Relationship Between the Gamma and Beta Functions

Recall that the gamma funciton is defined, for $\alpha > 0$, as

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx.$$

Recall that the beta function is defined, for $a, b > 0$, as

$$\mathcal{B}(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\, dx.$$

Claim: The gamma and beta functions are related as

$$\mathcal{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Proof of Claim:

$$\begin{aligned} \Gamma(a)\Gamma(b) &= \left(\int_0^\infty x^{a-1}e^{-x}\, dx\right)\left(\int_0^\infty y^{a-1}e^{-y}\, dy\right) \\ &= \int_0^\infty \int_0^\infty x^{a-1}y^{b-1}e^{-(x+y)}\, dy\, dx \end{aligned}$$

Now make the substitution $x = uv$, $y = u(1-v)$. Note that the Jacobian of this transformation is

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -u.$$

Also, since $u = x + y$ and $v = x/(x+y)$, we have that the limits of integration for u are 0 to ∞ and the limits of integration for v are 0 to 1.

Thus

$$\begin{aligned} \Gamma(a)\Gamma(b) &= \int_0^\infty \int_0^\infty x^{a-1}y^{b-1}e^{-(x+y)}\, dy\, dx \\ &= \int_0^1 \int_0^\infty (uv)^{a-1}[u(1-v)]^{b-1}e^{-[uv+u(1-v)]}\, |-u|\, du\, dv \\ &= \int_0^1 \int_0^\infty u^{a+b-1}v^{a-1}(1-v)^{b-1}e^{-u}\ du\, dv \\ &= \left(\int_0^1 v^{a-1}(1-v)^{b-1}\, dv\right)\left(\int_0^\infty u^{a+b-1}e^{-u}\, du\right) \\ &= \mathcal{B}(a,b) \cdot \Gamma(a+b) \end{aligned}$$

as desired!