

# Solutions Tutorial 3

①

(1) Clearly there is a subsequence  
 $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$

which converges to 0

There is a subsequence

$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$

which converges to 1.

Let sequence be  $r_n$ .

$$\text{Then } \limsup_{n \rightarrow \infty} r_n = 1$$

$$\liminf_{n \rightarrow \infty} r_n = 0$$

(2) Suppose that it is false, that for any  $\varepsilon > 0$ , there exists an  $N$  such that, for any  $n > N$ ,  $x_n < l + \varepsilon$ . Then for some  $\varepsilon > 0$  it is true that for each  $N$  we can find an  $n > N$  such that  $x_n \geq l + \varepsilon$ . Bolzano-Weierstrass implies that there is a convergent subsequence with limit  $\bar{l} \geq l + \varepsilon$ , which contradicts the definition of  $l$ .

For  $L$ , we can say that for any  $\varepsilon > 0$ , we can find an  $N$  such that for any  $n > N$ ,  $x_n > L - \varepsilon$ .

$$(3) |x_{n+1} - x_n| \leq r^n, \quad 0 < r < 1.$$

$$\text{Consider } |x_m - x_n| = |x_m - x_{m-1} + x_{m-1} - x_{m-2} + \dots + x_{n+1} - x_n|$$

$$\begin{aligned} &\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \dots + |x_{n+1} - x_n| \\ &= r^{m-1} + r^{m-2} + \dots + r^n \\ &= r^n (1 + r + \dots + r^{m-n-1}) \end{aligned}$$

(2)

$$= \frac{r^{m+2} - r^n}{r-1} \rightarrow 0 \text{ as } m, n \rightarrow \infty$$

since  $|r| < 1$ .

Thus the sequence is Cauchy.

(4) We have  $x_{n+2} = (x_{n+1} x_n)^{1/2}$   
and  $0 < a \leq x_1 \leq x_2 \leq b$ .

First note that

$$\begin{aligned} x_{n+2}^2 - x_{n+1}^2 &= x_{n+1} x_n - x_{n+1}^2 \\ &= x_{n+1} (x_n - x_{n+1}) \end{aligned}$$

$$\text{So } (x_{n+2} - x_{n+1})(x_{n+2} + x_{n+1}) = x_{n+1} (x_n - x_{n+1})$$

$$\text{or } x_{n+2} - x_{n+1} = \frac{x_{n+1} (x_n - x_{n+1})}{x_{n+2} + x_{n+1}}$$

Now  $a \leq x_n \leq b$  implies

$$|x_{n+2} - x_{n+1}| \leq \frac{b}{a+b} |x_n - x_{n+1}|$$

$$\text{Iterating we have } |x_{n+1} - x_n| \leq \left( \frac{b}{a+b} \right)^{n-1} |x_2 - x_1|$$

Now the proof is similar to the previous question.

The fact that  $a \leq x_n \leq b$  is obvious.

$$x_3 = (x_2 x_1)^{1/2} \leq (b^2)^{1/2} = b, \text{ so } x_4 = (x_3 x_2)^{1/2} \leq (b^2)^{1/2} = b$$

$$\text{etc. Similarly for } x_3 = (x_2 x_1)^{1/2} > (a^2)^{1/2} = a$$

etc

(5) We have  $x_{n+2} = \frac{1}{2}(x_{n+1} + x_n)$ ,  $x_1 = a, x_2 = b$ .  
We consider

$$\begin{aligned} x_{n+2} - x_{n+1} &= \frac{1}{2}(x_{n+1} + x_n) - x_{n+1} \\ &= \frac{1}{2}(x_n - x_{n+1}) \end{aligned}$$

(3)

$$\text{Thus } |x_{n+2} - x_{n+1}| = \frac{1}{2} |x_n - x_{n+1}| = \frac{1}{2^2} |x_n - x_{n-1}|$$

$$= \dots = \frac{1}{2^n} |x_2 - x_1| = \frac{1}{2^n} |b - a|$$

Hence if  $n > m$

$$\begin{aligned} |x_n - x_m| &= |x_n - x_{n-1} + x_{n-1} - x_{n-2} + \dots + x_{m+1} - x_m| \\ &\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \dots + |x_{m+1} - x_m| \\ &\leq \left( \frac{1}{2^{n-2}} + \frac{1}{2^{n-3}} + \dots + \frac{1}{2^{m-1}} \right) |b - a| \end{aligned}$$

$$= \frac{1}{2^{m-1}} \left( 1 + \frac{1}{2} + \dots + \frac{1}{2^{n-m-1}} \right) |b - a|$$

$$= \frac{1}{2^{m-1}} \frac{1 - \left(\frac{1}{2}\right)^{n-m}}{1 - \frac{1}{2}} |b - a| \leq \frac{1}{2^{m-2}} |b - a|$$

$$\rightarrow 0 \text{ as } m \rightarrow \infty$$

So  $\{x_n\}$  is Cauchy and thus convergent

(6) First suppose the sequence converges. Let  $x_n \rightarrow x$ . Then  $\lim_{n \rightarrow \infty} x_{n+1} = x = \lim_{n \rightarrow \infty} \frac{1}{2} \left( x_n + \frac{a}{x_n} \right)$

$$= \frac{1}{2} \left( x + \frac{a}{x} \right)$$

$$\text{So } x^2 = \frac{1}{2} x^2 + \frac{a}{2}$$

$$\text{or } x^2 = a \quad \text{sequence is}$$

positive so we take  $+\sqrt{\phantom{x}}$  and  $x_n \rightarrow \sqrt{a}$ .

Take  $a=2$ ,  $x_0=1$

$$x_1 = \frac{1}{2} (1 + 2) = \frac{3}{2} = 1.5$$

$$x_2 = \frac{1}{2} \left( \frac{3}{2} + 2 / \left( \frac{3}{2} \right) \right) = \frac{17}{12} = 1.41667$$

$$x_3 = \frac{1}{2} \left( \frac{17}{12} + 2 / \left( \frac{17}{12} \right) \right) = \frac{577}{408} = 1.41422$$

$$x_4 = \frac{1}{2} \left( \frac{577}{408} + 2 / \left( \frac{577}{408} \right) \right) = \frac{665857}{470832} = 1.41421$$

So only 4 steps gives a very accurate answer

(4)

To prove convergence let  $y_n = \frac{x_n}{\sqrt{a}}$ .  
 we show  $y_n \rightarrow 1$ .

Simple algebra shows that

$$y_{n+1} - 1 = \frac{(y_n - 1)^2}{2y_n}$$

$$y_{n+1} + 1 = \frac{(y_n + 1)^2}{2y_n}$$

$$\begin{aligned} \text{So } \frac{y_{n+1} - 1}{y_{n+1} + 1} &= \frac{(y_n - 1)^2}{(y_n + 1)^2} = \left( \frac{y_n - 1}{y_n + 1} \right)^2 \\ &= \dots = \left( \frac{y_0 - 1}{y_0 + 1} \right)^{2^n} \rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$  provided  $|y_0 - 1| < |y_0 + 1|$ .  
 This holds if  $x_0 > 0$ .

Thus  $y_{n+1} \rightarrow 1$  or  $x_n \rightarrow \sqrt{a}$ .

(7) Every rational number in the interval  $(0,1)$  occurs infinitely often as a term in the sequence. Let  $x \in [0,1]$  a term  $r_n$  of the sequence can be found such that

$$x - \frac{1}{n} < r_n < x + \frac{1}{n}$$

A term  $r_{n_2}$  with  $n_2 > n_1$  can be found such that  $x - \frac{1}{2} < r_{n_2} < x + \frac{1}{2}$

and so on. So we can a subsequence  $r_{n_k}$  such that  $x - \frac{1}{k} < r_{n_k} < x + \frac{1}{k}$

and so  $r_{n_k} \rightarrow x$ .

To understand this, write out the sequence to a high number of terms and pick some point in  $[0,1]$ . Try and identify  $r_n, r_m, \dots$  etc for your point.

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(8) Let  $(x_n)$  be a sequence of distinct points of  $S$ . Since  $\{x_n\}$  is bounded, it contains a convergent subsequence  $\{x_{n_r}\}$ . Suppose that  $x_{n_r} \rightarrow \xi$  as  $r \rightarrow \infty$ . Then  $\xi$  is a limit point of  $S$ .