Spring, 2022 - Mathematical and Physical Sciences Main Exam

UTS

SEAT NUMBER:

STUDENT NUMBER:

SURNAME:
(FAMILY NAME)

OTHER NAMES:

This paper and all materials issued must be returned at the end of the examination. They are not to be removed from the exam centre.

Examination Conditions:

It is your responsibility to fill out and complete your details in the space provided on all the examination material provided to you. Use the time before your examination to do so as you will not be allowed any extra time once the exam has ended.

You are **not** permitted to have on your desk or on your person any unauthorised material. This includes but not limited to:

- Mobile phones
- Smart watches and bands
- Electronic devices
- Draft paper (unless provided)
- Textbooks (unless specified)
- Notes (unless specified)

You are **not** permitted to obtain assistance by improper means or ask for help from or give help to any other person.

If you wish to **leave and be re-admitted** (including to use the toilet), you have to wait until **90 mins** has elapsed.

If you wish to **leave the exam room permanently**, you have to wait until **60 mins** has elapsed.

You are not permitted to leave your seat (including to use the toilet) during the final 15 mins.

During the examination **you must first seek permission** (by raising your hand) from a supervisor before:

- Leaving early
- Using the toilet
- Accessing your bag

Misconduct action will be taken against you if you breach university rules.

Declaration: I declare that I have read the advice above on examination conduct and listened to the examination supervisor's instructions for this exam. In addition, I am aware of the university's rules regarding misconduct during examinations. I am not in possession of, nor do I have access to, any unauthorised material during this examination. I agree to be bound by the university's rules, codes of conduct, and other policies relating to examinations.

Signature:

Date:

35007 Real Analysis

Time Allowed: 120 minutes.

Reading time: 0 minutes.

Reading time is for reading only. You are not permitted to write, calculate or mark your paper in any way during reading time.

Take-home exam
Open book
The exam window is 3 hours from the scheduled start time of your exam. You must submit your exam within the exam window

Permitted materials for this exam:

Your notes

Materials provided for this exam:

None

Students please note: All work must be shown. Online resources are not permitted.

Question 1 (5+2+3 = 10 marks)

(i) From the definition of convergence, prove that the sequence $a_n = \dfrac{n^2+1}{2n^2+4}$ converges to $1/2$.

(ii) Find $\limsup a_n$ and $\liminf a_n$ where $a_n = (-1)^n(1+(-1)^n)$.

(iii) If $a_n \to l$ and $b_n \to s$, prove that $2a_n + 3b_n \to 2l + 3s$.

Question 2 (3+3+4 = 10 marks)

(i) Prove that the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$ is convergent.

(ii) Use the ratio test to determine whether the series $\sum_{n=1}^{\infty} \frac{5^n}{n!}$ converges or diverges.

(iii) If the series $\sum_{n=1}^{\infty} b_n$ is absolutely convergent, prove that $\sum_{n=1}^{\infty} \frac{b_n}{n}$ is convergent. Hint: Let $c_n = b_n/n$ and use the most important convergence test.

Question 3 (3+3+4 = 10 marks)

(i) Prove that $f(x) = x^3 + 3x$ is continuous on $[0, 1]$ using the $\epsilon - \delta$ definition.

(ii) Calculate

$$\lim_{x \to 0} \frac{(1+x)^2 - 1}{x}.$$

(iii) Recall that $\lim_{x\to\infty} f(x) = l$ if given $\epsilon > 0$ we can find M such that $|f(x) - l| < \epsilon$ for all $x > M$. Suppose that f is continuous and $\lim_{x\to\infty} f(x) = l$. Prove that $\lim_{x\to\infty} f(e^x) = l$.

Question 4 (3+4+3 = 10 marks)

(i) From the definition of the derivative prove that $\frac{d}{dx}x^n = nx^{n-1}$.

(ii) Determine the Taylor series for $f(x) = \cos(2x)$ about the point $a = \frac{\pi}{4}$.

(iii) Prove that the series $\sum_{n=1}^{\infty} \frac{x^n}{n^2+1}$ converges uniformly on $[0, 1]$.

Question 5 (3+7 = 10 marks)

(i) Calculate

$$\frac{d}{dx}\int_a^{x^2} e^u du.$$

(1) Prove that the sequence $f_n(x) = \dfrac{x^4 \sin(n\pi x)}{x^3 + n^3}$ converges uniformly on $[0, 1]$. Hence calculate

$$\lim_{n\to\infty} \int_0^1 f_n(x) dx.$$