

Introduction to Real Analysis.

Tutorial One.

Manipulations with inf and sup .

Recall that $\inf(A)$ is the greatest lower bound of the set of real numbers A and \sup is the least upper bound of A .

- (1) Suppose that $\xi > 0$ and S is a non empty set of real numbers bounded above. Prove that

$$\sup_{x \in S} \xi x = \xi \sup_{x \in S} x.$$

- (2) Suppose that S is non empty, bounded above and that $S_0 \subseteq S$. (So S_0 is contained in S . It might equal S .) Prove that $\sup S_0 \leq \sup S$.

- (3) Suppose that S is non empty, bounded above and that ξ is any real number. Prove that $\sup_{x \in S} (x + \xi) = \xi + \sup_{x \in S} x$.

- (4) The distance between a point ξ and a set S is defined to be $d(\xi, S) = \inf_{x \in S} |\xi - x|$.

(a) If $\xi \in S$ prove that $d(\xi, S) = 0$.

(b) If S is bounded above and $\xi = \sup S$, prove that $d(\xi, S) = 0$. If S is bounded below, and $\xi = \inf S$ prove that $d(\xi, S) = 0$.

(c) If I is a closed interval, prove that $d(\xi, I) = 0$ implies that $\xi \in I$. If I is open, prove that we can always find an $\xi \notin I$ such that $d(\xi, I) = 0$.

Limits

Recall that a sequence $\{a_n\}_{n=1}^{\infty}$ converges to a limit a as $n \rightarrow \infty$ is for every $\epsilon > 0$ we can find an $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \epsilon$.

Use the definition of a limit to establish the following.

(5) $\lim_{n \rightarrow \infty} \frac{n}{2n+4} = \frac{1}{2}.$

(6) $\lim_{n \rightarrow \infty} \frac{2n+1}{3n+2} = \frac{2}{3}.$

(7) $\lim_{n \rightarrow \infty} \frac{n}{n^2+1} = 0.$

Use properties of limits to show that

(8) $\lim_{n \rightarrow \infty} \left(\frac{2n^3 - 3n}{5n^3 + 4n^2 - 2} \right) = \frac{2}{5}.$

(9) $\lim_{n \rightarrow \infty} (\sqrt{n^2 + 4} - n) = 0.$

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Tutorial Two.

The starred problems are harder.

Understanding Limits.

- (1) For what values of x does $\lim_{n \rightarrow \infty} \frac{x + x^n}{1 + x^n}$ exist?
- (2) Suppose that $\{y_n\}_{n=1}^{\infty}$ is a sequence of real numbers and $y_n \rightarrow 0$ as $n \rightarrow \infty$. Let $\{x_n\}_{n=1}^{\infty}$ be another sequence of real numbers. Suppose that for all n , $|x_n - l| \leq y_n$. Prove that $x_n \rightarrow l$.
- (3) Prove that the sequence $\left\{\left(1 + \frac{1}{n}\right)^n\right\}_{n=1}^{\infty}$ converges. Note that if a_1, \dots, a_n are positive, then $(a_1 a_2 \cdots a_n)^{\frac{1}{n}} \leq \frac{1}{n} \sum_{k=1}^n a_k$. (You will prove this later). Use your calculator to guess what the limit is.
- (4) (*) Let $x > 0$ and let N be the smallest natural number such that $N > x$. Prove that

$$\frac{x^n}{n!} \leq \frac{x^{N-1}}{(N-1)!} \left(\frac{x}{N}\right)^{n-N+1}, \quad n \geq N.$$

Conclude that $x^n/n! \rightarrow 0$ as $n \rightarrow \infty$. This result is essential for proving the convergence of power series.

- (5) (*) Let α be any positive rational number and let $|x| < 1$. Show that there exists a natural number N such that

$$(1 + 1/N)^{\alpha+1} |x| \leq 1.$$

Deduce that

$$|n^{\alpha+1} x^n| \leq |N^{\alpha+1} x^N|,$$

for $n \geq N$. Hence show that $n^{\alpha} x^n \rightarrow 0$ as $n \rightarrow \infty$. This is also important in establishing the convergence of certain kinds of series.

Subsequences.

- (6) Find a convergent subsequence of $\{\sin(\frac{\pi n}{2})\}_{n=1}^{\infty}$.
- (7) Suppose that $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence and for any N , we can find $n \geq N$, such that $x_n \geq b$. Show that x_n has a subsequence which converges to a limit $l \geq b$.
- (8) Find a convergent subsequence of $\left\{\frac{3^n + (-2)^n}{3^n - 2^n}\right\}_{n=1}^{\infty}$. What can you say about the limit in general?

- (9) It can be shown that $n^{1/n} \rightarrow 1$ as $n \rightarrow \infty$. Suppose that we know that the limit exists, but we do not know its value. Determine the limit by considering the behaviour of the subsequence $\{(2n)^{\frac{1}{2n}}\}$.

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Tutorial Three.

\limsup and \liminf .

- (1) Consider the sequence $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \dots$. Determine the \limsup and \liminf for this sequence.
- (2) Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence with limit superior given by l . Let the limit inferior be L . Show that for any $\epsilon > 0$ we can find $N > 0$ such that for all $n \geq N$, $x_n < l + \epsilon$. Formulate and prove the corresponding statement for x_n and L .

Cauchy Sequences.

- (3) Suppose that $|x_{n+1} - x_n| \leq r^n$ where $0 < r < 1$. Prove that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.
- (4) We have a sequence defined by the recursive formula

$$x_{n+2} = (x_{n+1}x_n)^{1/2}.$$

Suppose that $0 < a \leq x_2 \leq x_1 \leq b$. Prove that $a \leq x_n \leq b$ for all $n \geq 0$. Hence establish the inequality

$$|x_{n+1} - x_n| \leq \frac{b}{a+b} |x_n - x_{n+1}|.$$

Deduce that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence and hence converges.

- (5) Let $x_1 = a, x_2 = b$. Set $x_{n+2} = \frac{1}{2}(x_{n+1} + x_n)$ for all $n \geq 0$. Prove that the sequence $\{x_n\}_{n=1}^{\infty}$ converges.
- (6) How do calculators determine square roots? Most operations for determining function values are encoded into the hardware. Here is one algorithm. We let $x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n})$, $x_1 = x_0 > 0$, for $a > 0$. Prove that the sequence is Cauchy and that its limit is \sqrt{a} . Use this to obtain an approximation to the square root of 2.

The Bolzano-Weierstrass Theorem.

- (7) Show that every point in the interval $[0, 1]$ is the limit of a subsequence of the sequence defined in Question one.
- (8) (*) Given a set S of real numbers, let $S_{\xi} = \{x; x \in S, x \neq \xi\}$. We say that ξ is a limit point (or cluster point) of S if there is a sequence of points in S_{ξ} which converges to ξ . We can state a variation of the Bolzano-Weierstrass Theorem as follows. Every bounded set with an infinite number of elements contains at least one limit point. Prove this.

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Tutorial Four.

Finite Sums.

- (1) Evaluate the sum $\sum_{k=1}^N k$. Prove your result by induction.
- (2) Find a formula for the sum $\sum_{k=1}^N k^2$. Prove your formula by induction.
- (3) Sums of the form $\sum_{k=1}^N k^n$ can be shown to be given by polynomials in N of degree $n+1$. Use this to determine a formula for $\sum_{k=1}^N k^3$. You will need to solve a system of equations to find the coefficients of the fourth degree polynomial.

Infinite Sums.

- (4) Prove that the series $\sum_{n=1}^{\infty} \frac{1}{2n^2 + 3}$ is convergent.
- (5) Determine which of the following series converge and which diverge.
 - (i) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$, (ii) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n$, (iii) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n}$,
 - (iv) $\sum_{n=1}^{\infty} n^{\alpha} x^n, |x| < 1, \alpha > 0$, (v) $\sum_{n=1}^{\infty} \frac{x^n}{n!}, x \in \mathbb{R}$.
- (6) Prove that $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)(n+5)} = \frac{23}{480}$. Hint: Use partial fractions.
- (7) Prove that $\sum_{n=1}^{\infty} \frac{3n-2}{n(n+1)(n+2)} = 1$.
- (8) (*) The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is conditionally convergent. Let the sum be s . Let $S_N = \sum_{n=1}^N \frac{(-1)^{n+1}}{n}$. Now consider the rearranged series $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} - \frac{1}{7} \cdots$. Prove that this series converges to $\frac{1}{2}s$. (Hint: Look at the partial sum S_{3N} for this new series). The moral is that you cannot rearrange infinitely many terms in a conditionally convergent series and expect to get the same result.

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Tutorial Five.

Functions and their Properties.

- (1) Calculate the following limits.
 - (a) $\lim_{x \rightarrow 2} \frac{x}{x+3}$
 - (b) $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 4} - x)$
 - (c) $\lim_{x \rightarrow 0} \frac{\sin(2x)}{x}$
 - (d) $\lim_{x \rightarrow 2} \frac{x-2}{x^2-4}$
- (2) Prove Theorem 2.2 in the lecture notes.
- (3) Prove that the function $f(x) = x^2$ is continuous on any interval and that $\sin x$ is uniformly continuous on \mathbb{R} .
- (4) Prove that every polynomial is continuous everywhere.
- (5) A continuous function f is defined on an interval I and for every rational number $r \in I$, it satisfies $f(r) = r^2$. Prove that for all $x \in I$, $f(x) = x^2$.
- (6) Show that every polynomial of odd degree has at least one real root.
- (7) Let f be a continuous function on an interval $[a, b]$, where $-\infty < a < b < \infty$. Suppose that for every $x \in I$ there exists a $y \in I$ such that $|f(y)| \leq \frac{1}{2}|f(x)|$. Prove that there exists an $\xi \in I$ such that $f(\xi) = 0$.
- (8) Let $f : [a, b] \rightarrow [a, b]$ be continuous. Prove that f has a *fixed point*. That is, there exists $\xi \in [a, b]$ such that $f(\xi) = \xi$.
- (9) Prove that if I is an interval and f is continuous on I , then $f(I) = \{y \in \mathbb{R} : f(x) = y, x \in I\}$ is also an interval. So continuous functions map intervals to intervals.
- (10) Suppose that f is continuous on \mathbb{R} and that $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow -\infty} f(x) = 0$. Show that f attains its maximum and minimum values on \mathbb{R} .

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Tutorial Six.

The Derivative and its Applications.

- (1) Calculate the derivative of $f(x) = \cos x$ from first principles. Then determine it a second way, using the fact that $\frac{d}{dx} \sin x = \cos x$. (Hint: Use a trig identity).
- (2) Let $f(x) = \begin{cases} x, & x > 1 \\ x^2, & x \leq 1 \end{cases}$. Show that f is continuous everywhere, differentiable for $x \neq 1$, but not differentiable at $x = 1$.
- (3) Let $f(x) = \begin{cases} 2x, & x \geq 1 \\ x^2 + 1, & x < 1 \end{cases}$. Show that f is differentiable at $x = 1$ and $f'(1) = 2$.
- (4) Consider a polynomial P of degree n with the property that $P(\xi) = 0$ and $P'(\xi) = 0$. Prove that there is a polynomial Q of degree $n - 2$ such that $P(x) = (x - \xi)^2 Q(x)$.
- (5) Use induction to prove that $\frac{d^n}{dx^n} (fg) = \sum_{k=0}^n \binom{n}{k} \frac{d^k f}{dx^k} \frac{d^{n-k} g}{dx^{n-k}}$.
- (6) Suppose that f is such that $\frac{d}{dx}(f(x^2)) = \frac{d}{dx}(f(x))^2$. Prove that $f'(1) = 0$ or $f(1) = 1$.
- (7) Use the inverse function Theorem to give another proof of the fact that $\frac{d}{dx} e^x = e^x$.
- (8) Prove that $f(x) = (x + 1)^{1/n} - x^{1/n}$ decreases on $[0, \infty)$.
- (9) (*) Let f be differentiable, convex and bounded on \mathbb{R} . Prove that f is a constant.
- (10) (**) Let f be continuous on an interval I and satisfy

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2},$$

for each $x, y \in I$. Prove that for any set of points x_1, \dots, x_n contained in I ,

$$f\left(\frac{x_1 + x_2 + \dots + x_n}{n}\right) \leq \frac{1}{n}(f(x_1) + f(x_2) + \dots + f(x_n)).$$

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Tutorial Seven.

Taylor Series.

- (1) Prove that the Taylor series expansions about $a = 0$ for $\sin x$, $\cos x$ and e^x converge for all $x \in \mathbb{R}$.
- (2) Obtain the Taylor series expansions of $f(x) = \sin x$ and $g(x) = \cos x$ about the point $a = \frac{\pi}{2}$. What do you notice about the powers in the expansion?
- (3) Derive a Taylor series expansion for $f(x) = (1+x)^\alpha$, where α is not necessarily an integer. Prove that the series converges for $|x| < 1$.
- (4) Use the series in the previous question to obtain an approximation to $\sqrt{3/2}$.
- (5) Find a Taylor series expansion for $f(x) = \frac{x}{(1+x^2)^2}$ and determine its radius of convergence. Hint: There is an easy way to do this and a hard way.
- (6) Find a Taylor expansion for $f(x) = \ln(1+x)$ and determine its radius of convergence.
- (7) Determine the interval of convergence for the series $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$ and $\sum_{n=1}^{\infty} \frac{x^2}{n^2+1}$.
- (8) (*) Let $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Prove that f is infinitely differentiable at 0 and $f^{(n)}(0) = 0$ for all n . Hence the Taylor series expansion of f around $a = 0$ equals f only at $x = 0$.
- (9) Prove that the power series $\sum_{n=0}^{\infty} n^2 x^n$ is convergent for all $|x| < 1$ and calculate the sum. Hint compare with the geometric series.
- (10) Suppose that the power series $y = \sum_{n=0}^{\infty} a_n x^n$ is convergent everywhere and satisfies $y' = y$. Prove that it must be the series for the function $y = a_0 e^x$.

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Tutorial Eight.

The Riemann Integral.

- (1) Determine a formula for $\sum_{k=1}^n k^4$.
- (2) Use a Riemann sum to compute the value of the integral

$$\int_0^1 (x^4 + 3x^2 + 2x) dx.$$

- (3) Prove for any natural number n , that

$$\int_a^b x^n dx = \left[\frac{1}{n+1} x^{n+1} \right]_a^b.$$

- (4) Use a Riemann sum to show that for any positive rational number α that

$$\lim_{n \rightarrow \infty} \frac{1}{n^{\alpha+1}} (1^\alpha + 2^\alpha + \cdots + n^\alpha) = \frac{1}{1+\alpha}.$$

- (5) Suppose that f and g are continuous on $[a, b]$. Prove the Cauchy-Schwartz inequality

$$\left(\int_a^b f(x)g(x) dx \right)^2 \leq \int_a^b (f(x))^2 dx \int_a^b (g(x))^2 dx.$$

Hint: Consider the integral $\int_a^b (tf(x) + g(x))^2 dx$. Note that this is a quadratic in t and it is nonnegative. When is a quadratic nonnegative?

- (6) Let g be continuous on the interval $[a, b]$ and suppose that $g(x) \geq 0$ for all $x \in [a, b]$. If $\int_a^b g(x) dx = 0$ prove that g is identically equal to zero on $[a, b]$.
- (7) Suppose that f is twice differentiable on $[a, b]$ and that f'' is continuous on $[a, b]$. Prove the formula

$$\int_a^b x f''(x) dx = b f'(b) - f(b) - (a f'(a) - f(a)).$$

- (8) (*) Let f be positive and continuous on $[1, \infty)$. Now suppose that

$$F(x) = \int_1^x f(t) dt \leq (f(x))^2, \quad x \geq 1.$$

Prove that $f(x) \geq \frac{1}{2}(x-1)$ for $x \in [1, \infty)$. Hint: Consider the integral $\int_1^x \frac{F'(t)}{\sqrt{F(t)}} dt$.

- (9) Prove that the improper Riemann integral $\int_0^\infty \frac{dx}{1+x^2}$ converges and determine its value.
- (10) Prove that the improper Riemann integral $\int_0^1 \frac{dx}{\sqrt{x}}$ exists and determine its value.

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Tutorial Nine.

The Riemann Integral Continued.

- (1) (*) Let f be positive, continuous and decreasing on $[1, \infty)$. Prove that the sequence

$$\Delta_n = \sum_{k=1}^n f(k) - \int_1^n f(x) dx$$

is decreasing and bounded below by zero and so converges.

- (2) Use the previous question to prove that if $\int_1^\infty f(x) dx < \infty$, then the series $\sum_{n=1}^\infty f(n)$ is convergent. Conversely if the integral diverges, so does the infinite series.
- (3) Prove that $\sum_{n=1}^\infty \frac{1}{n^\alpha}$ converges if $\alpha > 1$ and diverges otherwise.
- (4) If f is continuous and increasing on $[0, \infty)$, prove that

$$\int_0^n f(x) dx \leq \sum_{k=1}^n f(k) \leq \int_1^{n+1} f(x) dx.$$

Show that $n \ln n - n \leq \ln(n!) \leq (n+1) \ln(n+1) - n$. Conclude that $\frac{n^n}{n!} \leq e^n \leq \frac{(n+1)^{n+1}}{n!}$.

- (5) Suppose that h is a positive continuous function on $[0, \infty)$. Let

$$H(x) = 1 + \int_0^x f(t) dt.$$

If $h(x) \geq H(x)$ show that for $x > 0$, $h(x) \geq e^x$.

Sequences of Functions.

- (6) Prove that $f_n(x) = x + 1/n$ converges uniformly to $f(x) = x$ on \mathbb{R} . Prove that $f_n^2 \rightarrow f^2$ pointwise on \mathbb{R} , but the convergence is not uniform.
- (7) Let $f_n(x) = x^{2n}(1 + x^{2n})^{-1}$. Let $f(x) = \lim_{n \rightarrow \infty} f_n(x)$. Show that $f(x) = 0$ for $|x| < 1$, $f(1) = f(-1) = 1/2$. and $f(x) = 1$ for $|x| > 1$. So that each f_n is continuous, but f is not continuous at $x = 1$ and $x = -1$.
- (8) Show that the series $\sum_{n=1}^\infty \frac{\cos(nx)}{n^4}$ is uniformly convergent on $[0, \infty)$.
- (9) Prove that the series $\sum_{n=1}^\infty \frac{x^n}{n^2}$ is uniformly convergent on $[0, 1]$.

- (10) Prove that $f_n(x) = nxe^{-nx^2} \rightarrow 0$ pointwise on \mathbb{R} , but the convergence is not uniform. Hint: Let $x = 1/\sqrt{n}$.
- (11) Let f be continuously differentiable on $[-\pi, \pi]$. Show that $\lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} f(x) \sin(n\pi x) dx = 0$. Hint: Integrate by parts.
- (12) Let $f_n(x) = \frac{x^2}{1 + e^{-xn}}$. Determine $\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx$.

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Tutorial Ten.

Review Problems.

- (1) Let $x_0 = 1$ and define the sequence $x_{n+1} = x_n + \frac{1}{x_n}$. This is an example of a *continued fraction*. Prove that the sequence converges and the limit is the so called *golden ratio*. (Look up the golden ratio to see that you have the correct answer).

- (2) Test the series $\sum_{n=1}^{\infty} \frac{n^2}{n^4+1}$ for convergence using the integral test.

- (3) If f and g are continuous on I , $a \in I$, prove that

$$\lim_{x \rightarrow a} (f \circ g)(x) = f(g(a)).$$

- (4) Prove that a Lipschitz continuous function on an interval $[a, b]$ is uniformly continuous on $[a, b]$.

- (5) If f is continuous and bounded on \mathbb{R} , prove that it is uniformly continuous on \mathbb{R} .

- (6) Calculate the Taylor series expansion of $f(x) = \cos^2 x$.

- (7) (*) Let $y'(x) = y(x)^2$ and suppose that $y(0) = 1$. By repeated applications of the chain rule, determine the first few terms of the Taylor series expansion of y .

- (8) Consider the integral $\int_0^1 x^2 dx$. Find numbers A, B and C such that

$$\int_0^1 x^2 dx = Af(0) + Bf(1/2) + Cf(1)$$

where $f(x) = x^2$. What happens if you apply this rule to $f(x) = x^3$?

- (9) The Gamma function is defined by $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$, $x > 0$. Prove that $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(n+1) = n!$ for n a natural number.

- (10) (i) Show that for $s > 1$,

$$\frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} e^{-nx} dx.$$

- (ii) Prove that for $s > 1$,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx.$$