

## Tutorial 6 - 35007 Real Analysis.

(Q1)

$$f(x) = \cos x$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\cos(x + \Delta x) = \cos x \cos \Delta x - \sin x \sin \Delta x.$$

$$\lim_{\Delta x \rightarrow 0} \frac{\cos(x + \Delta x) - \cos x}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} -\sin x \cdot \frac{\sin \Delta x}{\Delta x}$$

$$= -\sin x \cdot \frac{\sin \Delta x}{\Delta x} \rightarrow \frac{\Delta x}{\Delta x} = 1 \text{ as } \Delta x \rightarrow 0$$

$$\sin u \sim u - \frac{u^3}{3!} + \frac{u^5}{5!} - \dots$$

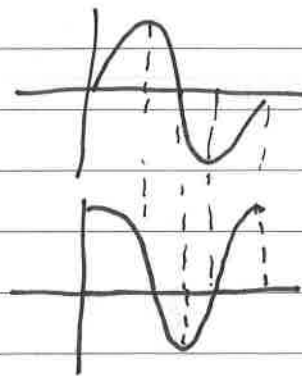
→ terms getting progressively smaller.

ii) Now  $\frac{d}{dx} (\sin(x+a)) = \cos(x+a)$

ie/  $\frac{d}{dx} (\sin(x + \frac{\pi}{2})) = \cos(x + \frac{\pi}{2})$

or  $\frac{d}{dx} (\cos x) = -\sin x$

as  $\frac{d}{dx} (x+a) = 1 = \frac{dz}{dx}$



(iii) Let  $h = \Delta x$ , take  $h \in [0, \pi]$ .

then  $\sin(h) \leq h \leq \tan(h)$  (Plot it!)

$$\nRightarrow 1 \leq \frac{h}{\sin(h)} \leq \frac{1}{\cos(h)}$$

$$\nRightarrow \cos(h) \leq \frac{\sin(h)}{h} \leq 1$$

$$\lim_{h \rightarrow 0} \cos(h) = \lim_{h \rightarrow 0} 1 = 1$$

So by the sandwich theorem  $\lim_{h \rightarrow 0} \frac{\sin(h)}{h} = 1$ .

(Q2)

$$f(x) = \begin{cases} x & x > 1 \\ x^2 & x \leq 1 \end{cases}.$$

(i) Continuous everywhere:

$f(x) \forall x \neq 1$  is continuous as  $f$  is a polynomial on this domain  $\nRightarrow$  polynomials are continuous  $\forall x \in \mathbb{R}$ .

$$\text{At } x = 1, \text{ we have } \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} x = 1.$$

$$\nRightarrow \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} x^2 = 1^2 = 1.$$

$$\text{hence } \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = f(1)$$

and the function defined by  $f(x)$  is continuous everywhere.

Q2 i) In  $x < 1$ , we have

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{x + \Delta x - x}{\Delta x} = 1.$$

if  $x > 1$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2\Delta x \cdot x + (\Delta x)^2 - x^2}{\Delta x} \\ &= 2x. \end{aligned}$$

At  $x=1$ , we have

$$\lim_{x \rightarrow 1^-} f'(x) = 1$$

$$\lim_{x \rightarrow 1^+} f'(x) = 2 \cdot 1 \neq \lim_{x \rightarrow 1^-} f'(x)$$

hence  $f'(x)$  is not well defined at  $x=1$ .

Q3)

$$f(x) = \begin{cases} 2x & x \geq 1 \\ x^2 + 1 & x < 1. \end{cases}$$

$$f'(x) = \begin{cases} 2 & x \geq 1 \\ 2x & x < 1. \end{cases}$$

$$\begin{aligned} \text{at } x=1, \text{ we have } \lim_{x \rightarrow 1^-} f'(x) &= 2 \cdot 1 = \lim_{x \rightarrow 1^+} f'(x) \\ &= 2. \end{aligned}$$

$$f(1^-) = 2 \cdot 1 = 2, \quad f(1^+) = 1 + 1 = 2.$$

$\therefore f(x)$  is continuous & differentiable at  $x=1$ .

(Q4)

$$p_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

$$\text{let } p(\xi) = 0, \quad p'(\xi) = 0.$$

$$\text{then } a_n \xi^n + a_{n-1} \xi^{n-1} + \dots + a_1 \xi + a_0 = 0.$$

$$p'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \dots + 2 a_2 x + a_1$$

$$p'(\xi) = 0 \text{ implies}$$

$$n a_n \xi^{n-1} + (n-1) a_{n-1} \xi^{n-2} + \dots + 2 a_2 \xi + a_1 = 0$$

$$p(\xi) = 0 \Rightarrow (x - \xi) Q(x) = p(x)$$

$$\text{then } \frac{dp}{dx} = \frac{d}{dx} (x Q(x)) - \xi \frac{dQ}{dx}.$$

$$= x \frac{dQ}{dx} + Q(x) - \xi \frac{dQ}{dx}.$$

$$= x Q'(x) + Q(x) - \xi Q'(x)$$

$$= (x - \xi) Q'(x) + Q(x).$$

$$p'(\xi) = Q(\xi) + (\xi - \xi) Q'(\xi) = Q(\xi) \\ = 0$$

$$\therefore Q(\xi) \text{ has a root } Q(\xi) = 0$$

$$\Rightarrow Q(x) = (x - \xi) R(x)$$

$$\therefore p(x) = (x - \xi)^2 R(x).$$

(Q5) Liebniz's Rule for Repeated Differentials:

$$y = f(x) g(x)$$

$$y' = fg' + gf'$$

$$\begin{aligned} y'' &= fg'' + f'g' + g'f' + gf'' \\ &= fg'' + 2f'g' + gf'' \end{aligned}$$

$\vdots$

$$y^{(n)} = \frac{d^n y}{dx^n} = \frac{d^n}{dx^n} (f(x)g(x))$$

We can see the binomial expansion applies here:

$$y^{(n)} = \sum_{k=0}^n {}^n C_k \frac{d^k f}{dx^k} \frac{d^{n-k} g}{dx^{n-k}}$$

Proof by induction:

Assume true for  $j=n$ , then

$$\begin{aligned} y^{(n+1)} &= \frac{d}{dx} \sum_{k=0}^n {}^n C_k f^{(k)} g^{(n-k)} \\ &= \sum_{k=0}^n {}^n C_k \left[ f^{(k+1)} g^{(n-k)} + f^{(k)} g^{(n-k+1)} \right] \\ &= \sum_{k=0}^n {}^n C_k f^{(k)} g^{(n-k+1)} + \sum_{k=1}^{n+1} {}^n C_{k-1} f^{(k)} g^{(n-k+1)} \quad \downarrow \text{index shift} \\ &= {}^n C_0 f^{(0)} g^{(n-1)} + {}^n C_n f^{(n+1)} g^{(0)} \\ &+ \sum_{k=1}^n \left[ {}^n C_k f^{(k)} g^{(n-k+1)} + {}^n C_{k-1} f^{(k)} g^{(n-k+1)} \right]. \end{aligned}$$

$$\text{Now } {}^nC_0 = {}^{(n+1)}C_0 \\ = {}^nC_n = {}^{(n+1)}C_{n+1} = 1.$$

$$\begin{aligned} & \nexists \quad {}^nC_k + {}^nC_{k-1} \\ &= \frac{n!}{(n-k)!k!} + \frac{n!}{(n-(k-1))!(k-1)!} \\ &= n! \left[ \frac{1}{(n-k)!k!} + \frac{1}{(n+1-k)!(k-1)!} \right] \\ &= \frac{n!}{(k-1)!} \left[ \frac{1}{(n-k)!k} + \frac{1}{(n+1-k)!} \right] \\ &= \frac{n!}{(k-1)!} \left[ \frac{1}{(n-k)!} \left[ \frac{1}{k} + \frac{1}{n+1-k} \right] \right] \\ &= \frac{n!}{(k-1)!(n-k)!} \left[ \frac{n+1-k+k}{k(n+1-k)} \right] \\ &= \frac{(n+1)n!}{(k-1)!k(n-k)! \cdot (n+1-k)} \\ &= \frac{(n+1)!}{k!(n+1-k)!} \end{aligned}$$

$$= {}^{n+1}C_k$$

$$\therefore \text{ we have } y^{(n+1)} = \sum_{k=0}^{n+1} ({}^{n+1}C_k) g^{(n+1-k)} f^{(k)}$$

so true for  $j=n+1$  , true for  $n=1,2,3,\dots$

$$(Q6) \quad \frac{d}{dx} f(x^2) = \frac{d}{dx} (\cancel{f} x^2) f'(x^2) \\ = 2x f'(x^2).$$

$$\frac{d}{dx} (f^2) = 2f \frac{df}{dx} \\ = 2ff'.$$

$$\therefore 2x f'(x^2) = 2f(x) f'(x).$$

$$x f'(x^2) = f(x) f'(x).$$

$$\text{If } x=1, \quad f'(1) = f(1) f'(1)$$

$$\Rightarrow \quad x = ax \text{ is only true for } a = \cancel{0} 1. \\ f'(1) = 1 \text{ or } 0.$$

$$(Q7) \quad f(x) = e^x \\ x = f^{-1}(y). \\ f(f^{-1}(x)) = f^{-1}(f(x)) = x. \\ e^{\ln x} = \ln(e^x) = x.$$

$$\ln x = \int_1^x \frac{dy}{y}.$$

$$\frac{d}{dx} (\ln(e^x)) = \frac{d}{dx} (x) = 1.$$

$$\therefore \frac{1}{e^x} \frac{d}{dx} (e^x) = 1 \Rightarrow \frac{d}{dx} (e^x) = e^x.$$

Alternate proof: Taylor Series (Next lecture...)

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$f'(x) = 1 + \frac{2}{2!} x + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \dots = 1 + x + \frac{x^2}{2!} + \dots = f(x).$$

(Q8)

$$f(x) = (1+x)^{\frac{1}{n}} - x^{\frac{1}{n}}$$

Prove that  $f(x)$  decreases on  $[0, \infty)$ .

$$f'(x) = \frac{1}{n} (1+x)^{\frac{1}{n}-1} \cdot 1 - \frac{1}{n} x^{\frac{1}{n}-1}$$

$$= \frac{1}{n} (1+x)^{\frac{1-n}{n}} - \frac{1}{n} x^{\frac{1-n}{n}}$$

Let  $n > 1$ , then  $f(0) = 1$ .

Now if  $n = 1$ ,  $f(x) = 1 + x - x = 1$ .

$$\forall x > 0 \quad (1+x)^n > x^n,$$

$$\therefore \frac{1}{x^n} > \frac{1}{(1+x)^n}$$

~~$$\frac{(x+1)}{(x+1)^n} < \frac{(x+1)}{x^n}$$~~

$$\frac{1}{(1+x)^n} < \frac{1}{x^n} \Rightarrow \frac{1}{(1+x)^{n-1}} < \frac{1}{x^{n-1}}$$

$$\Rightarrow \frac{(1+x)}{(1+x)^n} < \frac{x}{x^n}$$

$$(x+1)^{\frac{1}{n}-1} = \frac{(x+1)^{\frac{1}{n}}}{(x+1)}$$

$$\left[ \frac{(x+1)^{\frac{1}{n}}}{(x+1)} \right]^n = \frac{(x+1)}{(x+1)^n} < \frac{x}{x^n}$$



$$\text{so } f'(x) = \frac{1}{h} \left[ (1+x)^{\frac{1}{h}-1} - x^{\frac{1}{h}-1} \right].$$

$$= \frac{1}{h} \left[ \frac{(x+1)^{\frac{1}{h}}}{(x+1)} - \frac{x^{\frac{1}{h}}}{x} \right]$$

$$= \frac{1}{h} \left[ m \right] < 0$$

$$\frac{(1+x)}{(1+x)^n} < \frac{x}{x^n}$$

$$\frac{(1+x)^{\frac{1}{h}}}{(1+x)} < \frac{x^{\frac{1}{h}}}{x}$$

$$\therefore \frac{(1+x)^{\frac{1}{h}}}{(1+x)} - \frac{x^{\frac{1}{h}}}{x} < 0$$

hence  $f'(x) < 0$ ,  $f$  therefore decreases on  $[0, \infty)$ .