Final Exam

35007 Real Analysis

Spring 2023

School of Mathematical and Physical Sciences, UTS.

Instructions for Students.

Answer all 5 questions.

All working must be shown.

Questions are of equal value.

You may use your notes.

Wolfram alpha and other computational systems may not be used. This includes AI software.

You have a three hour upload window.

Question 1 (4+2+4=10 marks).

(i) From the definition, prove that the sequence $a_n = \dfrac{2n^2+4}{3n^2+5}$ converges to $2/3$.

(ii) Let $a_n = \sin(n\pi)$. Determine $\liminf a_n$ and $\limsup a_n$.

(iii) Suppose that $a_n \to 2$ and $b_n \to 1$. To what value does $a_n b_n$ converge? Prove your answer from the definition of convergence.

Question 2 (3+3+4=10 marks).

(i) Show that the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is conditionally convergent.

(ii) Prove that $\sum_{n=1}^{\infty} \frac{n}{2n^2+4}$ is divergent.

(iii) Use partial fractions to determine the value of the series

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n-1)} = \frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \cdots .$$

The partial fractions decomposition has the form

$$\frac{1}{(2n+1)(2n-1)} = \frac{A}{2n+1} + \frac{B}{2n-1}.$$

Question 3 (4+4+2=10 marks).

(i) Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose that there is a positive constant M such that $|f'(c)| \leq M$ for all c. Prove that f is uniformly continuous on $\mathbb{R}$. Hint: Mean Value Theorem.

(ii) Prove from the $\epsilon - \delta$ definition of continuity that $f(x) = x^4 + 2x^2$ is continuous on $[0, 3]$.

(iii) Calculate $\lim\limits_{x \to \infty} \dfrac{\ln x}{\ln(x^2 + 1)}$.

Question 4 (4+3+3=10 marks).

(i) Calculate the Taylor series expansion for $f(x) = \cos(2x)$ about the point $a = \frac{\pi}{4}$.

(ii) Determine the radius of convergence of the series $\displaystyle\sum_{n=1}^{\infty} nx^{2n}$.

(iii) Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^3}$ is uniformly convergent for $x \in [0, 1]$.

Question 5 (2+3+5=10 marks).

(i) Determine the value of the integral $\int_0^\infty e^{-x^4} dx$ in terms of the Gamma function.

(ii) Prove that $\int_0^1 t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}} dt = \pi$.

(iii) Let
$$f_n(x) = \frac{\cos(nx)}{x^4 + n^2}.$$
Prove that $f_n \to 0$ uniformly on $[0, 1]$. Hence determine
$$\lim_{n\to\infty} \int_0^1 f_n(x)dx.$$
Hint: You need to show that we can choose $N \in \mathbb{N}$ such that $n \geq N$ implies $|f_n(x) - 0| < \epsilon$ for all $x \in [0, 1]$. Observe that $x^4 + n^2 \geq n^2$.