

Tutorial 9 :

$$(Q1) \sum_{k=1}^n k^4 = An^5 + Bn^4 + Cn^3 + Dn^2 + En + F.$$

$$= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

$$(Q2) \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6} \quad (\text{See earlier tutes}).$$

$$Y = \int_0^1 (x^4 + 3x^2 + 2x) dx$$

$$[0,1] \rightarrow [0, \frac{1}{n}], [\frac{1}{n}, \frac{2}{n}], \dots, [\frac{n-1}{n}, 1].$$

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}.$$

$$Y = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) (x_i - x_{i-1})$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{i}{n}\right)^4 \left(\frac{i}{n} - \frac{(i-1)}{n}\right) + 3 \sum_{i=1}^n \left(\frac{i}{n}\right)^2 \left(\frac{i}{n} - \frac{(i-1)}{n}\right) + 2 \sum_{i=1}^n \left(\frac{i}{n}\right) \left(\frac{i}{n} - \frac{(i-1)}{n}\right).$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \left(\frac{i}{n}\right)^4 + \frac{3}{n} \sum_{i=1}^n \left(\frac{i}{n}\right)^2 + \frac{2}{n} \sum_{i=1}^n \frac{i}{n}.$$

$$= \lim_{n \rightarrow \infty} \frac{1}{5} + 1 + 1 = 2\frac{1}{5}.$$

$$(Q3) \quad \int_a^b x^n dx = \left[\frac{x^{n+1}}{n+1} \right]_a^b.$$

$$(Q4) \quad f(x) = x^\alpha$$

$$[0, 1] \rightarrow \left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \dots, \left[\frac{n-1}{n}, 1\right].$$

$$\int_0^1 x^\alpha dx = \frac{1}{1+\alpha}.$$

$$\sum_{i=1}^n f(x_i) (x_i - x_{i-1}) = \sum_{i=1}^n \left(\frac{i}{n}\right)^\alpha \left(\frac{i}{n} - \frac{(i-1)}{n}\right).$$

$$= \frac{1}{n^{\alpha+1}} \sum_{i=1}^n i^\alpha$$

$$= \frac{1}{n^{\alpha+1}} (1^\alpha + 2^\alpha + \dots + n^\alpha)$$

$$= \frac{1}{1+\alpha} \quad \text{as } n \rightarrow \infty.$$

$$\therefore \int_0^1 x^\alpha dx = \frac{1}{1+\alpha}.$$

(Q5)
$$\int_a^b (tf(x) + g(x))^2 dx \geq 0$$

$$\therefore \int_a^b [t^2 f^2(x) + 2t f(x)g(x) + g^2(x)] dx \geq 0.$$

$$= t^2 A + \cancel{2t} tB + C \geq 0.$$

$$\therefore B^2 \leq 4AC.$$

$$\therefore \left[2 \int_a^b f(x)g(x) dx \right]^2 \leq 4 \cdot \int_a^b (f(x))^2 dx \int_a^b (g(x))^2 dx$$

$$\text{or } \left[\int_a^b f(x)g(x) dx \right]^2 \leq \int_a^b (f(x))^2 dx \int_a^b (g(x))^2 dx$$

(Q6) $g(x) \geq 0 \quad \int_a^b g(x) dx = 0.$

If $g(x) \geq 0$, then $\int_a^b g(x) dx \geq 0.$

If $g(x) \geq 0$ then > 0 on some interval.

Suppose $g(x) > a > 0$ on $(\epsilon, \delta) \subset (a, b).$

where $\delta > 0$. Then we have:

$$\int_a^b g(x) dx \geq \int_{\epsilon}^{\delta} g(x) dx > a \int_{\epsilon}^{\delta} dx.$$

So if g is \neq non-zero, $\int_a^b g(x) dx = a(\delta - \epsilon) > 0.$

integral is non-zero. Since $\int_a^b g(x) dx = 0$ and g is cts, $g(x) = 0.$

$$\begin{aligned}
 (Q7) \quad \int_a^b x f''(x) dx &= [x f'(x)]_a^b - \int_a^b f'(x) dx \\
 &= b f'(b) - a f'(a) - [f(x)]_a^b \\
 &= b f'(b) - a f'(a) - f(b) + f(a) \\
 &= b f'(b) - f(b) - (a f'(a) - f(a)).
 \end{aligned}$$

$$(Q8) \quad \text{Let } F(x) = \int_1^x f(t) dt \leq (f(x))^2, \quad x \geq 1.$$

$$G(x) = \int_1^x \frac{F'(t)}{\sqrt{F(t)}} dt.$$

$$\text{So } F'(t) = f(t) \geq \sqrt{F(t)}.$$

$$\nexists \quad x-1 = \int_1^x 1 dt \leq \int_1^x \frac{F'(t)}{\sqrt{F(t)}} dt.$$

$$\nexists \quad \text{since } f(t) \leq g(t) \text{ on } [a, b],$$

$$\text{then } \int_a^b f(t) dt \leq \int_a^b g(t) dt.$$

$$\begin{aligned}
 \int_1^x \frac{F'(t)}{\sqrt{F(t)}} dt &= [2\sqrt{F(t)}]_1^x \quad F(1)=0 \\
 &= 2\sqrt{F(x)} \leq 2f(x).
 \end{aligned}$$

(Q9)

$$\lim_{R \rightarrow \infty} \int_1^R \frac{1}{1+t^2} dt$$

$$= \lim_{R \rightarrow \infty} \tan^{-1}(t) \Big|_0^R$$

$$= \tan^{-1}(\infty) - \tan^{-1}(0)$$

$$= \pi/2$$

(10)

$$\lim_{\epsilon \rightarrow 0} \int_{\epsilon}^1 \frac{dx}{\sqrt{x}} = \lim_{\epsilon \rightarrow 0} [2\sqrt{x}]_{\epsilon}^1$$

$$= 2 - \lim_{\epsilon \rightarrow 0} 2\sqrt{\epsilon} = 2$$