

# Tutorial One Solutions

①

(1) Let  $B = \sup S$ . Then for all  $x \in S$

$x \leq B$ .  
Let  $T = \{\frac{x}{3} : x \in S\}$ . Since  $\frac{1}{3} > 0$   
 $\frac{1}{3}x \leq \frac{1}{3}B$ , for any  $x \in S$ .

Hence  $T$  is bounded above by  $\frac{1}{3}B$ .

So  $T$  has a least upper bound, call this  $C$ .  
We show  $C = \frac{1}{3}B$ . Clearly  $C \leq \frac{1}{3}B$ .

Now reverse roles of  $S$  and  $T$ .  $C$  is the  
smallest number such that for any  $y \in T$ ,  
 $y \leq C$ . Since  $\frac{1}{3} > 0$

$$\frac{1}{3}y \leq \frac{1}{3}C$$

for any  $y \in T$ . But  $S = \{\frac{y}{\frac{1}{3}} : y \in T\}$ . Hence  
 $\frac{1}{3}C$  is an upper bound for  $S$ . But  $B$  is

the smallest upper bound for  $S$ . Thus  
 $\frac{1}{3}C \geq B$  or  $C \geq \frac{1}{3}B$ .

$$\text{So } C \geq \frac{1}{3}B, C \leq \frac{1}{3}B.$$

$\therefore C = \frac{1}{3}B$  and thus

$$\sup(\frac{x}{3}) = \frac{1}{3} \sup x.$$

(2)  $S$  is bounded above. Let  $\sup S = B$ .  
So for all  $x \in S$ ,  $x \leq B$ . But  $S_0 \subseteq S$ ,  
so for  $y \in S_0$   
 $y \leq \sup S_0 \leq B \leq \sup S$

(3) Let  $B = \sup S$ ,  $T = \{x+1 : x \in S\}$ . Since  
 $x \leq B$  for all  $x \in S$ ,  $x+1 \leq B+1$  for  $x \in S$ .  
Hence  $B+1$  is an upper bound for  $T$ . Let  
 $C$  be the smallest upper bound for  $T$ , then

(2)

$C \leq \xi + B$ , Now  $y \leq C$  for any  $y \in T$ ,  
So that

$$y - \xi \leq C - \xi \text{ for any } y \in T.$$

Since  $S = \{y - \xi : y \in T\}$  we see that  $C - \xi$  is an upper bound for  $S$ .

$$\therefore B \leq C - \xi.$$

$$B + \xi \leq C.$$

$$\therefore B + \xi \geq C \text{ and } B + \xi \leq C.$$

$$\text{So } C = B + \xi.$$

(4). (i) Let  $D = \{|\xi - x| : x \in S\}$ . This has zero as a lower bound, (It cannot have any negative elements). If  $\xi \in S$ , then the minimum is 0.

ii) Let  $\xi = \sup S$ . Then  $|\xi - x| = \xi - x$  for each  $x \in S$ . We show that no  $h > 0$  is a lower bound for  $D = \{\xi - x : x \in S\}$ . Suppose not, then we can find an  $h > 0$  such that

$\xi - x \geq h$  for all  $x \in S$ . But then  $x \leq \xi - h$  for all  $x \in S$ , hence  $\xi - h$  is an upper bound for  $S$ , smaller than the least upper bound. A contradiction. For the

second case ( $\xi = \inf S$ ), the proof is similar, with  $d(\xi, T)$ ,  $T = \{-x : x \in S\}$ .

ii) Since  $I$  is an interval,  $\xi \notin I \Rightarrow \xi$  is an upper or lower bound for  $I$ . Suppose it is an upper bound. Let  $B$  be the least upper bound of  $I$ . Then  $B \in I$  because  $I$  is closed.

$$\text{Given any } x \in I \quad |\xi - x| = \xi - x = \xi - B + B - x$$

$$= \xi - B + |B - x|$$

$$\text{Hence } \inf_{x \in B} |\xi - x| = \xi - B + \inf_{x \in B} |B - x| \quad (\text{Why?})$$

(3)

and therefore  $d(\bar{z}, S) = \bar{z} - B + d(B, S)$ . But

$$\bar{z} - B \geq 0, \quad d(B, S) \geq 0 \quad \text{and} \quad d(\bar{z}, S) = 0$$

It follows that  $\bar{z} = B$  and hence  $\bar{z} \in T$ .

If  $\bar{z}$  is a lower bound, the proof is similar.

# Limits

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(5)  $\lim_{n \rightarrow \infty} \frac{n}{2n+4} = \frac{1}{2}$ . Let  $\varepsilon > 0$  then

$$\left| \frac{n}{2n+4} - \frac{1}{2} \right| = \left| \frac{2n - (2n+4)}{2(2n+4)} \right| = \frac{2}{|2n+4|}$$

$$\text{So } \left| \frac{n}{2n+4} - \frac{1}{2} \right| < \varepsilon \Rightarrow \frac{1}{|2n+4|} < \frac{\varepsilon}{2}$$

$$\text{Since } n > 0, \quad 2n+4 > \frac{2}{\varepsilon}$$

$$\text{or } n > \frac{1}{2} \left( \frac{2}{\varepsilon} - 4 \right) = \frac{1}{\varepsilon} - 2.$$

Choose the smallest integer  $N$  such that  $N > 0$  and  $N \geq \frac{1}{\varepsilon} - 2$ .

Then for all  $n \geq N$ ,  $\left| \frac{n}{2n+4} - \frac{1}{2} \right| < \varepsilon \Rightarrow$

$$\lim_{n \rightarrow \infty} \frac{n}{2n+4} = \frac{1}{2}.$$

(6)  $\lim_{n \rightarrow \infty} \frac{2n+1}{3n+2} = \frac{2}{3}$ . Let  $\varepsilon > 0$ . Then

$$\left| \frac{2n+1}{3n+2} - \frac{2}{3} \right| = \left| \frac{3(2n+1) - 2(3n+2)}{3(3n+2)} \right|$$

$$= \left| \frac{3-4}{3(3n+2)} \right| = \frac{1}{3(3n+2)} < \varepsilon$$

$$\Rightarrow 3(3n+2) > \frac{1}{\varepsilon}, \text{ So } 3n+2 > \frac{1}{3\varepsilon}$$

$$\text{or } 3n > \frac{1}{3\varepsilon} - 2$$

$$\text{Hence } n > \frac{1}{9\varepsilon} - \frac{2}{3}.$$

Choose  $N > 0$ , such that  $N$  is an integer and  $N \geq \frac{1}{9\varepsilon} - \frac{2}{3}$ .

$$\text{Thus } n \geq N \Rightarrow \left| \frac{2n+1}{3n+2} - \frac{2}{3} \right| < \varepsilon.$$

(7)  $\lim_{n \rightarrow \infty} \frac{n}{n^2+1} = \lim_{n \rightarrow \infty} \frac{1}{n + \frac{1}{n}}$ . Now let  $\varepsilon > 0$

(4)

2 limit

(5)

Then  $\left| \frac{1}{n+1/n} \right| < \varepsilon$  implies  $n + \frac{1}{n} > \frac{1}{\varepsilon}$ .

Clearly  $n > \frac{1}{\varepsilon} \Rightarrow n + \frac{1}{n} > \frac{1}{\varepsilon}$ .

Choose  $N \geq \frac{1}{\varepsilon}$  then  $n \geq N \Rightarrow \left| \frac{n}{n^2+1} \right| < \varepsilon$

So  $\frac{n}{n^2+1} \rightarrow 0$ .

$$(8) \lim_{n \rightarrow \infty} \frac{2n^3 - 3n}{5n^3 + 4n^2 - 2} = \frac{2}{5} \lim_{n \rightarrow \infty} \frac{n^3}{n^3 + \frac{4}{5}n^2 - \frac{2}{5}}$$

$$= \frac{2}{5} \lim_{n \rightarrow \infty} \frac{n}{5n^3 + 4n^2 - 2}$$

$= \frac{2}{5}$  (Proof of the individual limits is an exercise)

$$(9) \lim_{n \rightarrow \infty} (\sqrt{n^2+4} - n) = \lim_{n \rightarrow \infty} (\sqrt{n^2+4} - n) \left( \frac{\sqrt{n^2+4} + n}{\sqrt{n^2+4} + n} \right)$$

$$= \lim_{n \rightarrow \infty} \frac{4}{\sqrt{n^2+4} + n} = 0$$

since  $\frac{1}{\sqrt{n^2+4} + n} \rightarrow 0$ , (Exercise)

$$\left| \frac{4}{\sqrt{n^2+4} + n} \right| < \varepsilon \Leftrightarrow \frac{4}{\sqrt{n^2+4} + n} < \varepsilon$$

$$4 < \varepsilon (\sqrt{n^2+4} + n) \Rightarrow \sqrt{n^2+4} + n > \frac{4}{\varepsilon}$$

$$\sqrt{n^2+4} > \frac{4}{\varepsilon} - n \Rightarrow \sqrt{n^2+4} > \frac{4}{\varepsilon} - n$$

$$\sqrt{n^2+4} > \frac{4}{\varepsilon} - n$$

Choose  $n > 0$  such that  $\frac{4}{\varepsilon} - n > 0$

$$\sqrt{n^2+4} > \frac{4}{\varepsilon} - n$$

$$\left| \frac{4}{\sqrt{n^2+4} + n} \right| < \varepsilon \Rightarrow \frac{4}{\sqrt{n^2+4} + n} < \varepsilon$$

Now let  $\varepsilon > 0$

Choose  $n > 0$  such that  $\frac{4}{\varepsilon} - n > 0$