

35007 Real Analysis.

Assignment. Part One

Full marks can be obtained by correct answers to five questions.

- (1) (i) Suppose that $Ax^2 + Bx + C \geq 0$. Show that $B^2 \leq AC$.

- (ii) Prove that if $a_1, \dots, a_n, b_1, \dots, b_n$ are any real numbers then
- $$(a_1b_1 + \dots + a_nb_n)^2 \leq (a_1^2 + \dots + a_n^2)(b_1^2 + \dots + b_n^2).$$

This is the Cauchy-Schwartz inequality. Hint: Consider an expression of the form

$$(a_1x + b_1)^2 + \dots + (a_nx + b_n)^2.$$

- (iii) Let a_1, \dots, a_n be positive real numbers. Let A_n denote the arithmetic mean and H_n the harmonic mean. ie. $A_n = \frac{1}{n} \sum_{k=1}^n a_k$ and $H_n = \left(\frac{1}{n} \left(\frac{1}{a_1} + \dots + \frac{1}{a_n} \right) \right)^{-1}$. Prove that $H_n \leq A_n$. Hint: Use (ii).

- (2) Let $\{y_n\}_{n=1}^\infty$ and $\{z_n\}_{n=1}^\infty$ be sequences such that $y_n \rightarrow l$ and $z_n \rightarrow l$. Suppose that for each n $y_n \leq x_n \leq z_n$ for all n . Prove that $x_n \rightarrow l$.

- (3) (i) Let

$$f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational.} \end{cases}$$

Prove that $\lim_{x \rightarrow 0} f(x)$ does not exist.

- (ii) Prove that $\lim_{x \rightarrow 0} xf(x) = 0$.

- (4) Suppose that $g(0) = g(1) = 0$ and that $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable at every point and $y = f(x)$ satisfies the differential equation $g(x)y'(x) + y(x) = 1$ for all $x \in [0, 1]$. Prove that $f(x) = 1$ for all $x \in [0, 1]$. Hint: Rolle's Theorem.

- (5) (*) Let $f : [0, 1] \rightarrow (0, \infty)$ be continuous on $[0, 1]$. Define a function $M : [0, 1] \rightarrow (0, \infty)$ by

$$M(x) = \sup_{0 \leq y \leq x} f(y), \quad (0 \leq x \leq 1).$$

Prove that if f is increasing then the function

$$\phi(x) = \lim_{n \rightarrow \infty} \left(\frac{f(x)}{M(x)} \right)^n,$$

is continuous. What happens if f is not increasing?

- (6) (*) Suppose that f is differentiable and convex on the open interval I . If $\xi \in I$, prove that $f(x) \geq f(\xi) + f'(\xi)(x - \xi)$ for $x \in I$.

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Assignment. Part Two

Full marks can be obtained by correct answers to five questions.

- (1) (i) Obtain a Taylor series expansion for the function $f(x) = \tan^{-1} x$. What is the radius of convergence?
(ii) Prove that $\pi = 16 \tan^{-1} \left(\frac{1}{5} \right) - 4 \tan^{-1} \left(\frac{1}{239} \right)$.
(iii) Use the first few terms of the series in (i) to obtain a rational approximation for π .
- (2) (*) Suppose that f has a continuous derivative on \mathbb{R} and that

$$f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right),$$

for all $xy < 1$. Prove that $f(x) = C \tan^{-1} x$ for some constant C . Hint: Look up the inverse tangent function to remind yourself of its properties.

- (3) (i) Prove that $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x$.
(ii) Find a polynomial that uniformly approximates $f(x) = e^x$ on $[0, 1]$ to an accuracy of less than .01.
- (4) Prove that if $n > m > 0$ then

$$\left| \int_m^n \frac{\sin t}{t} dt \right| < \frac{4}{m}.$$

Hence show that $\int_1^\infty \frac{\sin t}{t} dt$ exists.

- (5) (*) Suppose that g is continuous on $[a, b]$ and that f is differentiable on (a, b) and its derivative is continuous on $[a, b]$ with $f'(t) \geq 0$ for $t \in [a, b]$. Prove that for some $\xi \in [a, b]$ we can write

$$\int_a^b f(t)g(t)dt = f(a) \int_a^\xi g(t)dt + f(b) \int_\xi^b g(t)dt.$$

Hint: Integration by parts will help.

- (6) Let y be a solution of the differential equation $y'' + y = 0$ such that $y(0) = 1$ and $y'(0) = 0$. Use the differential equation to construct a series expansion of the solution. Show that you obtain the series expansion for $y = \cos x$. Hint:

$$y(x) = y(0) + y'(0)x + y''(0)\frac{x^2}{2} \cdots$$

Further $y''' = \frac{d}{dx}y''$ etc.