RA solutions . Tuesday

1) $a_n = \dfrac{n^2}{4n^2+1}$ $\quad |a_n - L| = \left|\dfrac{n^2}{4n^2+1} - \dfrac{1}{4}\right| = \left|\dfrac{4n^2-4n^2-1}{4(4n^2+1)}\right|$

$L = \frac{1}{4}$ $\quad = \dfrac{1}{16n^2+4}$

Now $16n^2+4 > n$.

So $\dfrac{1}{16n^2+4} < \dfrac{1}{n}$. Let $\varepsilon > 0$

Let $N \in \mathbb{N}$, $N > \frac{1}{\varepsilon}$

Then

$n \geq N \Rightarrow |a_n - L| = \dfrac{1}{16n^2+4} < \dfrac{1}{n} < \varepsilon$

So $a_n \to L$

2) $n+n^2 > n^2$ $\therefore \dfrac{1}{n+n^2} < \dfrac{1}{n^2}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$

Now $\displaystyle\sum_{n=1}^{N} \frac{1}{n+n^2} < \sum_{n=1}^{N} \frac{1}{n^2} \longrightarrow \infty$

So $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+n^2}$ is convergent. $\left[\text{in fact } \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n} = 1\right]$

(3) $f(x) = 2x^3+3$ $\quad |f(x)-f(y)| = 2|x^3-y^3|$

$= 2|x-y||x^2+xy+y^2|$

Now on $[0,3]$ $\quad 2|x^2+xy+y^2| \leq 2|9+9+9|$

$= 54.$

Let $\varepsilon > 0$. Choose $\delta = \dfrac{\varepsilon}{54}$.

Then $|x-y| < \delta \Rightarrow |f(x)-f(y)| \leq 2|x-y||x^2+xy+y^2|$

$< \dfrac{\varepsilon}{54} \cdot 54 = \varepsilon$

So f is continuous on $[0,3]$.

(4) $f(x) = \cos(2x)$. By the MVT

$\dfrac{f(x)-f(y)}{x-y} = f'(c)$ $\quad$ any $[x,y] \subset [0,\infty)$,

$f'(x) = -2\sin(2x)$ so $|f'(c)| \leq 2$

$\therefore\ |f(x)-f(y)| = |\cos(2x)-\cos(2y)| \leq 2|x-y|$.

(5) Let $f(x) = \sinh(2x)$

$f(0)=0,\ f'(0)=2,\ f''(0)=0,\ f'''(0)=2^3$

etc

So $\sinh(2x) = 2x + \dfrac{(2x)^3}{3!} + \dfrac{(2x)^5}{5!} + \cdots$

$$= \sum_{n=0}^{\infty} \frac{(2x)^{2n+1}}{(2n+1)!}$$

Hence $x\sinh(2x) = \displaystyle\sum_{n=0}^{\infty} \frac{(2x)^{2n+2}}{(2n+1)!}$