RA 2022 Final Exam Solutions

(1) (a)(i) $a_n = \dfrac{n^2+1}{2n^2+4}$, $L = \frac{1}{2}$

Let $\varepsilon > 0$. $|a_n - L| = \left|\dfrac{n^2+1}{2n^2+4} - \dfrac{1}{2}\right| = \left|\dfrac{2n^2+2-(2n^2+4)}{2(2n^2+4)}\right|$

$$= \left|\frac{-2}{2(2n^2+4)}\right| = \frac{1}{2n^2+4}.$$

Now $2n^2+4 > n$, so $\dfrac{1}{2n^2+4} < \dfrac{1}{n}$.

Let $N \in \mathbb{N}$ and $N \geq \frac{1}{\varepsilon}$

Then $n \geq N \Rightarrow |a_n - L| = \left|\dfrac{1}{2n^2+4}\right| < \dfrac{1}{n} < \varepsilon$.

So $a_n \to \frac{1}{2}$.

(b)(ii) $a_n = (-1)^n(1+(-1)^n)$ The largest this can be is when n is even

Then $a_n = (-1)^{2k}(1+(-1)^{2k}) = 2$, $n = 2k$

So $\limsup a_n = 2$

The smallest value is when n is odd

$$a_{2n+1} = (-1)^{2n+1}(1+(-1)^{2n+1})$$
$$= -(1-1) = 0$$

So $\liminf a_n = 0$

(iii) $a_n \to l$, $b_n \to s$

Then $|2a_n + 3b_n - 2l - 3s|$

$$= |2(a_n - l) + 3(b_n - s)|$$

$$\leq 2|a_n - l| + 3|b_n - s|.$$

Let $\varepsilon > 0$. Choose N_1 st $n \geq N_1 \Rightarrow |a_n - l| < \frac{\varepsilon}{2}$

Choose N_2 st $n \geq N_2 \Rightarrow |b_n - s| < \frac{\varepsilon}{3}$.

Take $N = \max\{N_1, N_2\}$ Then $n \geq N$

$$\Rightarrow |2a_n + 3b_n - 2l - 3s|$$
$$\leq 2|a_n - l| + 3|b_n - s| < \underbrace{2\tfrac{\varepsilon}{2} + 3\tfrac{\varepsilon}{3}}_{=\varepsilon}.$$

So $2a_n + 3b_n \to 2l + 3s$

Q2

(i) $a_n = \frac{1}{\sqrt{n+3}} \to 0$ as $n \to \infty$.

Also $a_{n+1} < a_n$

So by the alternating series test

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+3}} \text{ converges.}$$

ii) $a_n = \frac{5^n}{n!}$ $\quad \frac{a_{n+1}}{a_n} = \frac{5^{n+1}}{(n+1)!} \Big/ \frac{5^n}{n!}$

$$= \frac{5\,n!}{(n+1)!} = \frac{5}{n+1} \to 0 < 1$$

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, by ratio test, the series converges

iii) $\sum_{n=1}^{\infty} b_n$ is absolutely convergent

$|b_n| \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} |b_n| < \infty$.

Now let $c_n = \frac{b_n}{n}$.

$c_n < b_n$ Now $\sum_{n=1}^{\infty} |b_n|$ is convergent

So $\sum_{n=1}^{\infty} |c_n|$ is absolutely convergent.

Q3

(1) $f(x) = x^3 + 3x$. Now for $x, y \in [0,1]$

$$|f(x) - f(y)| = |x^3 + 3x - y^3 - 3y|$$
$$= |x^3 - y^3 + 3(x-y)|$$
$$= |(x-y)(x^2 + xy + y^2) + 3(x-y)|$$
$$= |x-y|\,\underbrace{|x^2 + xy + y^2 + 3|}$$
$$\le 6|x-y|.$$

Let $\delta = \frac{\varepsilon}{6}$

Then $|x-y| < \delta \Rightarrow |f(x) - f(y)| \le 6|x-y| < 6\,\frac{\varepsilon}{6} = \varepsilon$.

So f is continuous on $[0,1]$.

(ii) $\lim_{x\to 0} \frac{(1+x)^2-1}{x}$ has form $\frac{0}{0}$. Use LHôp

$= \lim_{x\to 0} \frac{2(1+x)}{1} = 2$

(iii) $\lim_{x\to\infty} f(x) = \ell$. Let $y = e^x$. The composition of functions is continuous

So $\lim_{x\to\infty} f(e^x) = \lim_{y\to\infty} f(y) = \ell$.

(4)(i) $\frac{d}{dx}x^n = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$

$= \lim_{h\to 0} \frac{x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 \ldots - x^n}{h}$

$= \lim_{h\to 0} nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdot \quad + \quad h^{n-1}$

$= nx^{n-1}$

(ii) $f(x) = \cos(2x)$. $f(\frac{\pi}{4}) = \cos(\frac{\pi}{2}) = 0$

$f'(x) = -2\sin(2x)$

$f'(\frac{\pi}{2}) = -2\sin(\frac{\pi}{2}) = -2$.

$f''(x) = -4\cos(2x)$. $f''(\frac{\pi}{4}) = -4\cos(\frac{\pi}{2}) = 0$

$f^{(iii)}(x) = 2^3\sin(2x)$. $f^{(iii)}(\frac{\pi}{4}) = 2^3$, $f^{(iv)}(\frac{\pi}{4}) = 0$

The pattern repeats with each term multiplied by 2 in the obvious sequence. So

$\cos(2x) = -2(x-\frac{\pi}{4}) + \frac{2^3(x-\frac{\pi}{4})^3}{3!} - \frac{2^5(x-\frac{\pi}{4})^5}{3!} + .$

(iii) $f_n(x) = \frac{x^n}{n^2+1}$. $|f_n(x)| \le \frac{|x^n|}{n^2+1} \le \frac{1}{n^2+1}$ $x \in [0,1)$

$\sum_{n=1}^{\infty} \frac{1}{n^2+1} < \infty$. So by M test $\sum f_n$ is uniformly convergent.

Q5

(i) $\frac{d}{dx}\int_a^{x^2} e^u\,du = e^{x^2}\frac{d}{dx}x^2$

$= 2xe^{x^2}$ by the Fundamental Theorem of Calculus

(ii) $f_n(x) = \frac{x^4\sin(n\pi x)}{x^3+n^3} \to 0$ as $n\to\infty$

and $x\in[0,1]$

Now $|f_n(x)| = \frac{|x^4\sin(n\pi x)|}{|x^3+n^3|} \leq \frac{1}{x^3+n^3}$

for $x\in[0,1]$

Now $x^3+n^3 \geq n^3$ for $x\in[0,1]$

So $\frac{1}{x^3+n^3} \leq \frac{1}{n^3}$.

Let $\varepsilon>0$. If $N \geq \frac{1}{\varepsilon^{1/3}}$, $N\in\mathbb{N}$

then $n\geq N \Rightarrow |f_n(x)-0| < \varepsilon$ all $x\in[0,1]$

So $f_n \to 0$ uniformly.

Since each f_n is Riemann integrable by the fact that the functions are continuous, the limit is integrable and by uniform convergence

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx$$

$$= 0.$$