

(Q1)

Tutorial 3

Consider sequence :

$$\left\{ \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \dots \right\}.$$

There exists a subsequence :

$$\left\{ \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-2}{n}, \frac{n-1}{n} \right\} = \{r_n\}.$$

If we take the limit as $n \rightarrow \infty$.

$$\begin{aligned} \text{we find } \lim_{n \rightarrow \infty} \sup(r_n) &= \lim_{n \rightarrow \infty} \frac{n-1}{n} \\ &= \lim_{n \rightarrow \infty} 1 - \frac{1}{n} = 1. \end{aligned}$$

$$\nexists \lim_{n \rightarrow \infty} \inf(r_n) = \lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

(Q2) let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence.

we have $L \leq x_n \leq l$ by assumption.

Now, assume the statement in the tutorial sheet is false. Then for some $\epsilon > 0$, we can find $n > N$ s.t. $x_n \geq l + \epsilon$.

Bolzano - Weierstrass theorem implies that \exists convergent subsequence with limit $\bar{l} \geq l + \epsilon$, however l is the supremum, hence contradiction. We must therefore have $x_n < l + \epsilon$.

Q2 cont:

The lower bound by a similar argument will be given by the inequality:

$$x_n > L - \epsilon$$

for any $n > N$.

Q3. Suppose $|x_{n+1} - x_n| \leq r^n$ with $0 < r < 1$.

Prove that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Proof:

$$|x_m - x_n| = |x_m - x_{m-1} + x_{m-1} - x_{m-2} + \dots + x_{n+1} - x_n|$$

$$\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \dots + |x_{n+1} - x_n|$$

$$= r^{m-1} + r^{m-2} + \dots + r^n$$

$$= r^n [r^{m-1-n} + \dots + 1]$$

$$\therefore \text{Let } S_{n,m} = r^{m-1} + \dots + r^n$$

$$r S_{n,m} = r^n [r^{m-n} + \dots + r]$$

$$\therefore (1-r) S_{n,m} = r^n (1 - r^{m-n})$$

$$\therefore S_{n,m} = \frac{r^{m-n} - r^n}{1-r}$$

$$\& \text{ as } 0 < r < 1, \text{ as } m, n \rightarrow \infty \quad r^{m-n} \rightarrow 0 \\ r^n \rightarrow 0$$

$$\text{So } |x_m - x_n| \rightarrow 0 \text{ as } m, n \rightarrow \infty$$

Cauchy's Criterion :

For a sequence $\{x_n\}$ to be convergent, it is necessary & sufficient that $\forall \epsilon > 0, \exists N$ s.t. for $n \geq N, m \geq N$ we satisfy $|x_n - x_m| < \epsilon$.

(Q4) $x_{n+2} = (x_{n+1} x_n)^{\frac{1}{2}}$

Let $0 < a \leq x_1 \leq x_2 \leq b$.

Consider now $x_{n+2}^2 - x_{n+1}^2$

$$\begin{aligned} x_{n+2}^2 - x_{n+1}^2 &= x_{n+1} x_n - x_{n+1}^2 \\ &= x_{n+1} (x_n - x_{n+1}). \end{aligned}$$

$$\therefore (x_{n+2} - x_{n+1})(x_{n+2} + x_{n+1}) = x_{n+1} (x_n - x_{n+1}).$$

$$\therefore x_{n+2} - x_{n+1} = \frac{x_{n+1} (x_n - x_{n+1})}{(x_{n+2} + x_{n+1})}.$$

Now, we have $a \leq x_n \leq b$ implies that:

$$|x_{n+2} - x_{n+1}| = \frac{|x_{n+1}|}{|x_{n+2} + x_{n+1}|} \cdot |x_{n+1} - x_n|.$$

Let us consider

$$\left| \frac{x_{n+1}}{x_{n+2} + x_{n+1}} \right|.$$

Then $\left| \frac{x_{n+1}}{x_{n+2} + x_{n+1}} \right| \leq \frac{b}{a+b}$

$$\& \quad |x_{n+2} - x_{n+1}| \leq \frac{b}{a+b} |x_{n+1} - x_n|.$$

So we may write

$$|x_3 - x_2| \leq \frac{b}{a+b} |x_2 - x_1|.$$

Q4 cont.:

$$\begin{aligned} |x_4 - x_3| &\leq \frac{b}{a+b} |x_3 - x_2| \\ &\leq \left(\frac{b}{a+b}\right)^2 |x_2 - x_1|. \end{aligned}$$

$$\therefore |x_5 - x_4| \leq \left(\frac{b}{a+b}\right)^3 |x_2 - x_1|$$

$$\vdots$$
$$|x_{n+2} - x_{n+1}| \leq \left(\frac{b}{a+b}\right)^{n-1} |x_2 - x_1|.$$

Assume $a > 0$, then $\frac{b}{a+b} < 1$, $\frac{b}{a+b} = r$.

$$\text{hence } |x_{n+2} - x_{n+1}| \leq r^{n-1} |x_2 - x_1|.$$

We can use the proof from q3 to establish that this is a Cauchy sequence.

(Q5) let $x_1 = a$, $x_2 = b$ & define recursive function

$$x_{n+2} = \frac{1}{2}(x_{n+1} + x_n).$$

$$\begin{aligned} \text{then } x_{n+2} - x_{n+1} &= \frac{1}{2}(x_{n+1} + x_n) - x_{n+1} \\ &= \frac{1}{2}(x_n - x_{n+1}). \end{aligned}$$

$$\begin{aligned} \therefore |x_{n+2} - x_{n+1}| &= \frac{1}{2} |x_{n+1} - x_n| = \frac{1}{2^2} |x_n - x_{n-1}| \\ &= \dots = \frac{1}{2^n} |x_2 - x_1| = \frac{1}{2^n} |b - a|. \end{aligned}$$

Then we can write :

$$|x_n - x_m| = |x_n - x_{n-1} + x_{n-1} - x_{n-2} + \dots + x_{m-1} - x_m|.$$

$$\text{So } |x_n - x_m| \leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \dots + |x_{m-1} - x_m|.$$

$$\leq \left(\frac{1}{2^{n-2}} + \frac{1}{2^{n-3}} + \dots + \frac{1}{2^{m-1}} \right) |b-a|.$$

$$\text{where we used } |x_{n+2} - x_{n+1}| = \frac{1}{2^n} |b-a|.$$

$$\text{then } |x_n - x_m| \leq \frac{1}{2^{n-1}} \left(1 + \frac{1}{2} + \dots + \frac{1}{2^{n-m-1}} \right) |b-a|.$$

$$S_n = \frac{a(1-r^n)}{1-r} \quad a=1, r=\frac{1}{2}.$$

$$S_{n-m-1} = \frac{\left(1 - \left(\frac{1}{2}\right)^{n-m}\right)}{1 - \left(\frac{1}{2}\right)}.$$

$$\therefore |x_n - x_m| \leq \frac{1}{2^{m-1}} \cdot \left[\frac{\left(1 - \left(\frac{1}{2}\right)^{n-m}\right)}{1 - \left(\frac{1}{2}\right)} \right] |b-a|.$$

$$\text{Now } \left(\frac{1}{2}\right)^{n-m} > 0$$

$$\begin{aligned} \therefore |x_n - x_m| &\leq \frac{1}{2^{m-1}} \cdot \frac{1}{\frac{1}{2}} |b-a| \\ &= \frac{1}{2^{m-2}} |b-a|. \end{aligned}$$

$$\text{hence } \lim_{m, n \rightarrow \infty} |x_n - x_m| = 0$$

So $\{x_n\}$ is Cauchy & the sequence converges.

(Q6) let $x_{n+1} = \frac{1}{2} \left(x_n + \frac{a}{x_n} \right)$.

If the sequence converges, we know that all subsequences converge to the same limit.

Then we can write $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} x_{n+1} = x$.

$$\therefore \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} \frac{1}{2} \left(x_n + \frac{a}{x_n} \right)$$

or $x = \frac{1}{2} \left(x + \frac{a}{x} \right)$

Solving the quadratic:

$$x^2 = \frac{1}{2} \left(x^2 + \frac{ax}{x} \right) \Leftrightarrow x^2 = \frac{1}{2} x^2 + \frac{a}{2}$$

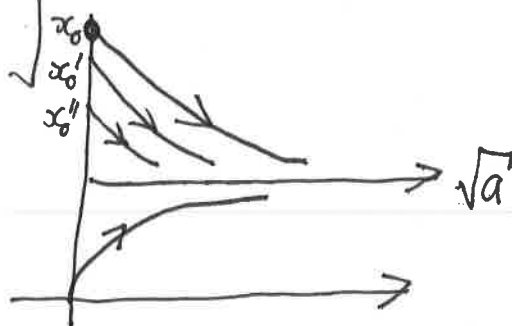
$$\therefore x^2 = a \text{ implying } x = \pm \sqrt{a}$$

Assume a & x strictly positive

$$x > 0, a > 0$$

Then $x = \sqrt{a} = \lim_{n \rightarrow \infty} \frac{1}{2} \left(x_n + \frac{a}{x_n} \right)$

$$\begin{aligned} x_1 &= x_1 \\ x_2 &= f(x_1) \\ x_3 &= f(x_2) \\ &\vdots \end{aligned} \left. \vphantom{\begin{aligned} x_1 &= x_1 \\ x_2 &= f(x_1) \\ x_3 &= f(x_2) \\ &\vdots \end{aligned}} \right\} \text{Recursive function.}$$



Calculating $\sqrt{2}' =$

we have $x_1 = 1$

$$x_2 = \frac{3}{2} = 1.5$$

$$x_3 = \frac{17}{12} =$$

$$x_4 = \frac{577}{408} =$$

Obviously, from this observation we can see that $\sqrt{2}'$ is not a rational, finite, decomposition of $\sqrt{2}'$. This question has a long illustrious history in number theory dating to the Ancient Greeks.

Proving convergence:

Let $y_n = \frac{x_n}{\sqrt{a'}}$, we have

$$\begin{aligned} y_{n+1} &= \frac{1}{2\sqrt{a'}} \left(x_n + \frac{a'}{x_n} \right) \\ &= \frac{1}{2} \left(\frac{x_n}{\sqrt{a'}} + \frac{\sqrt{a'}}{x_n} \right) \\ &= \frac{1}{2} \left(y_n + \frac{1}{y_n} \right). \end{aligned}$$

∴ we have

$$\begin{aligned}y_{n+1} - 1 &= \frac{1}{2} \left(y_n + \frac{1}{y_n} - 2 \right) \\&= \frac{1}{2y_n} (y_n^2 - 2y_n + 1) \\&= \frac{1}{2y_n} (y_n - 1)^2.\end{aligned}$$

$$\begin{aligned}y_{n+1} + 1 &= \frac{1}{2} \left(y_n + \frac{1}{y_n} + 2 \right) \\&= \frac{1}{2y_n} (y_n + 1)^2.\end{aligned}$$

$$\begin{aligned}\therefore \frac{y_{n+1} - 1}{y_{n+1} + 1} &= \left(\frac{y_n - 1}{y_n + 1} \right)^2 \\&= \left(\frac{y_{n-1} - 1}{y_{n-1} + 1} \right)^4 \\&= \dots = \left(\frac{y_0 - 1}{y_0 + 1} \right)^{2^{n+1}}\end{aligned}$$

If we have $\lim_{n \rightarrow \infty} \frac{y_{n+1} - 1}{y_{n+1} + 1} = 0$

implying $\lim_{n \rightarrow \infty} y_{n+1} = 1 \Leftrightarrow \lim_{n \rightarrow \infty} \frac{x_{n+1}}{\sqrt{a}} = 1$

then we must have $\left| \frac{y_0 - 1}{y_0 + 1} \right| < 1$.

Consider the sequence from Q1:

$$\left\{ \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, \dots \right\}$$

a_0, a_1, \dots

If we look at the sequence we have e.g.

$$\frac{1}{2} = a_0 = a_4 = a_{12} = \dots$$

$$\frac{1}{3} = a_1 = a_{11} = \dots$$

and we can see that each rational number occurs infinitely often for $0 < \frac{p}{q} < 1$ in this sequence.

This is a convergent subsequence; the interval is closed and bounded, so Bolzano Weierstrass Thm applies.

Then taking $r_n \in [0, 1]$ we can write

$$x - \frac{1}{k} < r_{n_k} < x + \frac{1}{k}$$

i.e. $\forall n_{k+n} > n_k$ we have

$$\lim_{k \rightarrow \infty} x - \frac{1}{k} < \lim_{k \rightarrow \infty} r_{n_k} < \lim_{k \rightarrow \infty} x + \frac{1}{k}, \text{ so } r_{n_k} \rightarrow x.$$

Q6 cont.

$$|y_0 - 1| < |y_0 + 1|.$$

This is true for $y_0 > 0$, which gives us

$$\frac{x_0}{\sqrt{a}} > 0, \text{ or } x_0 > 0.$$

As discussed, we have $\lim_{n \rightarrow \infty} y_{n+1} = 1$

$$\therefore \lim_{n \rightarrow \infty} x_{n+1} = \sqrt{a}.$$

(Q7) Bolzano-Weierstrass Theorem:

Every bounded sequence in \mathbb{R} has a convergent subsequence.

Alt:

If every sequence of points in X has a convergent subsequence converging to a point in X , we say the space X is sequentially compact.

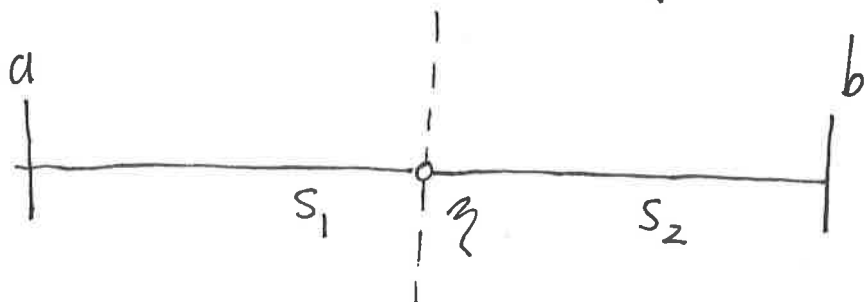
A subset of \mathbb{R} is sequentially compact if & only if it is closed and bounded.

(Q8)

Let $S_\xi = \{x; x \in S, x \neq \xi\}$.

If \exists a sequence of points in S_ξ which converges to ξ we say ξ is a limit point.

If we write out ξ explicitly, we have:



$$S_\xi = S_1 \cup S_2$$

$$S_1 = \{x \mid x < \xi\}, \quad S_2 = \{x \mid x > \xi\}.$$

Let $\{x_n\}$ be a sequence of points such that

$$x_n \neq x_m \quad \forall \quad n \neq m.$$

The sequence occurs on an interval which is closed & bounded at $x=a, b$. Under Bolzano-Weierstrass it contains a convergent subsequence $\{x_{n_r}\} \rightarrow \xi$.

We could e.g. halve the distance between $x_{n_r} \neq \xi$ at each step. Then

$$\lim_{r \rightarrow \infty} x_{n_r} = \xi.$$

On the interval $[a, b] / \{z\}$

we have infinite elements, see e.g. Q1 for a proof over $[0, 1]$, obviously there are "more" in $[a, b]$, and the deletion of a single point from a bounded interval in \mathbb{R} does not change this.

We therefore have shown that \forall bounded sets in \mathbb{R}
 \exists at least one limit point given by z ,

$$a < z < b$$

$$S_z = \{x; x \in S, x \neq z\}$$

and this has $\lim_{n \rightarrow \infty} x_n = z$.

