

(Q1) Let $B = \sup(S)$

Tutorial #1 Solution

Then $\forall x \in S, x \leq B$.

Let $T = \{ \frac{1}{2}x \mid x \in S \}$, with $\frac{1}{2} > 0$.

We have $\frac{1}{2}x \leq \frac{1}{2}B \quad \forall x \in S$.

So $\sup(\frac{1}{2}x) = C$.

We have $C \leq \frac{1}{2}B$.

Now let us reverse S, T . Then we have

$$\frac{1}{2}y \leq \frac{1}{2}C, \quad y \in T.$$

$$S = \{ \frac{y}{2} \mid y \in T \}.$$

$$\therefore \frac{1}{2}C \geq B.$$

So we have $C \geq 2B, C \leq \frac{1}{2}B$.

$$\text{or } C = \frac{1}{2}B.$$

$$\begin{aligned} \text{So } \sup_{x \in S} (\frac{1}{2}x) &= \frac{1}{2}B \\ &= \frac{1}{2} \sup_{x \in S} x. \end{aligned}$$

(Q2) S is bounded above.

$$\sup_{x \in S} x = B.$$

$$\forall x \in S, x \leq B.$$

Now $S_0 \subseteq S$, so $\forall y \in S_0$

$$y \leq \sup_{y \in S_0} (S_0) \leq B \leq \sup_{x \in S} (S).$$

$$(3) \text{ let } B = \sup_{x \in S} (S).$$

$$T = \{x + \gamma \mid x \in S\}.$$

then $x \leq B \forall x \in S$.

$$\gamma + x \leq \gamma + B \forall x \in S.$$

so $\gamma + B$ is an upper bound.

Let C be smallest upper bound, we have

$$C \leq \gamma + B.$$

$$\therefore \forall y \in T \text{ we have } y \leq C.$$

$$\text{So } y - \gamma \leq C - \gamma.$$

$$\therefore S = \{y - \gamma \mid y \in T\}.$$

$$\therefore B \leq C - \gamma.$$

$$\therefore C = B + \gamma.$$

(iii) Let I be an ^{closed} ~~open~~ interval, ~~$\xi \notin I$~~ $\xi \notin I$.

So ξ is either sup or inf of I .

Suppose $\xi = \sup(I) = B$.

$B \in I$, then $\forall x \in I$ we have:

$$|\xi - x| = \xi - x = \xi - B + B - x.$$

$$= |B - x| + \xi - B.$$

$$\therefore \inf_{x \in B} |\xi - x| = \inf_{x \in B} (\xi - B + |B - x|).$$

$$= \xi - B + \inf_{x \in B} |B - x|$$

as $\xi - B$ does not depend on x .

$$\text{then } d(\xi, S) = \xi - B + d(B, S).$$

$$\text{Now } \xi - B \geq 0, \quad d(B, S) \geq 0$$

$$d(\xi, S) = 0$$

So $\xi = B$, hence $\xi \in I$.

(04) i) Let $D = \{ |z - x| : x \in S \}$.

$$\min(D) = 0$$

Assume $z \in S$, then $\inf(D) = 0$

(ii) Let $z = \sup(S)$. $|z - x| = z - x$ as $z > x$.

Proof by contradiction.

Assume $z - x \geq \epsilon \quad \forall x \in S$.

then $x \leq z - \epsilon \quad \forall x \in S$.

so $z - \epsilon$ is an upper bound.

\nexists $z - \epsilon \leq z$, which contradicts our assumption that $z = \sup(S)$.

Case II :

Let $z = \inf(S)$.

$$d(-z, T) = |-z + x|$$

$$= x - z$$

$$T = \{ z - x \mid x \in S \}.$$

Let $z = \inf(S)$, then $\frac{z}{2} - x \leq \epsilon$.

$$x \geq z - \epsilon.$$

so $z - \epsilon$ is a lower bound.

$z - \epsilon \leq z$. so contradiction.

Limits :

(Q5)

$$\left| \frac{n}{2n+4} - \frac{1}{2} \right| = \left| \frac{n}{2n+4} - \frac{2(n+2)}{2(n+2)} \right|$$

$$= \left| \frac{-2}{2n+4} \right| = \frac{2}{2n+4}$$

$$\text{So let } \left| \frac{n}{2n+4} - \frac{1}{2} \right| < \epsilon$$

$$\frac{2}{2n+4} < \epsilon$$

$$2n+4 > \frac{2}{\epsilon}$$

$$2n > \frac{2}{\epsilon} - 4$$

$$n > \frac{1}{\epsilon} - 2$$

So choose $N \geq \frac{1}{\epsilon} - 2$, $N > 0$.

then $\forall n \geq N$ we have

$$\left| \frac{n}{2n+4} - \frac{1}{2} \right| < \epsilon$$

$$\text{So } \lim_{n \rightarrow \infty} \frac{n}{2n+4} = \frac{1}{2}.$$

$$(6) \quad \lim_{n \rightarrow \infty} \frac{2n+1}{3n+2} = \frac{2}{3}.$$

$$\text{let } \left| \frac{2n+1}{3n+2} - \frac{2}{3} \right|$$

$$= \left| \frac{3(2n+1) - 2(3n+2)}{3(3n+2)} \right|.$$

$$= \left| \frac{6n+3-6n-4}{3(3n+2)} \right|$$

$$= \left| \frac{-1}{3(3n+2)} \right| = \frac{1}{3(3n+2)} \quad n > 0.$$

$$\text{So let } \left| \frac{2n+1}{3n+2} - \frac{2}{3} \right| = \frac{1}{3(3n+2)} < \varepsilon.$$

$$\therefore 3(3n+2) > \frac{1}{\varepsilon}.$$

$$\therefore 9n+6 > \frac{1}{\varepsilon}.$$

$$\therefore 9n > \frac{1}{\varepsilon} - 6.$$

$$n > \frac{1}{9\varepsilon} - \frac{6}{9} = \frac{1}{9\varepsilon} - \frac{2}{3}.$$

Choose $N > 0$ s.t.

$$N \geq \frac{1}{9\varepsilon} - \frac{2}{3}.$$

then

$$\left| \frac{2n+1}{3n+2} - \frac{2}{3} \right| < \varepsilon.$$

(Q7)

$$\lim_{n \rightarrow \infty} \frac{n^2 + 1}{n} = \lim_{n \rightarrow \infty} \left(n + \frac{1}{n} \right) = \infty$$

Let

$$\left| n + \frac{1}{n} \right| < \varepsilon$$

$$n + \frac{1}{n} > \frac{\varepsilon}{2}$$

if

$$n > \frac{\varepsilon}{2}, \text{ then } n + \frac{1}{n} > \frac{\varepsilon}{2}$$

Choose

$$N \geq \frac{1}{\varepsilon}$$

we have:

$$\left| \frac{n^2 + 1}{n} \right| < \varepsilon \quad \forall n \geq N$$

(Q8)

$$\lim_{n \rightarrow \infty} \frac{2n^3 - 3n}{5n^3 + 4n^2 - 2}$$

Using

L'Hôpital rule:

$$f(n) = 2n^3 - 3n$$

$$g(n) = 5n^3 + 4n^2 - 2$$

$$f'(n) = 6n^2 \quad f''(n) = 12n \quad f'''(n) = 12$$

$$g'(n) = 15n^2 + 8n \quad g''(n) = 30n \quad g'''(n) = 30$$

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \lim_{n \rightarrow \infty} \frac{f'''(n)}{g'''(n)} = \frac{12}{30} = \frac{2}{5}$$

(9)

$$\lim_{n \rightarrow \infty} (\sqrt{n^2 + 4} - n)$$

$$\approx (\sqrt{n^2 + 4} - n) \times (\sqrt{n^2 + 4} + n)$$

$$= n^2 + 4 - n^2 = 4.$$

$$\therefore \lim_{n \rightarrow \infty} (\sqrt{n^2 + 4} - n)$$

$$= \lim_{n \rightarrow \infty} \left[(\sqrt{n^2 + 4} - n) \left(\frac{\sqrt{n^2 + 4} + n}{\sqrt{n^2 + 4} + n} \right) \right]$$

$$= \lim_{n \rightarrow \infty} \left[\frac{4}{\sqrt{n^2 + 4} + n} \right]$$

$$= \emptyset$$

