

$$(Q1)a) \quad \lim_{x \rightarrow 2} \frac{x}{x+3} = \frac{2}{2+3} = \frac{2}{5}.$$

$$(b) \quad \lim_{x \rightarrow \infty} (\sqrt{x^2+4} - x)$$

Intuitively, as $x \rightarrow \infty$, $\sqrt{x^2+4} \rightarrow \sqrt{x^2} = x$

So we expect $\lim_{x \rightarrow \infty} (\sqrt{x^2+4} - x) \rightarrow 0^+$.

Proof:

$$\begin{aligned} \lim_{x \rightarrow \infty} (\sqrt{x^2+4} - x) &= \lim_{x \rightarrow \infty} \left(\frac{[\sqrt{x^2+4} - x][\sqrt{x^2+4} + x]}{x + \sqrt{x^2+4}} \right) \\ &= \lim_{x \rightarrow \infty} \left(\frac{x^2+4 - x^2}{x + \sqrt{x^2+4}} \right) \\ &= \lim_{x \rightarrow \infty} \left(\frac{4}{x + \sqrt{x^2+4}} \right) = 0. \end{aligned}$$

$$(c) \quad \lim_{x \rightarrow 0} \frac{\sin(2x)}{x} = \frac{0}{0} \quad (\Rightarrow \text{use L'Hôpital!})$$

$$f(x) = \sin 2x, \quad f'(x) = 2 \cos 2x$$

$$g(x) = x, \quad g'(x) = 1.$$

$$\therefore \lim_{x \rightarrow 0} \frac{\sin(2x)}{x} = \lim_{x \rightarrow 0} \frac{2 \cos 2x}{1} = 2 \quad \square.$$

(c) cont.

An alternative method of proof might look at the approximation $\sin u \sim u$ for u small.

In which case we recover $\sin 2x \sim 2x$ as $x \rightarrow 0$

$$\therefore \lim_{x \rightarrow 0} \frac{\sin 2x}{x} = \lim_{x \rightarrow 0} \frac{2x}{x} = 2.$$

$$(d) \quad \lim_{x \rightarrow 2} \frac{x-2}{x^2-4} = \frac{\emptyset}{\emptyset}$$

→ Use L'Hôpital rule.

$$f(x) = x-2, \quad f'(x) = 1$$

$$g(x) = x^2-4, \quad g'(x) = 2x$$

$$\therefore \lim_{x \rightarrow 2} \frac{x-2}{x^2-4} = \lim_{x \rightarrow 2} \frac{1}{2x} = \frac{1}{4}.$$

(Q2) Thm 2.2 :

$f, g : X \rightarrow \mathbb{R}$, x_0 a limit point of X ,
 c is some ~~function~~ constant. If we have

$$\lim_{x \rightarrow x_0} f(x) = L \quad \& \quad \lim_{x \rightarrow x_0} g(x) = M$$

then (a) $\lim_{x \rightarrow x_0} cf(x) = cL$ (c) $\lim_{x \rightarrow x_0} (f(x)g(x)) = LM$

(b) $\lim_{x \rightarrow x_0} (f(x) + g(x)) = L + M$ (d) $\lim_{x \rightarrow x_0} \left(\frac{f(x)}{g(x)} \right) = \frac{L}{M}$.

(Q2a) A continuous function has

$$\lim_{x_n \rightarrow x_0} f(x_n) = f(x_0) = L.$$

So, $\forall \epsilon > 0$ we can find $n \geq N$ s.t.

$$|f(x_n) - f(x_0)| < \epsilon$$

then if we have $c f(x) = \tilde{f}$

$$|f(x_n) - f(x_0)| = |f(x_n) - L| < \frac{\epsilon}{c}$$

for some n , hence

$$|c f(x_n) - c L| < \epsilon$$

$\&$ we have $\lim_{x_n \rightarrow x_0} c f(x_n) = c f(x_0) = c L.$

(b) $\lim_{x \rightarrow x_0} (f(x) + g(x)).$

Consider $\lim_{x \rightarrow x_0} f(x) = L, \lim_{x \rightarrow x_0} g(x) = M$

So $\forall \epsilon > 0$, we have $\delta_1, \delta_2 > 0$ such that

$$|x - x_0| < \delta_1 \Rightarrow |f(x) - L| < \frac{\epsilon}{2}.$$

$$|x - x_0| < \delta_2 \Rightarrow |g(x) - M| < \frac{\epsilon}{2}.$$

Let $\delta_m = \min(\delta_1, \delta_2)$ then $|x - x_0| < \delta_m$ implies

$$\begin{aligned} |f(x) + g(x) - L - M| &\leq |f(x) - L| + |g(x) - M| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

$$(c) \quad \lim_{x \rightarrow x_0} f(x)g(x) = LM.$$

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)g(x) - Mf(x) + Mf(x) - LM| \\ &= |f(x)(g(x) - M) + M(f(x) - L)| \\ &\leq |f(x)| \cdot |g(x) - M| + |M| |f(x) - L|. \end{aligned}$$

Take $\epsilon > 0$. We have by assumption that $\lim_{x \rightarrow x_0} f(x) = L$.

This implies that $f(x)$ is bounded & that we have

$$|f(x_n)| \leq A \quad \forall x_n \in \text{Dom}(f).$$

Assume $A > 0$, if $A = 0$ $|f(x_n)| = 0$ & $f(x_n) = 0 \forall n$.

Then we can write, $n \geq N_1$

$$|g(x_n) - M| \leq \frac{\epsilon}{2|A|}$$

$$\& n \geq N_2$$

$$|f(x_n) - L| \leq \frac{\epsilon}{2|M|}.$$

Taking $N = \max \{N_1, N_2\} = \max \{N_1, N_2\}$, & $n \geq N$

$$\text{we have } |f(x_n)g(x_n) - LM|$$

$$\leq |f(x)| \cdot \frac{\epsilon}{2|A|} + |M| \cdot \frac{\epsilon}{2|M|}$$

$$= |A| \cdot \frac{\epsilon}{2|A|} + |M| \cdot \frac{\epsilon}{2|M|} = \epsilon, \quad \lim_{x \rightarrow x_0} f(x)g(x) = LM.$$

Q2d)

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \frac{L}{M}$$

Proof:

$$\begin{aligned} \left| \frac{f(x)}{g(x)} - \frac{L}{M} \right| &= \left| \frac{M f(x) - L g(x)}{M g(x)} \right| \\ &= \left| \frac{M f(x) - M L + M L - L g(x)}{M g(x)} \right| \\ &\leq \frac{1}{|M| |g(x)|} \cdot [|M| \cdot |f(x) - L| + |L| \cdot |g(x) - M|] \\ &= \frac{1}{|g(x)|} [|f(x) - L|] + \left| \frac{L}{M} \right| \cdot \frac{1}{|g(x)|} \cdot |g(x) - M|. \end{aligned}$$

$g(x)$ is bounded, so $|g(x)| \leq B$, $0 < B < \infty$.

hence

$$\left| \frac{f(x)}{g(x)} - \frac{L}{M} \right| \leq \frac{1}{B} |f(x) - L| + \frac{L}{MB} |g(x) - M|.$$

Choose $\delta > 0$ s.t. $|x - x_0| < \delta$ s.t. $\forall \epsilon > 0$

$$|f(x) - L| < \frac{\epsilon B}{2}, \quad |g(x) - M| < \frac{\epsilon MB}{2L}.$$

then we have:

$$\begin{aligned} \left| \frac{f(x)}{g(x)} - \frac{L}{M} \right| &< \frac{\epsilon B}{2} \cdot \frac{1}{B} + \frac{L}{MB} \cdot \frac{\epsilon MB}{2L} \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□.

Q3(i) $f(x) = x^2$.

Consider $|f(x) - f(y)| = |x^2 - y^2|$
 $= |x - y||x + y|$.

Let $x, y < \infty$ then $|x + y| \leq M$.

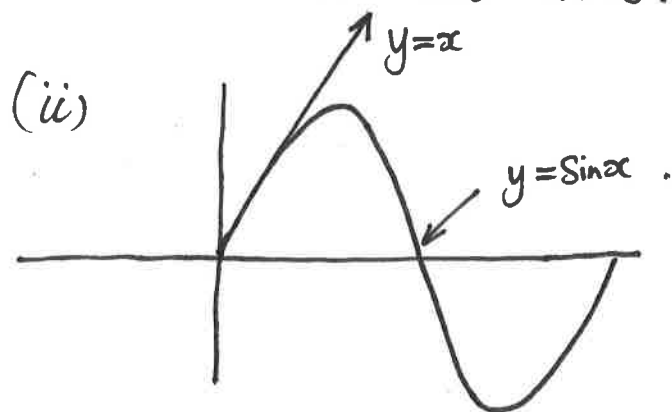
So $|f(x) - f(y)| \leq |x - y| \cdot M$.

Taking $|x - y| = \frac{\epsilon}{M}$, we have

$|x - y| = \frac{\epsilon}{M} < \delta$, so

$|f(x) - f(y)| \leq \frac{\epsilon}{M} \cdot M = \epsilon$.

$\therefore x^2$ is continuous.



$|f(x) - f(y)|$
 $= |\sin x - \sin y|$.

Double angle formula: $\sin A - \sin B = 2 \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$.

So $\sin x - \sin y = 2 \cos\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$.

$\therefore |\sin x - \sin y| = 2 \left| \cos\left(\frac{x+y}{2}\right) \right| \cdot \left| \sin\left(\frac{x-y}{2}\right) \right|$.

Using $\left| \cos\left(\frac{x+y}{2}\right) \right| \leq 1$, $\left| \sin\left(\frac{x-y}{2}\right) \right| \leq \frac{1}{2} |x - y|$

we have $|\sin x - \sin y| \leq 2 \cdot \frac{1}{2} \cdot 1 \cdot |x - y|$
 $= |x - y|$.

Q3ii) cont.

$$|\sin x - \sin y| \leq |x - y|.$$

So if $|x - y| < \delta$, we have

$$|\sin x - \sin y| \leq \delta, \text{ so if } \delta < \epsilon, \text{ we have}$$

$$|\sin x - \sin y| < \epsilon, \text{ hence } \sin x \text{ is uniformly cts.}$$

Discussion:

Examining the curve of $\sin x$, and remembering that $\sin(x + 2\pi) = \sin(x)$, if $\sin x$ is continuous over $[0, 2\pi]$, then it is cts over $[2\pi, 4\pi], [4\pi, 6\pi], \dots$ and similarly over $[-2\pi, 0], [-4\pi, -2\pi], \dots$ hence $\sin x$ is cts over \mathbb{R} .

(Q4) Continuity is preserved under linear combination of continuous functions.

$$\begin{aligned} \text{Consider } f(x) &= \sum_{n=0}^N a_n x^n \\ &= a_N x^N + a_{N-1} x^{N-1} + \dots + a_1 x^1 + a_0. \end{aligned}$$

We therefore need to prove that $f(x) = x^n$ is continuous.

$$\begin{aligned} |f(x) - f(y)| &= |x^n - y^n| \\ &= |x - y| |x^{n-1} + x^{n-2}y + \dots + y^{n-1}| \end{aligned}$$

Proof:

$$\begin{aligned} &(x - y)(x^{n-1} + x^{n-2}y + \dots + y^{n-1}) \\ &= x^n + x^{n-1}y + x^{n-2}y^2 + \dots + xy^{n-1} \\ &\quad - yx^{n-1} - y^2x^{n-2} - \dots - y^n = x^n - y^n. \end{aligned}$$

Assume $x, y < \infty$, then $\exists M > 0$, $M < \infty$

$$\text{s.t.} \quad |x^{n-1} + x^{n-2}y + \dots + y^{n-1}| \leq M.$$

$$\text{then} \quad |x^n - y^n| \leq |x - y| M.$$

\therefore If $\epsilon > 0$, if we take $|x - y| < \frac{\epsilon}{M}$ we have:

$$|x - y| < \delta \Rightarrow |x^n - y^n| < \frac{\epsilon}{M} \cdot M = \epsilon.$$

$\therefore \delta = \frac{\epsilon}{M}$, hence x^n is continuous.

Under linearity, the linear combination of monomials is polynomials is \therefore continuous.

(Q5) let f be continuous, $f(r) = r^2 \forall r \in \mathbb{I} \cap \mathbb{Q}$

Let $x \in \mathbb{I}$, $x \in \mathbb{Q}$, then $f(x) = x^2$. (all points on the interval that are rational).

Now assume we have $x \in \mathbb{I}$ but $x \notin \mathbb{Q}$.

Then x is irrational, & most importantly x is a limit point.

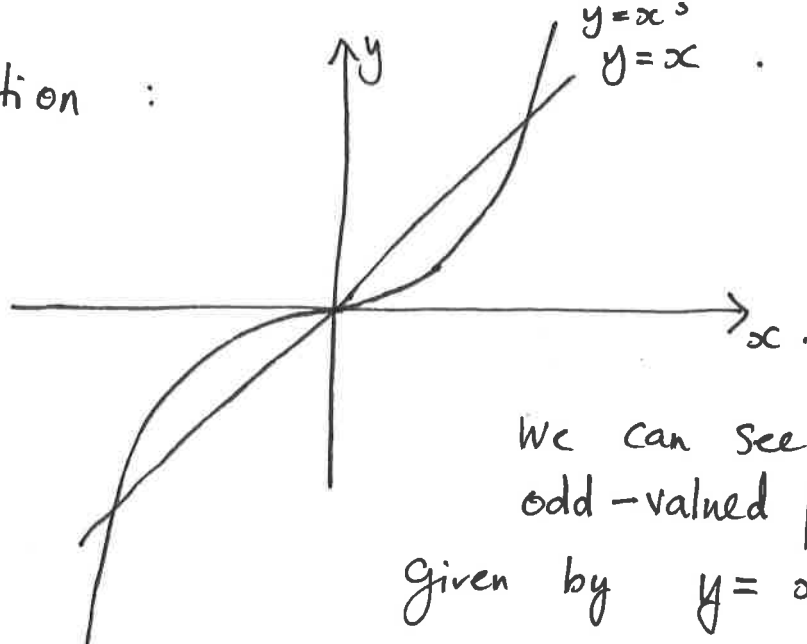
In the same way that in previous tutorials we were able to construct a ~~series~~ ^{sequence} of rationals to approximate $\sqrt{2}$, we can do likewise to approximate x .

then we have $r_n \rightarrow x$, and $\lim_{r_n \rightarrow x} f(r_n) = f(x)$

$$\text{or} \quad \lim_{r_n \rightarrow x} r_n^2 = x^2 \quad \square.$$

(Q6)

Intuition :



We can see that the odd-valued polynomials given by $y = x^{2n+1}$ cross the axis.

More generally, if we have the polynomial of odd degree defined through :

$$\begin{aligned} p_{2n+1}(x) &= a_{2n+1}x^{2n+1} + a_{2n}x^{2n} + \dots + a_1x + a_0 \\ &= a_{2n+1}x^{2n+1} + q_{2n}(x) \end{aligned}$$

then $q_{2n}(x)$ is of order $2n$.

Using the result from the previous question, we know that $p_{2n+1}(x)$ is continuous. Now, it is well known that continuous functions satisfy the intermediate value theorem.

If $f(a) < 0$ & $f(b) > 0$ then $\exists \xi \in [a, b]$ s.t. $f(\xi) = 0$.

We have
$$\lim_{|x| \rightarrow \infty} \frac{q_{2n}(x)}{a_{2n+1}x^{2n+1}} = 0$$

Assume that $a_{2n+1} > 0$, the alternative $a_{2n+1} < 0$ will follow by a sign change.

Then, by definition of the limit, if we take

$$x^{2n+1} > \left| \frac{q_{2n}(x)}{a_{2n+1}} \right| \quad \& \quad \bar{x} > x^{2n+1}$$

then $p_{2n+1}(\bar{x}) \Rightarrow 0$. On the other hand, by reversing the sign, we also can find

$$p_{2n+1}(\underline{x}) < 0$$

$$\text{where } |\underline{x}^{2n+1}| > \left| \frac{q_{2n}(\underline{x})}{a_{2n+1}} \right|.$$

Hence $\exists \xi \in [\underline{x}, \bar{x}]$ s.t. $p_{2n+1}(\xi) = 0$.

(Q7) let f be cts. fn. on $[a, b]$

$|b|, |a| < \infty$. Suppose $|f(y)| \leq \frac{1}{2} |f(x)|, y, x \in I$.

Consider the sequence y_i , then we can find

$$|f(y_1)| \leq \frac{1}{2} |f(y)| \leq \frac{1}{2^2} |f(x)|.$$

$\& y_2$ s.t.

$$|f(y_2)| \leq \frac{1}{2} |f(y_1)| \leq \frac{1}{2^3} |f(x)|$$

\therefore We have $|f(y_n)| \leq \frac{1}{2^{n+1}} |f(x)|$.

Now $y_n \in [a, b]$, hence it is bounded, and therefore contains a convergent subsequence.

(Q7) cont.

We can construct a limit, we have:

$$\lim_{n \rightarrow \infty} |f(y_n)| = 0 \text{ by inspection.}$$

We also have the convergent subsequence:

$$y_{n_k} \rightarrow \xi, \quad \xi \in I.$$

Now f is continuous, hence

$$\begin{aligned} \lim_{k \rightarrow \infty} |f(y_{n_k})| &= |f(\lim_{k \rightarrow \infty} y_{n_k})| \\ &= |f(\xi)| = 0. \end{aligned}$$

(Q8) Brouwer's fixed point theorem:

Let $f: [a, b] \rightarrow [a, b]$, so $\text{Im}(f) \subseteq \text{Dom}(f)$.

Assume f is continuous, then $f(a) \geq a$, $f(b) \leq b$.

Let $g(x) = f(x) - x$, $x \in [a, b]$.

$$g(a) = f(a) - a \geq 0$$

$$\nmid g(b) = f(b) - b \leq 0$$

$\therefore \exists \xi \in [a, b]$ s.t. $g(\xi) = 0$ under the intermediate value theorem.

Hence $g(\xi) = f(\xi) - \xi = 0$, or $f(\xi) = \xi$ \square .

(Q9) Let $J = f(I) = \{f(x) ; x \in I\}$.

Intervals are closed in that $\forall y_1, y_2 \in J$

$\nexists y_1 \leq \lambda \leq y_2$ then $\lambda \in J$.

Consider $S = \{x \mid f(x) \leq \lambda\}$, $T = \{x \mid f(x) \geq \lambda\}$.

Then S, T are non-empty. Every $x \in I$ is either in S or T .

Consider limit point $s \in S$, then $\{t_n\} \subset T$

$\nexists t_n \rightarrow s$ as $n \rightarrow \infty$. We then have

$$\lim_{n \rightarrow \infty} f(t_n) = f(s).$$

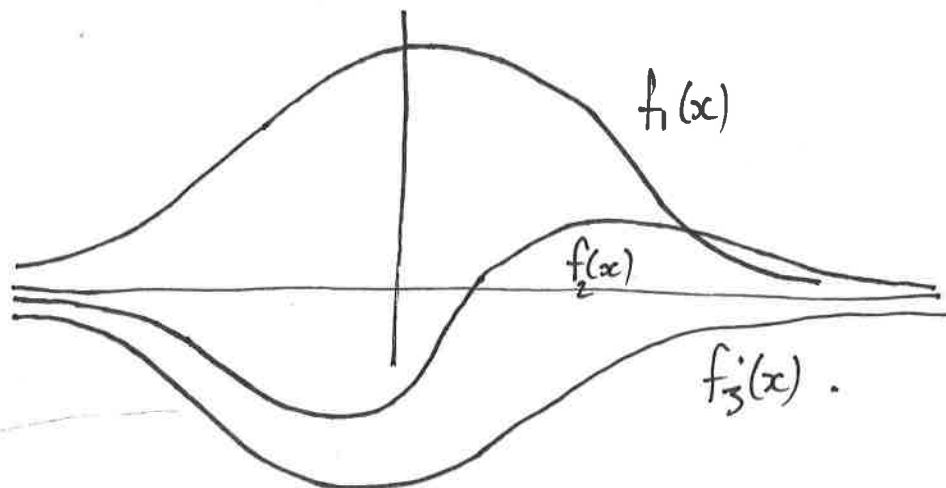
By definition, $f(t_n) \geq \lambda \forall n$, hence $f(s) \geq \lambda$.

However $s \in S$, so $f(s) \leq \lambda$.

We conclude that $f(s) = \lambda$.

(Q10) Let f be continuous on \mathbb{R} $\nexists \lim_{x \rightarrow \pm\infty} f(x) = 0$.

Picture



(Q16) Suppose we have $f(z) > 0$, then as $f(x) \rightarrow 0$ as $x \rightarrow \infty$, we have $b > z$ s.t. $\forall x > b$

$$|f(x)| < |f(z)|.$$

On the other hand, we have $\lim_{x \rightarrow -\infty} f(x) = 0$

Then $\exists a < z$ s.t. $|f(x)| < f(z)$.

hence \exists a maximum at z s.t. $f(x) < f(z) \forall x \in \mathbb{R}$.

If we take $f(z) < 0$, then we can perform the same calculation to prove \exists a minimum.