**35007 Real Analysis.**

**Spring 2023 Assignment**

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**All Working Must Be Shown.**

(1) Suppose that $\{a_n\}_{n\geq 1}$ is a sequence of real numbers such that $\lim_{n\to\infty} a_n = A$. Prove that

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = A.$$

Hint: By letting $a_n = b_n + A$, where $b_n \to 0$, you can reduce to the case when $A = 0$.

(2) The purpose of the question is to evaluate the infinite sum

$$S = \sum_{n=1}^{\infty} \frac{1}{(4n)^3 - 4n}.$$

(i) Calculate the sum $\sum_{n=1}^{\infty} \left(\frac{1}{4n} - \frac{1}{4n+2}\right)$. Hint: Write out the terms. It can be expressed in terms of a sum in the notes.

(ii) Use partial fractions, together with adding and subtracting a term to prove that

$$\sum_{n=1}^{\infty} \frac{1}{(4n)^3 - 4n} = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{4n-1} - \frac{1}{4n} + \frac{1}{4n+1} - \frac{1}{4n+2}\right) \tag{0.1}$$
$$- \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{4n} - \frac{1}{4n+2}\right).$$

(iii) Determine the value of $S$. You should see a familiar pattern in the sum on the right hand side of (0.1).

(3) Let $f(a^+) = \lim_{x\to a^+} f(x)$. That is, the limit as $x \to a$ from the right. Now suppose that $f$ is uniformly continuous in an open and bounded interval $(a, b)$. Prove that $f(a^+)$ exists. This should be challenging. Hint: You will need to prove that if $\epsilon > 0$ and $a < y_0 < a + \delta$ then

$$|f(x)| \leq |f(x) - f(y_0)| + |f(y_0)| < \epsilon + |f(y_0)|,$$

for $a < x \leq y_0$. Now consider a sequence $x_n$ such that

$$a < x_n < a + \frac{1}{n}.$$

Use the Bolzano-Weierstrass Theorem to introduce a convergent subsequence, and use some properties of continuous functions.

(4) (i) Use the inverse function theorem to prove that $\frac{d}{dx}e^x = e^x$.

(ii) Prove that there is no finite solution to the equation $e^x = 0$.

(iii) Let $a_0, ..., a_n$ be constants. Prove that the function $y = e^{rx}$ satisfies $\sum_{k=0}^n a_k y^{(k)}(x) = 0$, all $x$ if and only if

$$\sum_{k=0}^n a_k r^k = 0.$$

(5) In the numerical solution of differential equations, we often replace derivatives with finite differences. Let $I$ be an open interval and suppose that $f : I \to \mathbb{R}$ is at least three times differentiable on $I$. Prove that

$$\lim_{h\to 0} \frac{f(a+2h) - 2f(a+h) + f(a)}{h^2} = f''(a).$$

The expression on the left for small $h$ is often used to approximate $f''(a)$ in an equation involving the second derivative. If $f'(x) \neq 0$ prove also that

$$\lim_{h\to 0} \left[\frac{1}{f(a+h) - f(a)} - \frac{1}{hf'(a)}\right] = -\frac{f''(a)}{2(f'(a))^2}.$$

(6) Let $u_n(x)$ be continuous functions on the closed and bounded interval $[a, b]$ and suppose that $S(x) = \sum_{n=1}^\infty u_n(x)$ and the convergence is uniform on $[a, b]$. Prove that

$$\int_a^b S(x)dx = \sum_{n=1}^\infty \int_a^b u_n(x)dx.$$