RA solutions. Test Monday

(1) $a_n = \dfrac{2n-3}{4n+1}$ $L = \dfrac{1}{2}$.

Clearly $|a_n - L| = \left|\dfrac{2n-3}{4n+1} - \dfrac{1}{2}\right| = \left|\dfrac{4n-6-(4n+1)}{8n+2}\right|$

$= \left|\dfrac{-7}{8n+2}\right| = \dfrac{7}{8n+2}$

Now $8n+2 > n$, so $\dfrac{1}{8n+2} < \dfrac{1}{n}$

or $\dfrac{7}{8n+2} < \dfrac{7}{n}$.

Let $\varepsilon > 0$. Let $N \in \mathbb{N}$ and $N > \frac{7}{\varepsilon}$.

Then $n \geq N \Rightarrow |a_n - L| < \dfrac{7}{8n+2} < \dfrac{7}{n} < \dfrac{7\varepsilon}{7} = \varepsilon$.

So $a_n \to \frac{1}{2}$.

(2) Notice that $n + 2\sqrt{n} < 3n$.

So $\dfrac{1}{n+2\sqrt{n}} > \dfrac{1}{3n}$.

Now $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n} = \frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{n} = \infty$

Since $\displaystyle\sum_{n=1}^{N} \frac{1}{n+2\sqrt{n}} > \sum_{n=1}^{N}\frac{1}{3n} \to \infty$

we have $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n+2\sqrt{n}}$ is divergent.

3) $f(x) = 2x^4+3$. Now $|f(x)-f(y)| = |2x^4 - 2y^4|$

$= 2|x^2-y^2||x^2+y^2|$

$= 2|x-y||x+y||x^2+y^2|$

Now on $[0,3]$ so on $[0,3]$ $|x+y||x^2+y^2| \leq |3+3||9+9|$
$= 108$.

Let $\varepsilon > 0$.

Let $\delta = \frac{\varepsilon}{216}$. Then $|x-y| < \delta$

$\Rightarrow |f(x)-f(y)| = |x-y||x+y||x^2+y^2|$

$< 216\,\dfrac{\varepsilon}{216} = \varepsilon$. So f is continuous.

(4) $f(x) = e^{2x}$. By the MVT on $[0, \ln 2]$

$$\frac{f(x) - f(y)}{x-y} = f'(c), \quad c \in (0, \ln 2)$$

Now $f'(x) = 2e^{2x}$

So $|f'(c)| \le 2e^{2\ln 2} = 8$.

$\therefore |f(x) - f(y)| = |e^{2x} - e^{2y}| \le 8|x-y|$.

(5) $f(x) = \sinh(2x)$

$f(0) = 0, \quad f'(0) = 2, \quad f''(0) = 0, \quad f'''(0) = 2^3$

etc.

So

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \cdots$$

$$= 2x + \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \cdots$$

$$= \sum_{j=0}^{\infty} \frac{(2x)^{2j+1}}{(2j+1)!}$$