

Tutorial 7 - Real Analysis :

$$\begin{aligned} (Q1i) \quad e^x &= 1 + x + \frac{x^2}{2!} + \dots \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} a_n \end{aligned}$$

$$\therefore a_n = \frac{x^n}{n!}$$

→ Apply ratio test :

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x}{(n+1)} \right| = \lim_{n \rightarrow \infty} \frac{|x|}{(n+1)} = 0 < 1. \end{aligned}$$

So converges $\forall x \in \mathbb{R}$.

$$\begin{aligned} (ii) \quad \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} a_n. \end{aligned}$$

$$\therefore a_n = \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad a_{n+1} = \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!} \times \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x^2}{(2n+3)(2n+2)} \right| \\ &= 0 \quad \forall x \in \mathbb{R}, \text{ so converges } \forall x \in \mathbb{R}. \end{aligned}$$

$$(iii) \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} a_n.$$

$$\therefore \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{2(n+1)}}{(2(n+1))!} \times \frac{(2n)!}{x^{2n}} \right|.$$

$$= \lim_{n \rightarrow \infty} \left| \frac{x^2}{(2n+2)(2n+1)} \right| = 0 \quad \forall x \in \mathbb{R}.$$

Hence converges.

Indeed, a deeper result of Euler using complex analysis shows that

$$e^{i\theta} = \cos \theta + i \sin \theta$$

& this converges $\forall \theta \in \mathbb{C}$.

$$(Q2) \quad f(x) = \sin x, \quad f'(x) = \cos x, \quad f''(x) = -\sin x \dots$$

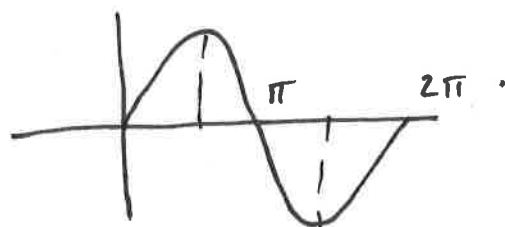
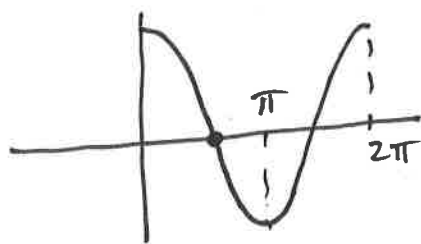
$$f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

$$f'\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

$$f''\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1$$

$$f'''\left(\frac{\pi}{2}\right) = -\cos\left(\frac{\pi}{2}\right) = 0$$

etc.



$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots$$

$$= 1 - \frac{(x - \frac{\pi}{2})^2}{2!} + \frac{(x - \frac{\pi}{2})^4}{4!} - \dots$$

$$= \sin x.$$

Also we have $\frac{d}{dx} (\sin x) = \cos x$.

$$\begin{aligned} \therefore \cos x &= \frac{-2 \left(x - \frac{\pi}{2}\right)}{2!} + \frac{4 \left(x - \frac{\pi}{2}\right)^3}{4!} - \frac{6 \left(x - \frac{\pi}{2}\right)^5}{6!} + \dots \\ &= -\left(x - \frac{\pi}{2}\right) + \frac{1}{3!} \left(x - \frac{\pi}{2}\right)^3 - \frac{1}{5!} \left(x - \frac{\pi}{2}\right)^5 + \dots \end{aligned}$$

(Q3) $f(x) = (1+x)^\alpha$ about $a=0$.

$$f(x) \sim f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots$$

$$f'(x) = \alpha(1+x)^{\alpha-1} \Rightarrow f'(0) = \alpha$$

$$f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2} \Rightarrow f''(0) = \alpha(\alpha-1)$$

\vdots

$$f(x) = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 + \dots$$

$$\therefore f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\prod_{j=0}^{n-1} \alpha(\alpha-1) \dots (\alpha-j) \right] x^n$$

$$a_n = \frac{1}{n!} x^n \cdot \alpha(\alpha-1) \dots (\alpha-n)$$

$$a_{n+1} = \frac{1}{(n+1)!} x^{n+1} \alpha(\alpha-1) \dots (\alpha-n+1)$$

$$\therefore \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x}{(n+1)} \cdot (\alpha-n) \right|$$

$$\& \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = |x| < 1 \quad \forall |x| < 1$$

(Q4)

$$\sqrt{\frac{3}{2}} = \left(1 + \frac{1}{2}\right)^{\frac{1}{2}} \quad x = \frac{1}{2}, \alpha = \frac{1}{2}.$$

$$\approx 1 + \frac{1}{2} \cdot \frac{1}{2} + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!} \cdot \left(\frac{1}{2}\right)^2 + \dots$$

$$= 1 + \frac{1}{4} - \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{2!} + \dots$$

$$= 1 + \frac{1}{4} - \frac{1}{32} \approx 1.21875.$$

$$\sqrt{\frac{3}{2}} \sim 1.2247.$$

(Q5) $f(x) = \frac{x}{(1+x^2)^2}.$

$$\frac{1}{1+r} = 1 - r + r^2 - r^3 + \dots \quad |r| < 1.$$

So if $r = x^2$, $\frac{1}{1+x^2} \sim 1 - x^2 + x^4 - x^6 + \dots$
 $|x|^2 < 1.$

$$\therefore \frac{d}{dx} \left(\frac{1}{1+x^2} \right) = \frac{-2x}{(1+x^2)^2}.$$

$$= \frac{d}{dx} (1 - x^2 + x^4 - x^6 + \dots)$$

$$= -2x + 4x^3 - 6x^5 + \dots$$

$$\therefore \frac{x}{(1+x^2)^2} = x - 2x^3 + 3x^5 - \dots$$

$$Q6) f(x) = \ln(1+x).$$

$$f'(x) = \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \dots$$

$$\begin{aligned} \therefore f(x) &= \int_0^x f'(t) dt \\ &= \int_0^x (1 - t + t^2 - t^3 + t^4 - \dots) dt \\ &= \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \right]. \end{aligned}$$

$$|x| < 1.$$

$$(Q7) \text{ i)} \quad \sum_{n=1}^{\infty} \frac{x^n}{\sqrt[n]{n}} \quad \therefore a_n = \frac{x^n}{\sqrt[n]{n}}$$

$$\begin{aligned} \therefore \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{\sqrt[n+1]{n+1}} \cdot \frac{\sqrt[n]{n}}{x^n} \right| \\ &= \lim_{n \rightarrow \infty} |x| \cdot \sqrt[n]{\frac{n}{n+1}} = |x|. \end{aligned}$$

So if series converges $|x| < 1$.

$$(ii) \quad \sum_{n=1}^{\infty} \frac{x^2}{n^2+1} \Rightarrow a_n = \frac{x^2}{n^2+1}.$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^2}{(n+1)^2+1} \cdot \frac{n^2+1}{x^2} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{n^2+1}{(n+1)^2+1} \right| = 1. \end{aligned}$$

\therefore Converges $\forall x \in \mathbb{R}$.

(Q8)

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$\begin{aligned} f'(0) &= \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x \rightarrow 0} \frac{e^{-1/x^2} - 0}{x} = \frac{0 - 0}{x} \\ &= 0. \end{aligned}$$

$$e^{-1/x^2} \sim 1 - \frac{1}{x^2} + \dots$$

$$f'(0) = 0.$$

$$f'(x) = \begin{cases} \frac{2}{x^3} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$f''(0) = \lim_{x \rightarrow 0} \frac{\frac{2}{x^3} e^{-1/x^2} - 0}{x - 0} = 0.$$

$$\therefore \rightarrow f^{(n)}(0) = 0.$$

$$\therefore f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots$$

$$= 0 + 0 + 0 + 0 + \dots$$

$$\Rightarrow \text{only true at } x=0.$$

$$(Q9) \quad \sum_{n=0}^{\infty} n^2 x^n \quad a_n = n^2 x^n.$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2 x^{n+1}}{n^2 x^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2}{n^2} \right| \cdot |x| \\ &= |x| \lim_{n \rightarrow \infty} \left| \left(1 + \frac{1}{n}\right)^2 \right| \\ &= |x|. \end{aligned}$$

\therefore Series converges for $|x| < 1$.

(Q10) cont.

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad (\text{Geometric series})$$

$$\therefore \frac{d}{dx} \sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} n x^{n-1} = \frac{d}{dx} \left(\frac{1}{1-x} \right)$$

$$= \frac{-1 \cdot -1}{(1-x)^2} = \frac{1}{(1-x)^2}.$$

$$\therefore \sum_{n=1}^{\infty} n x^n = x \cdot \sum_{n=1}^{\infty} n x^{n-1}$$

$$\therefore \frac{d}{dx} \sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n^2 x^{n-1}$$

$$= \frac{d}{dx} \left(\frac{x}{1-x^2} \right) = \frac{x+1}{(1-x)^3}.$$

$$\therefore \sum_{n=1}^{\infty} n^2 x^n = \frac{x(x+1)}{(1-x)^3}.$$

$$(Q10) \quad \frac{dy}{dx} = y.$$

$$\text{let } y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\therefore \frac{dy}{dx} = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

$$\S \quad a_0 + a_1 x + a_2 x^2 + \dots$$

$$= 1 \cdot a_1 x^{1-1} + 2a_2 x^{2-1} + 3a_3 x^{3-1} + \dots$$

$$= a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

$$\therefore a_0 = a_1$$

$$a_1 = 2a_2$$

$$a_2 = 3a_3$$

$$\vdots$$

$$a_{n-1} = n a_n.$$

$$a_0 = a_1$$

$$2a_2 = a_1 \Rightarrow a_2 = \frac{1}{2} a_1 = \frac{1}{2} a_0 = \frac{1}{2!} a_0$$

$$a_3 = \frac{1}{3} a_2 = \frac{1}{3 \times 2 \times 1} a_0 = \frac{1}{3!} a_0$$

$$\vdots$$

$$a_n = \frac{1}{n!} a_0$$

$$\S \quad y(x) = a_0 \sum_{n=0}^{\infty} \frac{1}{n!} x^n = a_0 e^x.$$