

# Tutorial 8 - Real Analysis.

$$(Q1) \quad (i) \quad \lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{\cos x}{1} = 1.$$

$$(ii) \quad Y = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \lim_{x \rightarrow 0} \frac{f(x)}{g(x)} \quad \text{Use L'Hôpital's Rule.}$$

$$f'(x) = \sin x \quad f''(x) = \cos x$$

$$g'(x) = 2x \quad g''(x) = 2.$$

$$Y = \lim_{x \rightarrow 0} \frac{\cos x}{2} = \frac{1}{2}.$$

$$(iii) \quad \lim_{z \rightarrow -1} \frac{z+1}{z^5+1} = \lim_{z \rightarrow -1} \frac{1}{5z^4} = \frac{1}{5} \cdot (-1)^{-4} = \frac{1}{5}.$$

$$(iv) \quad Y = \lim_{x \rightarrow 0} \frac{\tan x}{x} = \lim_{x \rightarrow 0} \frac{\sin x}{x \cos x}.$$

$$f(x) = \sin x$$

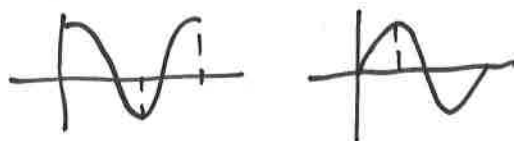
$$f'(x) = \cos x$$

$$g(x) = x \cos x.$$

$$g'(x) = \cos x - x \sin x.$$

$$\therefore Y = \lim_{x \rightarrow 0} \frac{\cos x}{\cos x - x \sin x} = 1.$$

$$(v) \quad \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{4x^2 - \pi^2}$$



$$= \lim_{x \rightarrow \frac{\pi}{2}} \frac{-\sin x}{8x} = -\frac{1}{8} \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x}{x} = -\frac{1}{8} \cdot \frac{1}{\pi/2} = -\frac{1}{4\pi}$$

(Q2)  $f(z) = \ln(z)$  .

Mean Value Theorem :

$$\exists c \in [z, w] \text{ s.t.}$$

$$f'(c) = \frac{f(z) - f(w)}{z - w} = \frac{\ln z - \ln w}{z - w} .$$

Now on the interval  $[\frac{1}{3}, 3]$  we have :

$$f'(z) = \frac{1}{z} \in \left[3, \frac{1}{3}\right] \leq 3 .$$

$$\therefore f'(c) \leq 3 .$$

$$\& \quad \frac{f(z) - f(w)}{z - w} = \frac{\ln z - \ln w}{z - w} \leq 3$$

$$\therefore \ln z - \ln w \leq 3(z - w) .$$

(Q3) Identical reasoning .

$$f(x) = \sin x, \quad f'(x) = \cos x \quad \therefore |f'(x)| \leq 1 .$$

$$\frac{\sin x - \sin y}{x - y} = f'(c) \quad c \in [x, y]$$

$$\therefore |\sin x - \sin y| \leq |x - y| .$$

(Q4)  $f(x)$  cts, diff'ble on  $[-7, 0]$ .

$$f(-7) = 3, \quad f'(x) \leq 2.$$

$$\frac{f(0) - f(-7)}{0 - (-7)} = f'(c).$$

$$\frac{f(0) - 3}{7} \leq 2.$$

$$\therefore f(0) - 3 \leq 14$$

$$f(0) \leq 17.$$

(Q5)  $f$  diff'ble on  $(1, 3)$ , cts on  $[1, 4]$ .

$$\therefore \frac{f(3) - f(1)}{3-1} = f'(c).$$

Now  $1 \leq f'(x) \leq 2$ . so  $f'(x) \leq 2$ .

$$\therefore f(3) - f(1) = 2f'(c) \leq 4.$$

$$\therefore f(3) - f(1) \leq 4.$$

$$f'(x) \geq 1.$$

$$\therefore f(3) - f(1) \geq 2.$$

$$\therefore 2 \leq f(3) - f(1) \leq 4.$$

(Q6)  $V_{\min} = 40 \text{ km/h}$  .  $T_{\max} = \frac{D}{V_{\min}} = \frac{200 \text{ km}}{40 \text{ km/h}} = 5 \text{ hr}$  .  
 $V_{\max} = 50 \text{ km/h}$  .  $T_{\min} = \frac{D}{V_{\max}} = \frac{200 \text{ km}}{50 \text{ km/h}} = 4 \text{ hrs}$  .

$\therefore T_{\min} \leq T \leq T_{\max}$

or  $4 \text{ hrs} \leq T \leq 5 \text{ hrs}$  .

(Q7)  $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  .

Let  $a$  be a double root,  $b$  be also double root. Then we have :

$$f(x) = (x-a)^2 (x-b)^2 p(x) .$$

$$f'(x) = (x-a)^2 (x-b)^2 p'(x) + 2(x-a)(x-b)^2 p(x) + 2(x-b)(x-a)^2 p(x) .$$

$a$  &  $b$  are roots, polynomial is cts in  $[a, b]$  .

So  $\exists c$  s.t.  $f'(c) = \frac{f(b) - f(a)}{b - a}$  .

Now  $f(b) = f(a)$  by construction, hence  $f'(c) = 0$  .

Then  $(c-a)^2 (c-b)^2 p'(c) + 2(c-a)(c-b)^2 p(c) + 2(c-b)(c-a)^2 p(c) = 0$  .

$$\therefore (c-a)(c-b)p'(c) + 2(c-b)p(c) + 2(c-a)p(c) = 0.$$

$$\therefore p'(c) = -2 \left[ \frac{1}{(c-a)} + \frac{1}{(c-b)} \right] p(c).$$

$$\text{or } p(c) = p'(c) = 0 \text{ hence } f(c) = 0$$

so  $\exists$  a 3<sup>rd</sup> root between the double roots.

(Q8) let  $x > y$  then  $\exists c \in [x, y]$  s.t.

$$\frac{f(x) - f(y)}{x - y} = f'(c) < 0$$

$$\text{then } f(x) - f(y) < 0$$

or  $f(x) < f(y)$  &  $f$  is strictly decreasing.

(Q9) let  $(\alpha, \beta, \gamma) \in [a, b]$   $\alpha < \beta < \gamma$ .

$$\exists c_1 \in (\alpha, \beta) \text{ s.t. } f'(c_1) = \frac{f(\beta) - f(\alpha)}{\beta - \alpha}$$

$$c_2 \in (\beta, \gamma) \text{ s.t. } f'(c_2) = \frac{f(\gamma) - f(\beta)}{\gamma - \beta}.$$

$$\therefore f'(c_1) = 0, f'(c_2) = 0.$$

Now apply MVT to  $f'(x)$  on  $(c_1, c_2)$ .

then

$$f''(c_3) = \frac{f'(c_1) - f'(c_2)}{c_1 - c_2} = 0.$$

$$\text{so } f''(c_3) = 0.$$

$$(Q10) \quad y = \sum_{n=0}^{\infty} a_n x^n.$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\therefore \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$\therefore \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + \sum_{n=2}^{\infty} n a_n x^{n-1}$$

$$= a_1 + \sum_{n=0}^{\infty} (n+2) a_{n+2} x^{n+1}$$

$$\therefore \sum_{n=0}^{\infty} a_n x^{n+1} = a_1 + \sum_{n=0}^{\infty} (n+2) a_{n+2} x^{n+1}$$

$$\sum_{n=0}^{\infty} [a_n - (n+2) a_{n+2}] x^{n+1} = a_1$$

$$\text{or } a_1 = 0 \quad a_n = (n+2) a_{n+2}$$

$$\Rightarrow a_{n+2} = \frac{a_n}{(n+2)}$$

Odd terms :

$$a_1 = 0, \quad a_{1+2} = a_3 = \frac{a_1}{1+2} = 0$$

$a_5 = 0 \dots$  , so all odd terms are zero.

Even powers :

$$a_2 = \frac{a_0}{(0+2)} = \frac{a_0}{2} = \frac{a_0}{1! 2^1}$$

$$a_4 = \frac{a_2}{2+2} = \frac{1}{4} \cdot \frac{a_0}{2} = \frac{a_0}{2^3} = \frac{a_0}{2! 2^2}$$

$$a_6 = \frac{a_4}{4+2} = \frac{1}{6} \frac{a_0}{2^3} = \frac{a_0}{3! 2^3}$$

$\vdots$

$$a_{2k} = \frac{a_0}{k! 2^k}$$

$$\begin{aligned} \S \quad y(x) &= \sum_{n=0}^{\infty} a_n x^n = \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1} \\ &= a_0 \sum_{k=0}^{\infty} \frac{1}{k! 2^k} x^{2k} \\ &= a_0 e^{x^2/2} \end{aligned}$$

We are provided  $y(0) = 1$ , hence  $a_0 = 1$

$\therefore y(x) = e^{x^2/2}$  which evidently solves

$$\frac{dy}{dx} = xy.$$