Real Analysis Assignment 2023 Solutions

Q1 Set $a_n = b_n + A$, $b_n \to 0$. Let $\varepsilon > 0$. Choose $K > 0$ such that $|b_n| < \frac{\varepsilon}{2}$ for all $n \geq K$.

Then

$$\frac{b_1 + b_2 + \cdots + b_n}{n} = \frac{b_1 + b_2 + \cdots + b_{K-1}}{n} + \frac{b_K + \cdots + b_n}{n}.$$

Now

$$\left|\frac{b_1 + b_2 + \cdots + b_n}{n} - 0\right| \leq \frac{|b_1 + \cdots + b_{K-1}|}{n} + \frac{|b_K + \cdots + b_n|}{n}$$

$$\leq \frac{B}{n} + \frac{|b_K| + \cdots + |b_n|}{n}$$

where $B = |b_1 + \cdots + b_{K-1}|$. B is fixed.
So we can choose N large enough so that for $n \geq N$

$$\left|\frac{B}{n}\right| < \frac{\varepsilon}{2}.$$

Now $\dfrac{|b_K| + \cdots + |b_n|}{n} < \dfrac{\left(\frac{\varepsilon}{2} + \cdots + \frac{\varepsilon}{2}\right)(n-K)}{n}$

$$\leq \frac{\varepsilon}{2}.$$

Thus for $n \geq N > K$

$$\left|\frac{b_1 + \cdots + b_n}{n} - 0\right| < \varepsilon$$

Hence $\dfrac{b_1 + \cdots + b_n}{n} \to 0$

Since $a_n = b_n + A$ it follows that

$$\frac{a_1 + \cdots + a_n}{n} = \frac{b_1 + \cdots + b_n + A + \cdots + A}{n}$$

$$= A + \frac{b_1 + \cdots + b_n}{n} \to A + 0 = A$$

Q2 (i) $\sum_{n=1}^{\infty}\left(\frac{1}{4n}-\frac{1}{4n+2}\right) = \frac{1}{4}-\frac{1}{6}+\frac{1}{8}-\frac{1}{10}+\frac{1}{12}-\frac{1}{14}+$

$= \frac{1}{2}\left(\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}+-\right)$

Now from lecture notes

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-- = \ln 2 \quad (*)$$

So $\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}---$

$= 1-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+-$

$= 1-\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\right)$

$= 1-\ln 2$

(ii) Partial fractions gives

$$\frac{1}{(4n)^3-4n} = \frac{1}{4n(4n-1)(4n+1)} = \frac{A}{n}+\frac{B}{4n+1}+\frac{C}{4n-1}$$

and we easily find

$$\frac{1}{(4n)^3-4n} = \frac{1}{2}\left(\frac{1}{4n-1}+\frac{1}{4n+1}-\frac{1}{2n}\right)$$

$$= \frac{1}{2}\left(\frac{1}{4n-1}-\frac{1}{4n}+\frac{1}{4n+1}-\frac{1}{4n}+\frac{1}{4n+2}-\frac{1}{4n+2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{4n-1}-\frac{1}{4n}+\frac{1}{4n+1}-\frac{1}{4n+2}-\frac{1}{4n}+\frac{1}{4n+2}\right)$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{(4n)^3-4n} = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{4n-1}-\frac{1}{4n}+\frac{1}{4n+1}-\frac{1}{4n+2}\right) - \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{4n}-\frac{1}{4n+2}\right)$$

We can split the sums up this way because they both converge and $\sum^{\infty}(a_n+b_n) = \sum^{\infty}a_n+\sum^{\infty}b_n$ if $\sum a_n<\infty$ and $\sum b_n<\infty$

Now $\sum_{n=1}^{\infty}\left(\frac{1}{4n-1}-\frac{1}{4n}+\frac{1}{4n+1}-\frac{1}{4n+2}\right)$.

$= \frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+-$

$= \ln 2 + \frac{1}{2} - 1 = \ln 2 - \frac{1}{2}$.

Thus $$\sum_{n=1}^{\infty} \frac{1}{(4n)^3 - 4n} = \frac{1}{2}\left(\ln 2 - \frac{1}{2}\right) - \frac{1}{4}(1 - \ln 2)$$
$$= \frac{3}{4}\ln 2 - \frac{1}{2}$$

(Q3) By uniform continuity, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

if $|x - y| < \delta$, $a < x < b$, $a < y < b$ we have
$$|f(x) - f(y)| < \varepsilon \qquad (*)$$

Now pick y_0 such that $a < y_0 < a + \delta$

Then $|f(x)| = |f(x) - f(y_0) + f(y_0)|$
$$\leq |f(x) - f(y_0)| + |f(y_0)| < \varepsilon + |f(y_0)|$$
from $(*)$

For $a < x \leq y_0$. Thus f is bounded on the interval $(a, y_0]$. Pick a sequence $\{x_n\}$ such that $a < x_n < a + \frac{1}{n}$. This is bounded so by the Bolzano-Weierstrass Theorem there is a convergent subsequence $\{x_{n_k}\}$ such that $\{f(x_{n_k})\}$ is a convergent subsequence of $\{f(x_n)\}$.

Let $\lim_{k \to \infty} f(x_{n_k}) = A$.

Choose an $\varepsilon > 0$ and a $\delta > 0$ so that $(*)$ is satisfied. If $a < x < a + \delta$ and $n_k \geq \frac{1}{\delta}$ then $|x - x_{n_k}| < \delta$

So $|f(x) - f(x_{n_k})| < \varepsilon$.

On the other hand $|f(x_{n_k}) - A| < \varepsilon$ if $n_k > N$ for some N large enough

Now $|f(x) - A| = |f(x) - f(x_{n_k}) + f(x_{n_k}) - A|$
$$\leq |f(x) - f(x_{n_k})| + |f(x_{n_k}) - A| < 2\varepsilon$$

Since ε is arbitrary $\lim_{x \to a^+} f(x) = A$ exists

(4)(i) Since $\ln(e^x) = x$ by the inverse function theorem if $y = e^x$

$$\frac{d}{dx} e^x = \frac{1}{\frac{d}{dy}(\ln y)} = \frac{1}{\frac{1}{y}} = y = e^x$$

(ii) Take $x = 0$. $e^0 = 1$. Now

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots$$
$$> 1 \quad \text{for all } x > 0$$

So e^x has no zeroes for $x > 0$.
Now e^x is finite for all finite x.
For $x < 0$ observe that

$$e^{-x} = \frac{1}{e^x} = \frac{1}{1 + x + \frac{x^2}{2!} + \cdots}$$
$$> 0 \text{ for all finite } x$$

(iii) Substitute into equation. We get

$$a_0 e^{rx} + a_1 r e^{rx} + \cdots + a_n r^n e^{rx}$$
$$= e^{rx}(a_0 + \cdots + a_n r^n) = 0$$

Now $e^{rx} \neq 0$. So we must have

$$a_0 + a_1 r + \cdots + a_n r^n$$

(5) By Taylor's Theorem

$$f(a+h) = f(a) + f'(a)h + \frac{f''(a)h^2}{2} + f'''(\xi)\frac{h^3}{3!}$$

$$f(a-h) = f(a) - f'(a)h + \frac{f''(a)h^2}{2} - f'''(\eta)\frac{h^3}{3!}$$

For some $\xi, \eta \in I$.

Thus $\dfrac{f(a+2h) - 2f(a+h) + f(a)}{h^2}$

(5) By Taylor's Theorem

$$f(a+2h) = f(a) + 2f'(a)h + 2f''(a)h^2 + \frac{8}{6}f'''(\xi)h^3$$

$$f(a+h) = f(a) + f'(a)h + f''(a)\frac{h^2}{2} + \frac{1}{6}f'''(\eta)h^3$$

for some $\eta, \xi \in I$.

Thus $\dfrac{f(a+2h) - 2f(a+h) + f(a)}{h^2}$

$$= \frac{1}{h^2}\Big(f(a) + 2f'(a)h + 2f''(a)h^2 + \frac{8}{6}f'''(\xi)h^3 - 2\big(f(a) + f'(a)h + \frac{1}{2}f''(a)h^2 + \frac{1}{6}f'''(\eta)h^3\big) + f(a)\Big)$$

$$= \frac{1}{h^2}\Big(f''(a) + \frac{1}{6}\big(8f'''(\xi) - \frac{1}{3}f'''(\eta)\big)h^3\Big)$$

$$= f''(a) + \big(8f'''(\xi) - \frac{1}{3}f'''(\eta)\big)h \to f''(a)$$

as $h \to 0$.

(b) $$\frac{1}{f(a+h) - f(a)} - \frac{1}{hf'(a)} = \frac{hf'(a) - (f(a+h) - f(a))}{hf'(a)(f(a+h) - f(a))}$$

$$= \frac{h\big(f'(a) - \frac{1}{h}(f(a+h) - f(a))\big)}{h\,f'(a)(f(a+h) - f(a))}$$

$$= \frac{f'(a) - \frac{1}{h}(f(a+h) - f(a))}{h\,f'(a)\left(\frac{f(a+h) - f(a)}{h}\right)}$$

$$= \frac{\Big(f'(a) - \frac{1}{h}\big(f(a) + f'(a)h + \frac{1}{2}f''(a)h^2 + \frac{1}{3!}f'''(\xi)h^3 - f(a)\big)\Big)}{h\,f'(a)\left(\frac{f(a+h) - f(a)}{h}\right)}$$

$$= \frac{-f''(a) - \frac{1}{3!}f'''(\xi)h}{2f'(a)\left(\frac{f(a+h) - f(a)}{h}\right)}$$

$$\to \frac{-f''(a)}{2(f'(a))^2} \quad \text{as } h \to 0$$

(6) Since each u_n is continuous and $S_N = \sum_{n=1}^{N} u_n(x) \to S(x)$ uniformly

$S(x)$ is continuous and hence Riemann integrable on $[a,b]$.

Now by uniform convergence

$$\lim_{N\to\infty} \sum_{n=1}^{N} \int_a^b u_n(x)\,dx = \int_a^b \sum_{n=1}^{\infty} u_n(x)\,dx$$

$$= \int_a^b S(x)\,dx$$

or $$\sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx = \int_a^b S(x)\,dx$$