

Tutorial 4 Solutions :

$$(Q1) \quad \sum_{k=1}^n k = an^2 + bn.$$

$$\text{First term } n=1, \quad \sum_{k=1}^1 k = a + b = 1.$$

$$n=2 \quad \sum_{k=1}^2 k = 1 + 2 = 3.$$

$$= a \cdot 2^2 + 2 \cdot b$$

$$= 4a + 2b.$$

$$a + b = 1 \quad \therefore b = -a + 1$$

$$\hookrightarrow 4a + 2b = 4a + 2(1-a)$$

$$= 2a + 2.$$

$$= 3.$$

$$\therefore 2a = 3 - 2 \text{ or } a = \frac{1}{2} \Rightarrow b = \frac{1}{2}.$$

$$\therefore \sum_{k=1}^n k = \frac{1}{2}(n^2 + n) = \frac{1}{2}n(n+1).$$

Let us prove using induction that this is true
 \forall positive integer n , then we have:

(1) Case $n=1$, Shown as above, also case $n=2$.

Assume true for $n=N$, so

$$\sum_{k=1}^N k = \frac{1}{2} N(N+1)$$

Now for $n=N+1$, we have:

$$\sum_{k=1}^{N+1} k = \sum_{k=1}^N k + N+1$$

$$= \frac{1}{2} N(N+1) + (N+1)$$

$$= \frac{(N+1)}{2} [N+2]$$

→ so true for $n=N+1$.

↳ Hence true for $n=1, 2, 3, \dots$

$$Q2) \quad \sum_{k=1}^N k^2 = aN^3 + bN^2 + cN.$$

$$N=1, \quad \sum_{k=1}^1 k^2 = 1$$
$$= a + b + c.$$

$$N=2, \quad \sum_{k=1}^2 k^2 = 1^2 + 2^2 = 5$$
$$= 8a + 4b + 2c.$$

$$N=3, \quad \sum_{k=1}^3 k^2 = 1^2 + 2^2 + 3^2 = 14$$
$$= 3^3 a + 3^2 b + 3c$$
$$= 27a + 9b + 3c.$$

We have a set of 3 equations in 3 unknowns,
writing as a matrix:

$$\begin{bmatrix} 1 & 1 & 1 \\ 8 & 4 & 2 \\ 27 & 9 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 14 \end{bmatrix}$$

$$\hat{A} \underline{x} = \underline{b}.$$

$$\therefore \underline{x} = \hat{A}^{-1} \underline{b}$$

$$\hat{A}^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{6} \\ -\frac{5}{2} & 2 & -\frac{1}{2} \\ 3 & -\frac{3}{2} & \frac{1}{3} \end{bmatrix}$$

$$\therefore \underline{x} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{2} \\ \frac{1}{6} \end{bmatrix} \Rightarrow a = \frac{1}{3}, b = \frac{1}{2}, c = \frac{1}{6}.$$

$$\begin{aligned} \therefore \sum_{k=1}^N k^2 &= \frac{1}{3} N^3 + \frac{1}{2} N^2 + \frac{1}{6} N \\ &= \frac{1}{6} (N + 2N^3 + 3N^2) \\ &= \frac{N}{6} [1 + 2N^2 + 3N] \\ &= \frac{N}{6} (2N+1)(N+1) \quad // \end{aligned}$$

Q2 cont.

Suppose we have

$$\sum_{k=1}^N k^2 = \frac{N}{6} (2N+1)(N+1).$$

then for $n = N+1$ we have :

$$\sum_{k=1}^{N+1} k^2 = \sum_{k=1}^N k^2 + (N+1)^2$$

$$= \frac{N}{6} (2N+1)(N+1) + (N+1)^2$$

$$= \frac{(N+1)}{6} [N(2N+1) + 6(N+1)]$$

$$= \frac{(N+1)}{6} [2N^2 + N + 6N + 6]$$

$$= \frac{(N+1)}{6} [2N^2 + 7N + 6]$$

$$= \frac{(N+1)}{6} [2N+3][N+2]$$

$$= \frac{(N+1)}{6} [2(N+1) + 1][(N+1) + 1].$$

So true for $n = N+1$.

We have shown true for $n=1, 2$, if true for $n=N$, then true for $n=N+1$, hence true for $n=1, 2, 3, 4, \dots$ by induction.

$$(Q3) \quad \sum_{k=1}^N k^n = a_{n+1} N^{n+1} + a_n N^n + \dots + a_1 N.$$

$$\text{So } \sum_{k=1}^N k^3 = a_4 N^4 + a_3 N^3 + a_2 N^2 + a_1 N.$$

Calculating for $N=1, 2, 3, 4$ we have:

$$S_n = \sum_{k=1}^n k^3$$

$$S_1 = 1, \quad S_2 = 1^3 + 2^3 = 9$$

$$S_3 = 1^3 + 2^3 + 3^3 = 36$$

$$S_4 = 1^3 + 2^3 + 3^3 + 4^3 = 100.$$

The right hand sides of these formulae are given by

$$1 = a_4 + a_3 + a_2 + a_1,$$

$$9 = 16a_4 + 8a_3 + 4a_2 + 2a_1,$$

$$36 = 81a_4 + 27a_3 + 9a_2 + 3a_1,$$

$$100 = 256a_4 + 64a_3 + 16a_2 + 4a_1.$$

Writing as a matrix equation:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 16 & 8 & 4 & 2 \\ 81 & 27 & 9 & 3 \\ 256 & 64 & 16 & 4 \end{bmatrix} \begin{bmatrix} a_4 \\ a_3 \\ a_2 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 36 \\ 100 \end{bmatrix}.$$

(Q3) cont.

$$\hat{A} \underline{x} = \underline{b} \Rightarrow \underline{x} = \hat{A}^{-1} \underline{b}.$$

Inverting the matrix & evaluating we find

$$\begin{bmatrix} a_4 \\ a_3 \\ a_2 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \\ 0 \end{bmatrix}.$$

$$\therefore \sum_{k=1}^N k^3 = \frac{1}{4} N^4 + \frac{1}{2} N^3 + \frac{1}{4} N^2.$$

$$= \frac{N^2}{4} [N^2 + 2N + 1]$$

$$= \frac{N^2}{4} (N+1)^2$$

$$= \left[\frac{N(N+1)}{2} \right]^2.$$

\hookrightarrow implies

$$\sum_{k=1}^N k^3 = \left(\sum_{k=1}^N k^2 \right)^2.$$

(Q4)

Prove $\sum_{n=1}^{\infty} \frac{1}{2n^2+3}$ is convergent.

Comparison test: If $\sum_{n=1}^{\infty} b_n$ converges, and we have $a_n < b_n$, then a_n converges.

So consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

We have $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, so this series converges.

Now $2n^2 + 3 > n^2 \quad \forall n \geq 1$.

then $\frac{1}{n^2} > \frac{1}{2n^2+3}$, hence $b_n = \frac{1}{n^2}$, $b_n > a_n$.

hence as $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2} > \sum_{n=1}^{\infty} \frac{1}{(2n^2+3)}$.

we obtain $\sum_{n=1}^{\infty} \frac{1}{(2n^2+3)} < \frac{\pi^2}{6} < \infty$.

So $\sum_{n=1}^{\infty} \frac{1}{(2n^2+3)}$ converges.

(Q5i) Determine convergence:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$$

Ratio test:

If a series given by $\sum_{n=1}^{\infty} a_n$ has

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

the series converges. Evaluating:

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(2n)!}{(2(n+1))!} \times \frac{[(n+1)!]^2}{(n!)^2} \right|$$

Assume $n > 0$, we have:

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \left[\frac{(2n)!}{(2n+2)(2n+1)(2n)!} \times \left[\frac{(n+1) \cdot n!}{n!} \right]^2 \right] \\ &= \left[\frac{(n+1)^2}{(2n+2)(2n+1)} \right] = \frac{n^2 + 2n + 1}{4n^2 + 6n + 2} \\ &= \frac{1 + \frac{2}{n} + \frac{1}{n^2}}{4 + \frac{6}{n} + \frac{2}{n^2}} \end{aligned}$$

$\therefore \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{4} < 1$, hence series converges.

$$(ii) \quad \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n = \sum_{n=1}^{\infty} a_n$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)^2}{(2n+2)(2n+1)} \left| \frac{x^{n+1}}{x^n} \right|$$

$$= \frac{(n+1)^2}{(2n+2)(2n+1)} |x|$$

$$\therefore \lim_{n \rightarrow \infty} \frac{(n+1)^2}{(2n+2)(2n+1)} |x| = \frac{|x|}{4}$$

So $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ for $0 < |x| < 4$.

ie converges. For $|x| > 4$, series diverges.

$$(iii) \quad \sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n} = \sum_{n=1}^{\infty} \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{n(\sqrt{n+1} + \sqrt{n})}$$

$$= \sum_{n=1}^{\infty} \frac{(n+1 - n)}{n(\sqrt{n+1} + \sqrt{n})}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n(\sqrt{n+1} + \sqrt{n})}$$

Now, $n(\sqrt{n+1} + \sqrt{n}) > n^{1+1/2} \quad \forall n \geq 1$.

$$\therefore \frac{1}{n(\sqrt{n+1} + \sqrt{n})} < \frac{1}{n^{1+1/2}}$$

$\therefore \sum_{n=1}^{\infty} a_n < \sum_{n=1}^{\infty} \frac{1}{n^{1+1/2}} < \infty$, hence $\sum_{n=1}^{\infty} a_n$ converges.

$$(iv) \sum_{n=1}^{\infty} n^{\alpha} x^n, \quad |x| < 1 \text{ \& \# } \alpha > 0.$$

Ratio test:

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)^{\alpha} x^{n+1}}{n^{\alpha} x^n} \right|$$

$$= \left(\frac{n+1}{n} \right)^{\alpha} |x|.$$

$$\begin{aligned} \text{Then } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= |x| \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^{\alpha} \\ &= |x| < 1 \text{ (by assumption).} \end{aligned}$$

Hence series converges.

$$(v) \sum_{n=1}^{\infty} \frac{x^n}{n!} = \sum_{n=1}^{\infty} a_n.$$

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{n!}{(n+1)!} \frac{x^{n+1}}{x^n} \right| \\ &= \frac{1}{n+1} |x|. \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = |x| \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0.$$

So converges $\forall x$. In fact,

$$e^x = \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

(Q6)
$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)(n+5)}$$

Partial fractions:

$$\frac{1}{(n+1)(n+3)(n+5)} = \frac{A}{(n+5)} + \frac{B}{(n+3)} + \frac{C}{(n+1)}.$$

Multiplying out, find:

$$A(n+3)(n+1) + B(n+5)(n+1) + C(n+5)(n+3) = 1.$$

or

$$A(n^2 + 4n + 3) + B(n^2 + 6n + 5) + C(n^2 + 8n + 15) = 1.$$

We can write this in matrix form as:

$$\begin{aligned} & (A + B + C) n^2 \\ & + (4A + 6B + 8C) n \\ & + (3A + 5B + 15C) = 1. \end{aligned}$$

So

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 6 & 8 \\ 3 & 5 & 15 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\hat{A} \underline{x} = \underline{b} \Rightarrow \hat{A}^{-1} \underline{b} = \underline{x}.$$

finding $\underline{x} = \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1/8 \\ -1/4 \\ 1/8 \end{bmatrix}.$

∴ We can write :

$$\frac{1}{(n+1)(n+3)(n+5)} = \frac{1}{8} \frac{1}{(n+5)} - \frac{1}{4} \cdot \frac{1}{(n+3)} + \frac{1}{8} \cdot \frac{1}{(n+1)}$$

$$= \frac{1}{8} \left[\frac{1}{n+5} - \frac{2}{n+3} + \frac{1}{n+1} \right]$$

$$= \frac{1}{8} \left[\left(\frac{1}{n+5} - \frac{1}{n+3} \right) + \left(\frac{1}{n+1} - \frac{1}{n+3} \right) \right]$$

$$\therefore \sum_{h=1}^{\infty} \frac{1}{(h+1)(h+3)(h+5)}$$

$$= \frac{1}{8} \sum_{h=1}^{\infty} \frac{1}{h+5} - \frac{1}{h+3} \quad (=A)$$

$$+ \frac{1}{8} \sum_{h=1}^{\infty} \frac{1}{h+1} - \frac{1}{h+3} \quad (=B)$$

$$A = \sum_{h=1}^{\infty} \frac{1}{h+5} - \frac{1}{h+3}$$

$$= \frac{1}{6} - \frac{1}{4} + \frac{1}{7} - \frac{1}{5} + \frac{1}{8} - \frac{1}{6} + \dots$$

$$= -\frac{1}{4} - \frac{1}{5} = -\frac{9}{20}$$

$$B = \sum_{h=1}^{\infty} \frac{1}{h+1} - \frac{1}{h+3}$$

$$= \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \dots$$

$$= \frac{1}{2} + \frac{1}{3} + \left(\frac{1}{4} - \frac{1}{4} \right) + \left(\frac{1}{5} - \frac{1}{5} \right) + \dots$$
$$= \frac{5}{6}$$

Q6

$$\begin{aligned}\therefore \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)(n+5)} &= \frac{1}{8} \left(\frac{5}{6} - \frac{9}{20} \right) \\ &= \frac{23}{480}.\end{aligned}$$

(Q7)

$$\frac{3n-2}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2}.$$

$$3n-2 = A(n+1)(n+2) + Bn(n+2) + C(n)(n+1).$$

$$= A(n^2 + 3n + 2) + B(n^2 + 2n) + C(n^2 + n).$$

$$= n^2(A + B + C) + n(3A + 2B + C) + 2A.$$

$$\therefore A + B + C = 0$$

$$3A + 2B + C = 3.$$

$$2A = -2.$$

$$\therefore \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 5 \\ -4 \end{bmatrix}$$

hence $\frac{3n-2}{n(n+1)(n+2)} = \frac{-1}{n} + \frac{5}{n+1} - \frac{4}{n+2}.$

Q7 cont.

$$\therefore \sum_{n=1}^{\infty} \frac{3n-2}{n(n+1)(n+2)}$$

$$= \sum_{n=1}^{\infty} \frac{5}{n+1} - \frac{1}{n} - \frac{4}{n+2}.$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n} \right) + 4 \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right)$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n} \right) = \frac{1}{2} - 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{4} - \frac{1}{3} + \dots$$

$$= -1 + \left(\frac{1}{2} - \frac{1}{2} \right) + \left(\frac{1}{3} - \frac{1}{3} \right) + \left(\frac{1}{4} - \frac{1}{4} \right) + \dots$$

$$= -1.$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(n+1)} - \frac{1}{(n+2)} \right] = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \dots$$

$$= \frac{1}{2} + \left(\frac{1}{3} - \frac{1}{3} \right) + \left(\frac{1}{4} - \frac{1}{4} \right) + \dots$$

$$= \frac{1}{2}.$$

$$\therefore \sum_{n=1}^{\infty} \frac{3n-2}{n(n+1)(n+2)} = -1 + \frac{4}{2} = 1.$$

(Q8)

$$S_N = \sum_{n=1}^N \frac{(-1)^{n+1}}{n}$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

$$S = \lim_{N \rightarrow \infty} S_N = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

Consider

$$\tilde{S}_N = 1 - \frac{1}{2} - \frac{1}{4}$$

$$+ \frac{1}{3} - \frac{1}{6} - \frac{1}{8}$$

$$+ \frac{1}{5} - \frac{1}{10} - \frac{1}{12} - \frac{1}{7}$$

$$+ \dots$$

$$= \left(1 + \frac{1}{3} + \dots + \frac{1}{2N-1} \right)$$

$$- \frac{1}{2} \left(1 + \frac{1}{3} + \dots + \frac{1}{2N-1} \right)$$

$$- \frac{1}{2} \left(\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2N} \right)$$

$$= \frac{1}{2} \left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \right]$$

$$\text{ie } \lim_{N \rightarrow \infty} \tilde{S}_N = \frac{1}{2} \lim_{N \rightarrow \infty} S_N = \frac{1}{2} S$$