

Tutorial Two Solutions

①

(1) If $x=1$, $\lim_{n \rightarrow \infty} \frac{x+x^n}{1+x^n} = \frac{1+1^n}{1+1^n}$ each n

\therefore limit = 1. If $x=-1$, sequence diverges

If $|x| > 1$,
If $|x| < 1$, then for some $h > 0$

$$|x|^n = \frac{1}{(1+h)^n} = \frac{1}{1+nh+\frac{n(n-1)}{2}h^2+\dots+h^n} < \frac{1}{nh}$$

Now $\frac{1}{n} \rightarrow 0$ as $n \rightarrow \infty$. So $|x|^n \rightarrow 0$

and hence $x^n \rightarrow 0$. For $|x| < 1$

$$\lim_{n \rightarrow \infty} \frac{x+x^n}{1+x^n}$$

(2) $y_n \rightarrow 0$. So given $\varepsilon > 0$ we can find $N \in \mathbb{N}$ such that $n \geq N \Rightarrow y_n < \varepsilon$.

(Clearly question implies $y_n \geq 0$)

Let $n \geq N$, then $|x_n - 1| \leq y_n < \varepsilon$

So $x_n \rightarrow 1$.

(3) $x_n = \left(1 + \frac{1}{n}\right)^n$. The limit is actually e .

First we show sequence is bounded above

$$\left(1 + \frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2}\left(\frac{1}{n}\right)^2 + \dots + \left(\frac{1}{n}\right)^n$$

$$= 1 + 1 + \left(1 - \frac{1}{n}\right)\frac{1}{2!} + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\frac{1}{3!} + \dots + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\dots\left(1 - \frac{n-1}{n}\right)\frac{1}{n!}$$

$$\leq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}$$

$$\leq 1 + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}}\right) \quad (2^{n-1} \leq n!)$$

$$= 1 + \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} = 1 + 2\left(1 - \left(\frac{1}{2}\right)^n\right) < 3$$

(2)

Let $a_n = 1$, $a_1 = a_2 = \dots = a_{n-1} = 1 + \frac{1}{n-1}$. By the inequality

$$\text{Then } \left(1 + \frac{1}{n-1}\right)^{n-1} \leq \frac{(n-1)\left(1 + \frac{1}{n-1}\right) + 1}{n} = \left(1 + \frac{1}{n}\right)$$

$$\text{Hence } \left(1 + \frac{1}{n-1}\right)^{n-1} \leq \left(1 + \frac{1}{n}\right)^n, n=2,3,\dots$$

So the sequence is increasing and bounded above.

Thus it converges.

$$\text{If we take } n=10, \left(1 + \frac{1}{10}\right)^{10} = 2.5937$$

$$n=100, \left(1 + \frac{1}{100}\right)^{100} = 2.7048$$

$$n=1000, \left(1 + \frac{1}{1000}\right)^{1000} = 2.7169 \text{ etc.}$$

(4) Let N be the smallest natural number such that $N > x$. Then for $n \geq N$

$$\frac{x^n}{n!} \leq \frac{x}{2} \cdot \frac{x}{N-1} \cdot \frac{x}{N} \dots \frac{x}{N}$$

$$= \frac{x^{N-1}}{(N-1)!} \left(\frac{x}{N}\right)^{n-N+1}$$

Since $\frac{x}{N} < 1$, $\left(\frac{x}{N}\right)^n \rightarrow 0$ as $n \rightarrow \infty$. Now we

Question 2.

(5) Suppose that $\left(1 + \frac{1}{n}\right)^{\alpha+1} |x| > 1$ for all $n \in \mathbb{N}$.
Then $\frac{1}{n} > \left(\frac{1}{|x|}\right)^{\frac{1}{\alpha+1}} - 1 > 0$

for all $n \in \mathbb{N}$ and this is a contradiction.
Hence, for some $N \in \mathbb{N}$, $\left(1 + \frac{1}{N}\right)^{\alpha+1} |x| \leq 1$.
If $x \neq 0$, consider the expression

(3)

$$\left| \frac{(n+1)^{\alpha+1} x^{n+1}}{n^{\alpha+1} x^n} \right| = \left(1 + \frac{1}{n}\right)^{\alpha+1} |x|. \text{ If } n \geq N$$

$$|(n+1)^{\alpha+1} x^{n+1}| \leq |n^{\alpha+1} x^n|$$

It follows that for $n \geq N$ $|n^{\alpha+1} x^n| \leq |N^{\alpha+1} x^N|$
 So that for $n \geq N$

$$|n^{\alpha} x^n| \leq \frac{1}{n} |N^{\alpha+1} x^N|$$

and thus $n^{\alpha} x^n \rightarrow 0$ as $n \rightarrow \infty$ by Q2.

6)

we have $\left\{ \sin\left(\frac{\pi n}{2}\right) \right\}_{n=1}^{\infty}$
 There are many convergent subsequences

Take $n_k = (4k+1)$

$$\begin{aligned} \text{Then } a_{n_k} &= \sin\left(\frac{(4k+1)\pi}{2}\right) = \sin\left(2k\pi + \frac{\pi}{2}\right) \\ &= \sin(k\pi) \cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right) \cos(2k\pi) \\ &= 1 \end{aligned}$$

So subsequence is $\{1, 1, 1, \dots\}$

which is clearly convergent.

(7) The sequence is bounded above and below.
 The Bolzano-Weierstrass Theorem says that it has a convergent subsequence. We can take the lower bound to be b for the subsequence. Apply BW again to the subsequence and there is a sub-subsequence which converges and the limit is $\geq b$.

(8) Let n_k be any sequence of odd numbers. $a_{n_k} = 1$.

$$\text{Now } \frac{3^n + (-2)^n}{3^n - 2^n} = \frac{3^n \left(1 + \left(-\frac{2}{3}\right)^n\right)}{3^n \left(1 - \left(\frac{2}{3}\right)^n\right)} \rightarrow 1 \text{ as } n \rightarrow \infty.$$

So all subsequences converge to 1.

(4)

(9) If the sequence converges, all subsequences converge to the same limit and this is the limit of the sequence.

$(2n)^{\frac{1}{2n}} \rightarrow l$, the limit of the sequence.

Now we can show $\{2^{\frac{1}{2n}} \rightarrow 1 \text{ as } n \rightarrow \infty$.
Thus

$$n^{\frac{1}{2n}} \rightarrow l \text{ as } n \rightarrow \infty$$

Since $(2n)^{\frac{1}{2n}} = 2^{\frac{1}{2n}} n^{\frac{1}{2n}}$

Hence $n^{\frac{1}{n}} = n^{\frac{1}{2n}} n^{\frac{1}{2n}} \rightarrow l \cdot l = l^2 = l$.

So either $l=1$ or $l=0$. We cannot have $l=0$, so $l=1$ and $n^{\frac{1}{n}} \rightarrow 1$.