RA. Solutions Thursday

(1) $a_n = \frac{2n^2+3}{5n^2+9}$, $L = \frac{2}{5}$

$|a_n - L| = \left| \frac{2n^2+3}{5n^2+9} - \frac{2}{5} \right| = \left| \frac{10n^2+15-10n^2-18}{5(5n^2+9)} \right|$

$= \frac{3}{25n^2+45}$.

$25n^2+45 > n$. So $\frac{3}{25n^2+45} < \frac{3}{n}$.

Let $\varepsilon > 0$. Let $N \in \mathbb{N}$, $N > \frac{3}{\varepsilon}$

Then $n \geq N \Rightarrow |a_n - L| = \frac{3}{25n^2+45} < \frac{3\varepsilon}{3} = \varepsilon$

So $a_n \to L$

2) $a_n = \frac{n!}{e^n}$, $\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{e^{n+1}} \Big/ \frac{n!}{e^n}$

$= \frac{n}{e} \to \infty$ as $n \to \infty$

So series diverges

3) $f(x) = x^3 + 4x$

$|f(x) - f(y)| = |x^3 - y^3 + 4x - 4y|$

$\leq |x^3 - y^3| + 4|x-y|$

$= |x-y|(|x^2+xy+y^2| + 4)$

$\leq |x-y|(12+4) = 16|x-y|$.

Let $\varepsilon > 0$, $\delta = \frac{\varepsilon}{16}$. $|x-y| < \delta$

$\Rightarrow |f(x) - f(y)| \leq 16|x-y| = 16\frac{\varepsilon}{16} = \varepsilon$

So f is continuous

(4) $f(x) = \sin(4x)$ on $[0,\infty)$ $|f'(x)| \le 4$

Let $(x,y) \in [0,\infty)$

Then by MVT $\frac{f(x)-f(y)}{x-y} = \frac{\sin(4x)-\sin(4y)}{x-y}$

$= f'(c)$

$c \in [x,y)$. Since $|f'(c)| \le 4$

$$|\sin(4x) - \sin(4y)| \le 4|x-y|$$

(5) $\frac{1}{1+z} = 1 - z + z^2 - z^3 + \ldots$

$\therefore \frac{x}{1+x^2} = x\left(1 - x^2 + x^4 - x^6 + \ldots\right)$