RA Assignment

Zachary Zerafa · 24557656

1) $\{a_n\}_{n\geq 1}$ $\quad \lim_{n\to\infty} a_n = A \quad a_n = b_n + A \quad \lim_{n\to\infty} b_n = 0$

$\forall \varepsilon \ \exists N : |b_n| < \varepsilon \quad \forall n > N$

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} a_k}{n}$$

$$= \lim_{n\to\infty} \frac{\sum_{k=1}^{n} b_k + A}{n}$$

$$= \lim_{n\to\infty} \frac{nA + \sum_{k=1}^{n} b_k}{n}$$

$$= \lim_{n\to\infty} A + \frac{\sum_{k=1}^{n} b_k}{n}$$

$$= A + \lim_{n\to\infty} \frac{\sum_{k=1}^{n} b_k}{n}$$

$$= A$$

$$\left| \frac{\sum_{k=1}^{n} b_k}{n} \right|$$

$$= \left| \frac{\sum_{k=1}^{N} b_k + \sum_{k=N+1}^{n} b_k}{n} \right|$$

$$< \left| \frac{\sum_{k=1}^{N} b_k + n\frac{\varepsilon}{2}}{n} \right|$$

$$< \frac{\sum_{k=1}^{N} |b_k|}{n} + \frac{\varepsilon}{2}$$

Let $\sum_{k=1}^{N} |b_k| = K$ (showing its nature as a constant)

$$= \frac{K}{n} + \frac{\varepsilon}{2}$$

Now choose n large enough such that $\frac{K}{n} \leq \frac{\varepsilon}{2}$ so that

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\therefore \left| \frac{\sum_{k=1}^{n} b_k}{n} \right| < \varepsilon \quad \forall \varepsilon \ \exists M : \forall n > M$

$$\therefore \lim_{n\to\infty} \frac{\sum_{k=1}^{n} b_k}{n} = 0$$

2)

i) $\sum_{n=1}^{\infty} \frac{1}{4n} - \frac{1}{4n+2}$

$= \frac{1}{4} - \frac{1}{6} + \frac{1}{8} - \frac{1}{10} + \frac{1}{12} - \frac{1}{14} + \dots$

$= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \dots\right)$

$= \frac{1}{2}\left(1 - 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \dots\right)$

$= \frac{1}{2}\left(1 - \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots\right)\right)$

$= \frac{1}{2}(1 - \ln 2)$

ii) $\sum_{n=1}^{\infty} \frac{1}{(4n)^3 - 4n}$

$\frac{1}{(4n)^3 - 4n} = \frac{1}{4n(4n+1)(4n-1)}$

$\frac{1}{4n(4n+1)(4n-1)} = \frac{A}{4n} + \frac{B}{4n+1} + \frac{C}{4n-1}$

$0 = A(4n+1)(4n-1) + B(4n)(4n-1) + C(4n)(4n+1) - 1$

$n=0 \Rightarrow A=-1 \qquad n=-\frac{1}{4} \Rightarrow B=\frac{1}{2} \qquad n=\frac{1}{4} \Rightarrow C=\frac{1}{2}$

$\therefore \frac{1}{(4n)^3-4n} = -\frac{1}{4n} + \frac{1}{2}\frac{1}{4n+1} + \frac{1}{2}\frac{1}{4n-1}$

$= -\frac{1}{2}\frac{1}{4n} - \frac{1}{2}\frac{1}{4n} + \frac{1}{2}\frac{1}{4n+1} + \frac{1}{2}\frac{1}{4n-1}$

$= \frac{1}{2}\left(-\frac{1}{4n} - \frac{1}{4n} + \frac{1}{4n+1} + \frac{1}{4n-1}\right)$

$= \frac{1}{2}\left(-\frac{1}{4n} - \frac{1}{4n} + \frac{1}{4n+1} + \frac{1}{4n-1} + \frac{1}{4n+2} - \frac{1}{4n-2}\right)$

$= \frac{1}{2}\left(-\frac{1}{4n} + \frac{1}{4n+1} + \frac{1}{4n-1} - \frac{1}{4n-2}\right) - \frac{1}{2}\left(\frac{1}{4n} - \frac{1}{4n-2}\right)$

$\therefore \sum_{n=1}^{\infty} \frac{1}{(4n)^3-4n} =$

$\frac{1}{2}\sum_{n=1}^{\infty}\left(-\frac{1}{4n} + \frac{1}{4n+1} + \frac{1}{4n-1} - \frac{1}{4n+2}\right) - \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{4n} - \frac{1}{4n+2}\right)$

iii) $S = \sum_{n=1}^{\infty} \frac{1}{(4n)^3 - 4n}$

$S = \frac{1}{2}\sum_{n=1}^{\infty}\left(-\frac{1}{4n} + \frac{1}{4n+1} + \frac{1}{4n-1} - \frac{1}{4n-2}\right) - \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{4n} - \frac{1}{4n+2}\right)$

$S = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{4n-1} - \frac{1}{4n} + \frac{1}{4n+1} - \frac{1}{4n-2}\right) - \frac{1}{2}\left(\frac{1}{2}(1-\ln 2)\right)$

$S = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \dots\right) - \frac{1}{2}\left(\frac{1}{2}(1-\ln 2)\right)$

$S = \frac{1}{2}\left(-\left(1-\frac{1}{2}\right) + \ln 2\right) - \frac{1}{2}\left(\frac{1}{2}(1-\ln 2)\right)$

$S = \frac{1}{2}\left(\frac{3\ln 2}{2} - 1\right)$

$S = \frac{1}{4}(3\ln 2 - 2)$

$S = \frac{1}{4}(\ln 8 - 2) \approx 0.019860$

3) f is uniformly continuous on (a,b), so...

$\forall \varepsilon \ \exists \delta : |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon \qquad x,y\in(a,b)$

Let $\lim_{x\to a^+} f(x) = f(a^+) = A$

$\therefore \lim_{x\to a^+} f(x) = A \Leftrightarrow \left(a<x<a+\delta \Rightarrow |f(x)-A|<\varepsilon\right)$

Let $a<x\leq y_0<a+\delta \ \therefore \ |x-y_0|<\delta$

$\therefore |f(x)| = |f(x)+f(y_0)-f(y_0)|$

$< \cancel{|f(x)|}\, |f(x)-f(y_0)| + |f(y_0)|$ (Tri ineq)

$< \varepsilon + |f(y_0)|$ (because $|x-y_0|<\delta$)

$\therefore a<x\leq y_0<a+\delta \Rightarrow |f(x)|<\varepsilon+|f(y_0)|$ so $|f(x)| \leq |f(y_0)|$

Let $x_n : a<x_n<a+\frac{1}{n}$

By BWT, x_{n_k} is convergent $\because x_n$ is bounded, Let $f(x_{n_k})\to A$

$\therefore |f(x)-A| = |f(x)-f(x_{n_k})+f(x_{n_k})-A|$ (inject $f(x_{n_k})$)

$< |f(x)-f(x_{n_k})| + |f(x_{n_k})-A|$ (Tri ineq)

$< |f(x)-f(x_{n_k})| + \varepsilon$

$\leq |f(x_{n_k})-f(x_{n_k})| + \varepsilon$ (Since when $a<x\leq x_{n_k}<a+\delta$, $\cancel{|f(x)|\leq|f(x_{n_k})|}$ $f(x)\leq f(x_{n_k})$ when $f(x_{n_k})>0$)

$= \varepsilon$

But for $|f(x_{n_k})|>0$...

$|f(x)-A| < |f(x)-f(x_{n_k})| + \frac{\varepsilon}{2}$

$< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}$ (since $|x-x_{n_k}|<\delta$ for $a<x\leq x_{n_k}<a+\delta$, uniform continuity says $|f(x)-f(x_{n_k})|<\frac{\varepsilon}{2}$)

$= \varepsilon$

4)

i) $f(x) = e^x \quad f^{-1}(x) = \ln x$

$f'(x) = ? \quad (f^{-1})'(x) = \frac{1}{x}$

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

$$(f^{-1})'(f(x)) = \frac{1}{(e^x)} = e^{-x}$$

$$\therefore \frac{1}{f'(x)} = \frac{1}{e^x}$$

$$\therefore f'(x) = e^x$$

iii) $y = e^{rx} \quad y^{(k)}(x) = r^k e^{rx}$

$$0 = \sum_{k=0}^{n} a_k y^{(k)}(x)$$

$$0 = \sum_{k=0}^{n} a_k r^k e^{rx}$$

$$0 = e^{rx} \sum_{k=0}^{n} a_k r^k$$

$\because ab = 0 \Leftrightarrow (a=0) \vee (b=0)$

$\because e^{rx} \neq 0$ (see ii)

$\therefore \sum_{k=0}^{n} a_k y^{(k)}(x) = 0 \Leftrightarrow \sum_{k=0}^{n} a_k r^k = 0$

ii) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

$x > 0$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$\forall x>0 \quad \frac{x^n}{n!} > 0 \quad (n \in \mathbb{N})$

$\therefore \forall x>0 \ \exists z \in \mathbb{R}_+^* : e^x = 1 + z$

$\therefore \forall x>0 \ e^x \neq 0$

$x = 0 \Rightarrow e^x = 1 \ \therefore x = 0 \Rightarrow e^x \neq 0$

$x < 0$

$e^x : x \in (0, \infty) \Leftrightarrow e^{-x} : x \in (0, \infty) = \frac{1}{e^x}$

above we proved $\forall x>0 \ e^x \in \mathbb{R}_+^*$, therefore $\frac{1}{e^x}$ is never 0 for all Positive-real-nonzero numbers and hence e^x is not 0 for all negative-real-non zero numbers

$\therefore \forall x<0 \ e^x \neq 0$

$\therefore \forall x \in \mathbb{R} \ e^x \neq 0$

5) $f: I \to \mathbb{R}$ $f \in C^3(I)$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

$$f''(a) = \lim_{h\to 0} \frac{f'(a+h)-f'(a)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{f(a+2h)-f(a+h)}{h}\right) - \left(\frac{f(a+h)-f(a)}{h}\right)}{h}$$

$$= \lim_{h\to 0} \frac{f(a+2h)-2f(a+h)+f(a)}{h^2}$$

5)

ii) $\lim_{h\to 0}\left[\frac{1}{f(a+h)-f(a)} - \frac{1}{f'(a)h}\right] = \lim_{h\to 0} \frac{f'(a)h - (f(a+h)-f(a))}{hf'(a)(f(a+h)-f(a))}$

$$= \lim_{h\to 0} \frac{f'(a)h-(f(a+h)-f(a))}{h^2 f'(a)^2}$$

$$f(a+h) = f(a)+f'(a)h + \frac{f''(a)h^2}{2!} + \frac{f'''(a)h^3}{3!} + \dots$$

$$= \lim_{h\to 0} \frac{\cancel{f'(a)h} + \cancel{f(a)} - (\cancel{f(a)} + \cancel{f'(a)h} + \frac{f''(a)h^2}{2} + \frac{f'''(a)h^3}{6} + \dots)}{h^2 f'(a)^2}$$

$$= \lim_{h\to 0} \frac{-f''(a)\cancel{h^2}}{2\cancel{h^2}f'(a)^2} + \frac{-\sum_{n=3}^{\infty} \frac{f^{(n)}(a)h^{n-2}}{n!}}{f'(a)^2}$$

$$= \lim_{h\to 0} \frac{-f''(a)}{2f'(a)^2} - \lim_{h\to 0} \sum_{n=3}^{\infty} \frac{f^{(n)}(a)h^{n-2}}{n!f'(a)^2}$$

$$= \frac{-f''(a)}{2f'(a)^2}$$

6) $u_n(x)$ is continuous on $[a,b]$

$S(x)=\sum_{n=1}^{\infty} u_n(x)$ is uniformly convergent on $[a,b]$

$\therefore \int_a^b S(x)dx = \int_a^b \lim_{K\to\infty} \sum_{n=1}^{K} u_n(x)\, dx = \lim_{K\to\infty} \int_a^b \sum_{n=1}^{K} u_n(x)dx$ (by uniform convergence)

$\lim_{K\to\infty} \int_a^b \sum_{n=1}^{K} u_n(x)dx = \lim_{K\to\infty} \sum_{n=1}^{K} \int_a^b u_n(x)dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)dx$

$\therefore \int_a^b S(x)dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)dx$