(x), g(x)$  are any two Riemann-integrable functions, then if  $f(x) < g(x)$  it is true that

$$\int_a^b f(x)dx < \int_a^b g(x)dx \quad (50)$$

**Q24.**

Show that:

$$\int_2^6 \frac{1}{5} x^2 dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left( 2 + \frac{4k}{n} \right)^2 \frac{4}{5n} \quad (51)$$

Evaluate the limit of the series.

**Q25.**

Prove that:

$$\int_1^3 (3x + 2)dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left[ 3 \left( 1 + \frac{2k}{n} \right) + 2 \right] \frac{2}{n} \quad (52)$$

Evaluate the limit of the series.

**Q26.**

Find the limit for the following functions, or prove it does not exist.

$$\lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{1 - x} \quad (53)$$

A:  $1/2$

$$\lim_{x \rightarrow 0} \frac{x \sin x}{2 - 2 \cos x} \quad (54)$$

A:  $1$

$$\lim_{x \rightarrow 3} \frac{\lfloor x \rfloor}{x} \quad (55)$$

B: Does not exist.

$$\lim_{x \rightarrow 2^-} \frac{x^2(4x - 8)}{x - 2} \quad (56)$$

A:  $-16$

$$\lim_{x \rightarrow \infty} \frac{x + \sin x}{x + \cos x} \quad (57)$$



A: 1

$$\lim_{x \rightarrow 0} \frac{\tan 6x}{x} \quad (58)$$

A: 6

**Q27.**

Use the Mean Value Theorem to prove that  $|\cos x - \cos y| \leq |x - y|$ .

**Q28.**

Use the Intermediate Value Theorem to prove that the function defined by:

$$f(x) = \sqrt{1+x} + \sin x - \frac{1}{2} \quad (59)$$

has a zero in the interval  $[-\frac{\pi}{4}, 0]$ .

**Q29.**

Take the function:

$$f(x) = \frac{x}{x-1} \quad (60)$$

- a) Evaluate  $\lim_{x \rightarrow \infty} f(x) = L$ .
- b) Draw a neat graph of the function, showing clearly all limiting behaviour.
- c) For a chosen  $\epsilon > 0$ , find a value  $M = M(\epsilon)$  such that for all  $x > M(\epsilon)$ ,  $|f(x) - L| < \epsilon$ .
- d) Show that there exists no  $\xi$  in  $(0, 2)$  such that:

$$f'(\xi) = \frac{f(2) - f(0)}{2 - 0} \quad (61)$$

and explain why this does not contradict the Mean Value Theorem.

A: a) 1, c)  $1 + \frac{1}{\epsilon}$

**Q30.**

Prove that if  $\sum_{n=1}^{\infty} a_n(x)$  is uniformly convergent for  $x$  on some interval, and  $|b_n(x)| \leq a_n(x)$  for all  $x \in I$ , then  $\sum_{n=1}^{\infty} b_n(x)$  is uniformly convergent on the same interval.

**Q31.**

Given the Taylor series defined by  $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ , show that  $\frac{df}{dx} = f(x)$ .

**Q32.**

Given the Taylor series defined by  $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{2n+1} x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ , show that  $\frac{d^2 f}{dx^2} = -f(x)$ . Can you construct a second expression which satisfies the same equation? Calculate both of the limits at the origin using the Taylor series expansion.

**Q33.**

Newton's method for approximation of roots relies on the following recursion algorithm:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (62)$$

Apply Newton's method to approximate  $\pi$  to 10 decimal places by using the function  $f(x) = \sin x$ , with initial point  $x = 3$ . Prove that the sequence of values converges.

**Q34.**

Does the sequence  $a_n = \frac{\ln n}{e^n}$  converge?

**Q35.**

Show that the series  $\sum_n \frac{1}{\sqrt{4n^2 - 7}}$  diverges by comparing it to the harmonic series.

**Q36.**

Prove the infinite series formula below, using the method of partial fractions.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n(n+1)(n+2)} = \frac{(x+1)^2}{2x^2} \ln(1+x) - \frac{3x+2}{4x} \quad (63)$$

You may assume the Taylor series for the logarithm defined through:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad (64)$$

**Q37.**

Prove that

$$(a+x)^n = x^n \left( 1 + n \frac{a}{x} + \frac{n(n-1)}{2!} \left( \frac{a}{x} \right)^2 + \dots \right) \quad (65)$$

**Q38.**

For this question, you shall be using the arithmetic mean and geometric mean defined below.

$$A_n = \frac{1}{n} (a_1 + a_2 + \dots + a_n) \quad (66)$$

$$G_n = \sqrt[n]{a_1 a_2 \dots a_n} \quad (67)$$

Using induction, show that the geometric mean is bounded by the arithmetic mean, i.e.

$$A_n \geq G_n \quad (68)$$

**Q39.**

Assuming the Cauchy-Schwartz inequality, given below:

$$\left| \int f(x)g(x)dx \right|^2 \leq \left| \int f(x)dx \right|^2 \left| \int g(x)dx \right|^2 \quad (69)$$

This theorem applies to real vectors, where we take the function into a vector via:

$$\int a(x)dx \rightarrow \sum_{k=1}^n a_i \quad (70)$$

Show that this implies

$$\left| \sum_{k=1}^n a_i b_i \right|^2 \leq \left| \sum_{k=1}^n a_i \right|^2 \left| \sum_{k=1}^n b_i \right|^2 \quad (71)$$

Prove that

$$\frac{(\sum_{k=1}^n x_i)^2}{\sum_{k=1}^n y_i} \leq \sum_{k=1}^n \frac{x_i^2}{y_i} \quad (72)$$

**Q40.**

Consider the function  $f(x) = \sin x$ . Prove, using the trapezoidal rule, that we can write the integral

$$\int_a^b \sin x dx = \lim_{h \rightarrow 0} (h \cdot [\sin(a+h) + \sin(a+2h) + \dots + \sin(a+nh)]) = \lim_{h \rightarrow 0} S_h \quad (73)$$

where  $h = \frac{b-a}{n}$ . Use the trigonometric law of addition  $2 \sin u \sin v = \cos(u-v) - \cos(u+v)$  to reduce this to:

$$S_h = \frac{h}{2 \sin \left( \frac{h}{2} \right)} \left[ \cos \left( a + \frac{h}{2} \right) - \cos \left( a + \frac{(2n+1)h}{2} \right) \right] \quad (74)$$

and hence the integral is solved

$$\int_a^b \sin x dx = \cos a - \cos b \quad (75)$$

**Q41.**

Show that if  $f(x)$  has a continuous derivative in the interval  $a \leq x \leq b$ , then  $f(x)$  can be represented as the difference of two monotonic functions.

**Q42.**

The silver means are defined through the following sequence:

$$x_{n+1} = 2x_n + \frac{1}{x_n} \quad (76)$$

Assume  $x_0 = 1$ . Prove that

$$x_{n+1} = 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{\dots}}} \quad (77)$$

Show that the sequence is bounded and convergent. Find the decimal value of the silver ratio to 10 decimal places, both by recursion, and by solving the quadratic equation.

A: 2.41421356237

**Q43.**

A space with a metric may be defined by the following property: if a Cauchy sequence has a convergent subsequence, then the sequence itself converges. If the distance between two points is represented by a distance function  $d(x, y) = |x - y|$  show that

$$d(x_n, x) \leq 2\epsilon \quad (78)$$

**Q44.**

Let  $\{a_n\}$  be a bounded sequence and define the set  $S = \{x \in \mathbb{R} | x < a_n \text{ for infinitely many terms}\}$ . Show that there exists a subsequence  $a_{n_k}$  such that it converges to  $\sup S$ .

**Q45.**

Prove that if a function  $f(x)$  is continuous on a closed interval  $[a, b]$ , there exist numbers  $x_{min}, x_{max} \in [a, b]$  such that

$$f(x_{max}) = \max_{x \in [a, b]} \{f(x)\} \quad (79)$$

$$f(x_{min}) = \min_{x \in [a, b]} \{f(x)\} \quad (80)$$

Show, by using the example of a hyperbola, that these theorems do not hold true if a function is not differentiable.

A: Calculus, with Analytical Geometry (Purcell). pp 757 Thm A.2.4

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