

(Q1)

Tutorial 2 :

$$\lim_{n \rightarrow \infty} \frac{x + x^n}{1 + x^n}$$

Let $x=1$, we have $\lim_{n \rightarrow \infty} \frac{1 + 1^n}{1 + 1^n} = 1$.

For $x=-1$, we have $\lim_{n \rightarrow \infty} \frac{(-1) + (-1)^n}{1 + (-1)^n}$.

Now consider $n=2m$, $m \in \mathbb{N}$.

$$\lim_{m \rightarrow \infty} \frac{(-1) + (-1)^{2m}}{1 + (-1)^{2m}} = 0$$

but for $n=2m+1$, we have:

$$\begin{aligned} \lim_{m \rightarrow \infty} a_{2m+1} &= \lim_{m \rightarrow \infty} \frac{(-1) + (-1)^{2m+1}}{1 + (-1)^{2m+1}} \\ &= \frac{-2}{0} = -\text{infinity} = -\infty. \end{aligned}$$

Since \nexists no common limit for odd and even terms, for $x=-1$ the series diverges.

If $|x| > 1$, $\lim_{n \rightarrow \infty} \frac{x + x^n}{1 + x^n} = \lim_{n \rightarrow \infty} \frac{x}{1 + x^n} + \frac{x^n}{1 + x^n}$.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{x}{1 + x^n} &= x \lim_{n \rightarrow \infty} \frac{1}{1 + x^n} + \lim_{n \rightarrow \infty} \frac{x^n}{1 + x^n} \\ &= x \cdot 0 + 1 = 1. \end{aligned}$$

Q1 cont)

If $|x| < 1$ then for some $h > 0$ we may write:

$$|x|^n = (1+h)^{-n}$$

$$= \frac{1}{1 + nh + \frac{n(n-1)}{2!}h^2 + \dots + h^n} < \frac{1}{nh}.$$

So $|x|^n \rightarrow 0$ as $n \rightarrow \infty$ for $|x| < 1$.

$$\therefore \text{ we have } \lim_{n \rightarrow \infty} \frac{x + x^n}{1 + x^n}$$

$$= x \lim_{n \rightarrow \infty} \frac{1}{1 + x^n} + \lim_{n \rightarrow \infty} \frac{x^n}{1 + x^n}.$$

$$= x \cdot 1 + 0 = x.$$

(Q2) let $\{y_n\}_{n=1}^{\infty}$ be a decreasing sequence

such that $y_n \rightarrow 0$ as $n \rightarrow \infty$.

let $\{x_n\}_{n=1}^{\infty}$ be another sequence such that

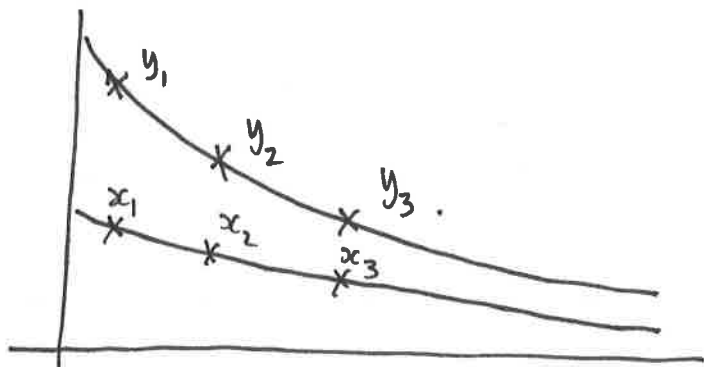
$$|x_n - l| \leq y_n.$$

Assume that $y_n \geq 0$. We can then find $N \in \mathbb{N}$

such that $\forall n \geq N, y_n < \epsilon$.

let $n \geq N$, then $|x_n - l| \leq y_n < \epsilon$.

So as $n \rightarrow \infty, |x_n - l| \rightarrow 0, \epsilon \rightarrow 0$.



(Q3)

$$\text{let } y_n = \left(1 + \frac{1}{n}\right)^n.$$

By the binomial theorem:

$$\begin{aligned} \left(1 + \frac{1}{n}\right)^n &= 1 + n \left(\frac{1}{n}\right) + \frac{n(n-1)}{2!} \left(\frac{1}{n}\right)^2 + \dots + \left(\frac{1}{n}\right)^n \\ &= 1 + 1 + \left(1 - \frac{1}{n}\right) \frac{1}{2!} + \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \frac{1}{3!} \\ &\quad + \dots + \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \times \dots \times \left(1 - \frac{(n-1)}{n}\right) \frac{1}{n!} \\ &\leq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}. \end{aligned}$$

Since $2^{n-1} \leq n!$ (prove this!)

We have:

$$\left(1 + \frac{1}{n}\right)^n \leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}}.$$

$$\text{So } \left(1 + \frac{1}{n}\right)^n \leq 1 + 1 + \frac{1}{2} + \dots + \frac{1}{2^{n-1}}.$$

$$S_n = \frac{a(1-r^n)}{1-r}.$$

$$\text{let } a=1, \text{ then } 1 + \frac{1}{2} + \dots + \frac{1}{2^{n-1}}$$

$$= \frac{1\left(1 - \left(\frac{1}{2}\right)^n\right)}{1 - \frac{1}{2}} = 2\left(1 - \left(\frac{1}{2}\right)^n\right).$$

$$\& \left(1 + \frac{1}{n}\right)^n \leq 1 + 2\left(1 - \left(\frac{1}{2}\right)^n\right) \leq 3 \text{ for } n > 0.$$

Part II - (Q3) .

Show convergence .

$$\text{let } a_{n-1} = 1 + \frac{1}{n-1} .$$

$$\begin{aligned} \text{then } \left(1 + \frac{1}{n-1}\right)^{\frac{n-1}{n}} &\leq \frac{(n-1)\left[1 + \frac{1}{n-1}\right] + 1}{n} \\ &= 1 + \frac{1}{n} . \end{aligned}$$

$$\text{So } \left(1 + \frac{1}{n-1}\right)^{n-1} \leq \left(1 + \frac{1}{n}\right)^n .$$

$$\text{or } a_{n-1} \leq a_n .$$

The sequence is bounded above, and increasing.

It therefore converges to a limit ("e").

So

$$\left(1 + \frac{1}{n}\right)^n \leq 1 + 1 + \frac{1}{2} + \dots + \frac{1}{2^{n-1}}$$

$$= 1 + \frac{(1 - (\frac{1}{2})^n)}{1 - \frac{1}{2}}$$

$$= 1 + 2(1 - (\frac{1}{2})^n)$$

$$< 3.$$

Second part: Show convergence.

$$\text{Let } a_{n-1} = 1 + \frac{1}{n-1}$$

$$\begin{aligned} \text{Now } \left(1 + \frac{1}{n-1}\right)^{\frac{n-1}{n}} &\leq \frac{(n+1)\left(1 + \frac{1}{n-1}\right) + 1}{n} \\ &= 1 + \frac{1}{n} \end{aligned}$$

$$\text{so } \left(1 + \frac{1}{n-1}\right)^{n-1} \leq \left(1 + \frac{1}{n}\right)^n$$

by taking powers, using exponents & positive functions, so no need to change inequality.

$$\text{hence } a_{n-1} \leq a_n.$$

The series is therefore bounded above & increasing, it therefore converges to some limit ("e" = 2.7169...)

(Q4) Consider $\frac{1}{n!}$, we have $\forall n \geq N$

$$\begin{aligned}\frac{1}{n!} &\leq \frac{1}{(N-1)!} \cdot \frac{1}{N^{n-N+1}} \\ &= \frac{1}{N!} \cdot \frac{1}{N^{n-N}}\end{aligned}$$

Assume $x > 0$.

then
$$\frac{x^n}{n!} \leq \frac{x^n}{(N-1)!} \cdot \frac{1}{N^{n-N+1}}$$

$$\forall x^n = x^{N-1} \cdot x^{n-N+1}$$

$$\therefore \frac{x^n}{n!} \leq \frac{x^{N-1}}{(N-1)!} \cdot \left(\frac{x}{N}\right)^{n-N+1}$$

Now, assume $N > x$, $n \geq N$.

We have $\frac{x}{N} < 1$ $\forall \left(\frac{x}{N}\right)^n \rightarrow 0$ as $n \rightarrow \infty$.

Then we can conclude

$$\frac{x^n}{n!} \leq 0 \quad \forall \quad \frac{x^n}{n!} \geq 0 \quad \text{as } n \rightarrow \infty$$

hence
$$\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0$$

(Q5)

let

$$\left(1 + \frac{1}{n}\right)^{\alpha+1} |x| > 1 \quad \text{for } |x| > 1.$$

then

$$\frac{1}{|x|} < \left(1 + \frac{1}{n}\right)^{\alpha+1}.$$

\nRightarrow

$$\left(\frac{1}{|x|}\right)^{\frac{1}{\alpha+1}} < 1 + \frac{1}{n}.$$

or

$$\left(\frac{1}{|x|}\right)^{\frac{1}{\alpha+1}} - \frac{1}{n} < 1.$$

or

$$\left(\frac{1}{|x|}\right)^{\frac{1}{\alpha+1}} - 1 < \frac{1}{n}.$$

but $|x| > 1$, so the left hand side must be > 1 , and this is a contradiction.

For any number $a \in \mathbb{R}$ we have the two mutually exclusive propositions, $x_0 > a$ vs $x_0 \leq a$, as $x_0 > a$ results in a contradiction we therefore conclude $x_0 \leq a$, $x_0 = \left(1 + \frac{1}{n}\right)^{\alpha+1} |x|$.

$$\therefore \left(1 + \frac{1}{n}\right)^{\alpha+1} |x| \leq 1.$$

Q5 cont.

Then we can conclude:

$$\left| \frac{(n+1)^{\alpha+1} x^{n+1}}{n^{\alpha+1} x^n} \right| = \left| \left(1 + \frac{1}{n}\right)^{\alpha+1} x \right|$$
$$= \left(1 + \frac{1}{n}\right)^{\alpha+1} |x|.$$

Now $\forall n \geq N$, we have

$$\left| (n+1)^{\alpha+1} x^{n+1} \right| \leq \left| n^{\alpha+1} x^n \right| \quad \text{where } |x| < 1.$$

$$\left| (n+1)^{\alpha+1} x^{n+1} \right| \leq \left| n^{\alpha+1} x^n \right|$$

implied by

$$\left| (n+1)^{\alpha+1} x^{n+1} \right| = \left(1 + \frac{1}{n}\right)^{\alpha+1} |x| \cdot \left| n^{\alpha+1} x^n \right|.$$

$$\& \quad \left(1 + \frac{1}{n}\right)^{\alpha+1} |x| \leq 1.$$

hence we conclude that $\forall n \geq N$

$$\left| n^{\alpha+1} x^n \right| \leq \left| N^{\alpha+1} x^N \right|.$$

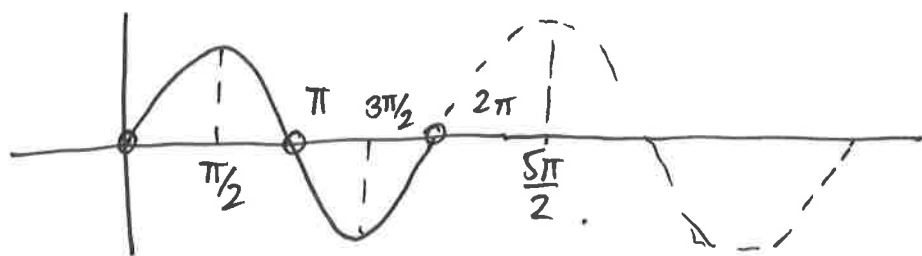
$$\therefore \left| n^{\alpha} x^n \right| \leq \frac{1}{n} \left| N^{\alpha+1} x^N \right|$$

As $n \rightarrow \infty$ the RHS $\rightarrow 0$ $\therefore \lim_{n \rightarrow \infty} n^{\alpha} x^n = 0$
for $|x| < 1$

(Q6)

Convergent subsequences.

Consider $a_n = \sin\left(\frac{n\pi}{2}\right)$, $n \in \mathbb{N}$.



The subsequences $n = 4m+1$, $m \in \mathbb{N}$, yield

$$a_n = \{a_1, a_5, a_9, \dots\}_{m=0}^{\infty}$$

$$= \left\{ \sin\left(\frac{\pi}{2}\right), \sin\left(\frac{5\pi}{2}\right), \sin\left(\frac{9\pi}{2}\right), \dots \right\}.$$

$$= \{1, 1, 1, \dots\} \text{ which obviously}$$

converges.

(Q7) Suppose $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence.

and for any N , we can find $n \geq N$

such that $x_n \geq b$.

By assumption, the sequence is bounded above & below. By the Bolzano-Weierstrass theorem (see lecture notes/textbook), there exists a convergent subsequence.

(Q7 cont.)

Let b be the lower bound of the subsequence.

Then the iterate of the process has a further subsequence which converges by another application of Bolzano-Weierstrass Thm.

We have that \exists a subsequence which converges to a limit $\geq b$ (by assumption + BW), which means we satisfy the question as req'd.

$$\begin{aligned} \text{(Q8)} \quad \text{let } a_n &= \frac{3^n + (-2)^n}{3^n - 2^n} \\ &= \frac{3^n \left(1 + \left(\frac{-2}{3}\right)^n\right)}{3^n \left(1 - \left(\frac{2}{3}\right)^n\right)}. \end{aligned}$$

$$\text{Then } \lim_{n \rightarrow \infty} \left(\frac{2}{3}\right)^n = 0$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\frac{-2}{3}\right)^n &= \lim_{n \rightarrow \infty} (-1)^n \cdot \left(\frac{2}{3}\right)^n \\ &= 0. \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} (a_n) = \lim_{n \rightarrow \infty} \frac{1}{1} = 1.$$

There is a result that states:

If a sequence converges, then every subsequence converges to the same limit.

So this implies that every subsequence in this case converges to 1. \square .

(Q9) Statement :

If a sequence converges, all subsequences converge to the same limit

So consider $(2n)^{\frac{1}{2n}} \rightarrow l$.

Now $2^{\frac{1}{2n}} \rightarrow 2^{0^+}$ as $n \rightarrow \infty$.

So we must have $2^{\frac{1}{2n}} \cdot n^{\frac{1}{2n}} \rightarrow l$

or $n^{\frac{1}{2n}} \rightarrow l$ as $n \rightarrow \infty$.

$$\therefore n^{\frac{1}{n}} = n^{\frac{1}{2n}} \cdot n^{\frac{1}{2n}}$$

hence if $\lim_{n \rightarrow \infty} n^{\frac{1}{2n}} \rightarrow l$

then $\lim_{n \rightarrow \infty} n^{\frac{1}{n}} \rightarrow l^2$.

but our hypothesis was $(2n)^{\frac{1}{2n}} \rightarrow l$

so we conclude $l^2 = l$, true only for $l = 0, 1$. It is straightforward to see that $l \neq 0$, hence $\lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1$.