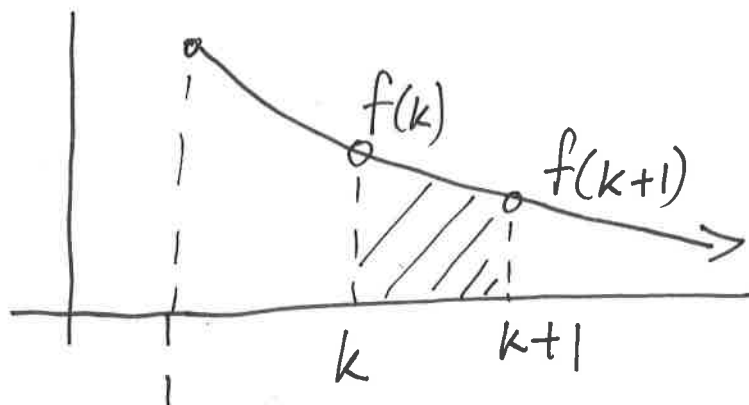


## Tutorial 10 Solutions:

(Q1)  $f$  cts, positive & decreasing on  $[1, \infty)$ .



From the graph we can see that

$$f(k+1) \leq \int_k^{k+1} f(x) dx \leq f(k)$$

Define now  $\Delta_n = \sum_{k=1}^n f(k) - \int_1^n f(x) dx$ .

$$\text{then } \Delta_{n+1} - \Delta_n = \sum_{k=1}^{n+1} f(k) - \int_1^{n+1} f(x) dx$$

$$\begin{aligned} &+ \int_1^n f(x) dx - \sum_{k=1}^n f(k) \\ &= f(n+1) - \int_n^{n+1} f(x) dx \\ &\leq f(n+1) - f(n+1) = 0. \end{aligned}$$

$\therefore \Delta_{n+1} \leq \Delta_n$  &  $\Delta_n$  decreases.

$$\Delta_n = \sum_{k=1}^n f(k) - \sum_{k=1}^{n-1} \int_k^{k+1} f(x) dx$$

$$\geq \sum_{k=1}^n f(k) - \sum_{k=1}^{n-1} f(k) = f(n) > 0$$

$\therefore \Delta_n$  is bounded below.

(Q2). If  $f$  is positive, cts & decreasing on  $[1, \infty)$  then we must have

$$\sum_{k=1}^n f(k) < \infty \quad \& \quad \int_1^{\infty} f(x) dx < \infty$$

or

$$\sum_{k=1}^n f(k) = \infty \quad \& \quad \int_1^{\infty} f(x) dx = \infty.$$

(Q3). Prove that  $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$  converges if  $\alpha > 1$ .  
diverges if  $\alpha \leq 1$

$$\int_1^{\infty} \frac{dx}{x^p} = \frac{1}{1-p} x^{1-p} \Big|_1^{\infty}$$

$$= \frac{1}{1-p} < \infty \quad p > 1$$

$\therefore \forall p > 1 \quad \sum_{n=1}^{\infty} \frac{1}{n^p} < \infty$

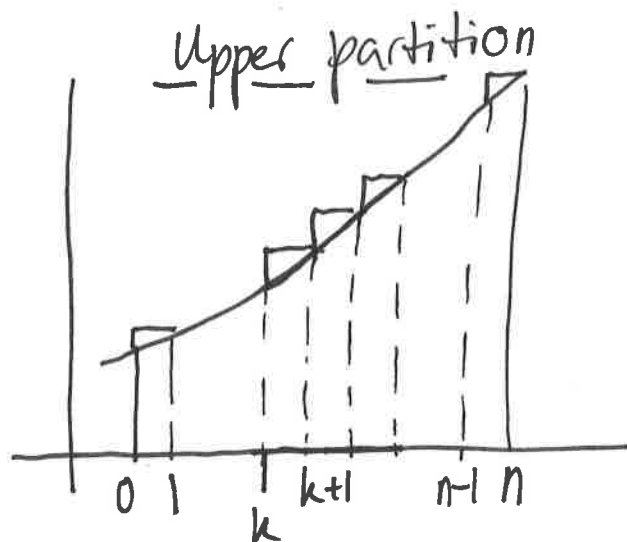
$p=1 \quad \int_1^{\infty} \frac{dx}{x} = \ln x \Big|_1^{\infty} = \ln \infty = \infty$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n} = \infty$

$$p < 1 \quad \int_1^{\infty} \frac{dx}{x^p} = \frac{1}{1-p} x^{1-p} \Big|_1^{\infty} = \infty$$

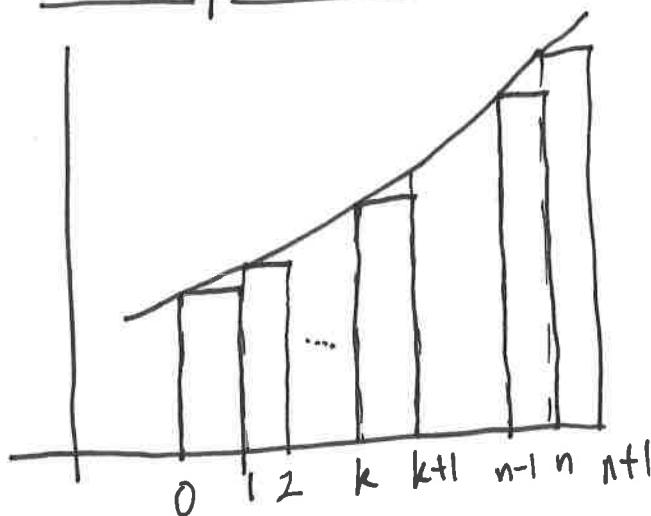
$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^p} = \infty.$$

(Q4)



$$\int_0^n f(x) dx \leq \sum_{k=1}^n f(k).$$

Lower partition



$$\sum_{k=1}^n f(k) \leq \int_1^{n+1} f(x) dx.$$

$$\therefore \int_0^k f(x) dx \leq \sum_{k=1}^n f(k) \leq \int_1^{n+1} f(x) dx.$$

$$\int_0^n \ln x \, dx = \left[ x \ln x - x \right]_0^n$$

$$= n \ln n - n.$$

$$\lim_{x \rightarrow 0} x \ln x = 0. \quad f(x) = \ln x.$$

$$\ln(n!) = \ln 1 + \ln 2 + \dots + \ln(n).$$

$\therefore$  under the conditions of the theorem we have:

$$n \ln(n) - n \leq \sum_{k=1}^n \ln(k) = \ln(n!)$$

$$\leq \int_1^{n+1} \ln x \, dx = (n+1) \ln(n+1) - n$$

$$\therefore \ln(n^n) - n \leq \ln(n!) \leq \ln[(n+1)^{n+1}] - n$$

$$\therefore n^n e^{-n} \leq n! \leq (n+1)^{n+1} e^{-n}$$

$$\therefore \frac{n^n}{n!} \leq e^n \leq \frac{(n+1)^{n+1}}{n!}.$$

(Q5) Suppose  $h$  is +ve, cts fn on  $[0, \infty)$

$$\& \quad H(x) = 1 + \int_0^x h(t) dt.$$

$h(x) \geq H(x)$ , show that  $h(x) \geq e^x$ .

$$H(x) = 1 + \int_0^x h(t) dt$$

$$\frac{dH}{dx} = h(x) \geq h(x).$$

$$\int_0^x d(\ln(H(t))) \geq \int_0^x dt = x.$$

$$H(0) = 1 \Rightarrow H(x) \geq e^x.$$

$$\therefore h(x) \geq H(x) \geq e^x \quad \text{Q.E.D.}$$

$$(Q6) \quad f_n(x) = x + \frac{1}{n} \quad \lim_{n \rightarrow \infty} f_n(x) = x = f(x)$$

$$|f(x) - f_n(x)| = \left| \frac{1}{n} \right| = \frac{1}{n}.$$

$$\therefore \text{if } n \geq N \geq \frac{1}{\epsilon}$$

$$|f_n(x) - f(x)| < \frac{1}{1/\epsilon} < \epsilon$$

$$\therefore f_n \rightarrow f \text{ uniformly.}$$

$$f_n^2(x) = \left(x + \frac{1}{n}\right)^2$$

$$= x^2 + \frac{2x}{n} + \frac{1}{n^2}$$

$$|f_n^2(x) - f^2(x)|$$

$$= \left| \frac{2x}{n} + \frac{1}{n^2} \right|$$

let  $\epsilon > 0, x > 0$ .

Triangle inequality:-

$$\left| \frac{2x}{n} + \frac{1}{n^2} \right| \leq \left| \frac{2x}{n} \right| + \left| \frac{1}{n^2} \right|$$

$$= \frac{2x}{n} + \frac{1}{n^2} < \epsilon$$

$$\therefore 2xn + 1 < \epsilon n^2$$

$$\epsilon n^2 - 2xn - 1 > 0$$

$\therefore$  ~~Roots~~ Roots are

$$n = 2x \pm \sqrt{4x^2 + 4\epsilon}$$

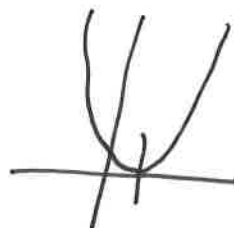
positive root only  $2\epsilon$ .

$$\Rightarrow n = \frac{2x + \sqrt{4x^2 + 4\epsilon}}{2\epsilon}$$

$$n > \frac{2x + \sqrt{4x^2 + 4\epsilon}}{2\epsilon}$$

$$n \geq g(x)$$

$\rightarrow$  convergence is NOT uniform.



(\*)

$$(Q7) \quad f_n(x) = \frac{x^{2n}}{1+x^{2n}} \quad |x| < 1 \quad x^{2n} \rightarrow 0 \quad n \rightarrow \infty$$

$$\therefore |x| < 1 \Rightarrow f_n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

for  $|x| = 1$ , we have

$$f_n(1) = \frac{1}{1+1} = \frac{1}{2}.$$

$$f_n(-1) = \frac{1}{1+1} = \frac{1}{2}.$$

$$|x| > 1$$

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{x^{2n}}{1+x^{2n}}$$

$$= \lim_{n \rightarrow \infty} \frac{1+x^{2n}-1}{1+x^{2n}}$$

$$= 1 - \lim_{n \rightarrow \infty} \frac{1}{1+x^{2n}}$$

$$= 1.$$

$\rightarrow 0$  as  $n \rightarrow \infty$

$\forall |x| > 1.$

$\therefore f_n$  is cts, but  $f$  is not.

(Q8).

Show that  $\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^4}$  is uniformly convergent on  $[0, \infty)$ .

$$|f_n(x)| = \left| \frac{\cos(nx)}{n^4} \right| \leq \frac{1}{n^4}.$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^4} < \infty$$

$\therefore$  Weierstrass M-test implies that  $\sum_{n=1}^{\infty} f_n$  is uniformly ~~conv~~ <sup>conv</sup> ~~div~~ <sup>gent</sup>.

(Q9)

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2} = \sum_{n=1}^{\infty} f_n(x)$$

$$|f_n(x)| = \left| \frac{x^n}{n^2} \right| \leq \frac{1}{n^2} \quad \forall x \in [0, 1].$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty$$

hence  $\sum_{n=1}^{\infty} f_n(x)$  is uniformly convergent.



Q10)  $f_n(x) = nx e^{-nx^2}$

$$f_n(0) = 0.$$

$$\begin{aligned} \lim_{n \rightarrow \infty} f_n(x) &= \lim_{n \rightarrow \infty} \frac{nx}{e^{nx^2}} \cdot \frac{\infty}{\infty} \Rightarrow \frac{x}{x^2 e^{nx^2}} \\ &= 0 = f_n(0) = f(0) \quad \frac{1}{x e^{nx^2}}. \end{aligned}$$

$$\begin{aligned} |f_n(x) - f(0)| &= |nx e^{-nx^2}| \\ &\leq |nx| < \epsilon. \end{aligned}$$

$\therefore$   $n$  is a function of  $x$  & convergence is not uniform.

Q11) let  $f$  be cts diff'ble on  $[-\pi, \pi]$ .

$$\text{let } b_n = \int_{-\pi}^{\pi} f(x) \sin(n\pi x) dx$$

$$= -\frac{f(x)}{n\pi} \cos(n\pi x) \Big|_{-\pi}^{\pi}$$

$$+ \int_{-\pi}^{\pi} f'(x) \frac{\cos(n\pi x)}{n\pi} dx.$$

$$\nexists |b_n| \leq \frac{1}{n\pi} 2|f(x)| + \int_{-\pi}^{\pi} \frac{|f'(x)|}{n\pi} dx.$$

$$\leq \frac{1}{n\pi} 2 \cdot \max(f(x)) + \frac{1}{n\pi} \cdot 2\pi \cdot \max(|f'(x)|)$$

$\therefore$  as  $f(x)$  is cts, diff'ble

$$\max(f(x)) = A < \infty \quad \text{on } [-\pi, \pi]$$

$$\max(|f'(x)|) = B < \infty \quad \text{on } [-\pi, \pi]$$

$$\forall |b_n| \leq \frac{2}{n\pi} A + \frac{2\pi}{n\pi} B$$

$$\forall \lim_{n \rightarrow \infty} |b_n| \rightarrow 0 \quad \text{as}$$

$$\therefore \lim_{n \rightarrow \infty} b_n = 0.$$

$$(Q12) \quad f_n(x) = \frac{x^2}{1 + e^{-x^n}} \leq x^2$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx &= \int_0^1 \lim_{n \rightarrow \infty} f_n(x) dx \\ &= \int_0^1 x^2 dx = \frac{1}{3}. \end{aligned}$$